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MASATOSHI FUKUSHIMA
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Preface

Professor Masatoshi Fukushima is one of the most influential probabilists of our times. His fundamental work on Dirichlet forms and Markov processes made Hilbert space methods a tool in stochastic analysis and by this he opened the way to several new developments. His impact on a new generation of probabilists in his native country as well as in many other countries can hardly be overstated.

In publishing a selection of his seminal papers we aim to serve the community, and at the same time we want to express our appreciation of a highly respected, humane scholar.

All owners of copyrights of papers being included in the Selecta (see the list at the end of this volume) followed the old and good tradition to grant permission for a reproduction in a Selecta without any charge. Unfortunately this is no longer a policy adopted by all publishers. Therefore we are especially grateful to those who by their generosity continue to support the mathematical community in keeping the tradition of publishing selected or collected works alive.

The editors' thanks go to all who supported us in our enterprise, in particular we want to mention Professor T. Uemura (Kansai University), Dr. K. P. Evans (Swansea University) as well as S. Albroscheit and Dr. R. Plato (Walter de Gruyter).

Swansea, Kumanoto and Sendai
Fall 2009

Niels Jacob
Yōichi Ōshima
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Curriculum vitae

1935, August 23	Born in Osaka, Japan
1959	Graduated from Kyoto University, Faculty of Science
1961	Master degree from Kyoto University, Faculty of Science
1962	Research Associate of Nagoya University, Faculty of Science
1963	Lecturer of Kyoto University, College of General Education
1966	Associate Professor of Tokyo University of Education, Faculty of Science
1967	Doctor of Science from Osaka University
1969–1971	Post Doctral Fellow, University of Illinois, Department of Mathematics, Urbana-Champaign
1972	Associate Professor of Osaka University, Faculty of Science
1977	Professor of Osaka University, College of General Education
1980	Visiting Professor Bielefeld University, Faculty of Physics
1990	Professor of Osaka University, Faculty of Engineering Sciences
1998	Professor Emeritus, Osaka University
1998	Professor of Kansai University, Faculty of Engineering
2003	Awarded Analysis Prize of Mathematical Society of Japan
2006	Retired from Kansai University
2007	Honorary Professor Swansea University

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Professor Masatoshi Fukushima – Scholar and Mentor*

Let me start with some remarks on the scientific achievement of Professor Fukushima. First investigations of Professor Fukushima were related to diffusions under boundary conditions and this led him to consider related Hilbert spaces. From here the road was open to Dirichlet forms, Beurling's and Deny's axiomatisation of the notion of energy in potential theory. The breakthrough was the 1971 paper in the Transactions of the American Mathematical Society where a Hunt process was constructed associated to a given regular Dirichlet form. This result was immediately recognised by experts as an outstanding one, already 1978 Professor Fukushima was an invited speaker in the session on Probability Theory in the ICM in Helsinki. Adding to this I would like to mention two other honours Professor Fukushima received: The Analysis Prize from the Japanese Mathematical Society and being an Invited Lecturer of the London Mathematical Society.

The 1971 paper and the book "Dirichlet forms and Markov Processes" published in 1980 in English changed the landscape of modern probability theory. Of course there are many other contributions of Professor Fukushima's worth mentioning:

- exceptional sets and refinements
- plurisubharmonic functions (especially the Acta Mathematica Paper with M. Okada)
- stochastic analysis on fractals
- boundary behaviour and traces of Markov processes

and many more. Of particular importance was and is the influence of his work to mathematical physics. The construction of diffusion processes on infinite dimensional state spaces highly depend on his seminal contribution. The impact of the "new" book "Dirichlet Forms and Symmetric Markov Processes" written jointly with Y. Oshima and M. Takeda can hardly be overstated.

In 1990 when participating in a conference organised by Professor Kunita in Nagoya, I once had a coffee with Professor Shinzo Watanabe. In our discussion Professor Watanabe stated that for him in the 1970's there had been two major breakthroughs in probability theory: Malliavin calculus and Fukushima's theory of Hunt processes

*This is a slightly modified version of a banquet speech given during a meeting to celebrate Professor Fukushima's 70th birthday.

associated with Dirichlet forms – Professor Watanabe is a very modest person – he should have added his own contributions too. However there is no doubt, Professor Fukushima's work is of lasting impact.

Mathematics is an international subject and Professor Fukushima was and is acting on the international stage. This is natural to all the outstanding Japanese probabilists raised in Professor K. Itô's school. Professor Fukushima's work on the 1971 paper was partly done when being in the U.S.A. with the late Professor Doob. Here he also established contacts to Martin Silverstein and he, Professor Fukushima, always emphasised Professor Silverstein's contributions to our subject. Professor Fukushima was very engaged in the series of Japanese-Russian seminars on probability theory and a frequent visitor to European countries. In addition he was a great help and supporter of many young non-Japanese mathematicians, Y. Lejan, J. Kim, Z.-M. Ma, Z.-Q. Chen, J. Ying, ..., and of course I have to mention myself.

Being a world-open mathematician, a scholar who has visited (partly for longer periods) many countries is one aspect of Professor Fukushima. There is another one: As many cosmopolitans he is deeply rooted in his own culture, i.e. in the traditional Japanese culture. This makes any encounter with him also an encounter with Japan. Through him I myself as well as quite a few of my students and colleagues learnt to appreciate Japan's great culture.

Professor Fukushima belongs to the generation whose childhood was in war-time – I recommend everyone to read Kappa Senoh's "A Boy Called H" to get a feeling of what this meant to his generation. He as many other Japanese scientists, writers and artists of his generation took on the difficult task to assure his country a respected place in the modern post-war world – and they were rather successful.

A final more personal word. Due to his relations to the late Professor Heinz Bauer we met first in Erlangen. I am very happy that I could build on Professor Bauer's contacts and could even extend them. This refers to the Oberwolfach meetings on Dirichlet forms or a German-Japanese exchange programme supported by DFG and JSPS. Moreover I am grateful that several of my own (former) PhD students could not only visit Japan but could start to build up their own contacts bringing the collaboration to the next generation.

You, Professor Fukushima, have made lasting contributions to Mathematics and you have been over the years of great support to many of us. For this we are grateful and we are looking forward to many further years to come with your company.

Niels Jacob

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Osaka J. Math.
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A CONSTRUCTION OF REFLECTING BARRIER BROWNIAN MOTIONS FOR BOUNDED DOMAINS

MASATOSHI FUKUSHIMA

(Received February 20, 1967)

1. Introduction

Let D be an arbitrary bounded domain of the N-dimensional Euclidean space R^N.

We will call a function $G_\alpha(x, y)$ $(\alpha>0,\ x,\ y\in D,\ x\neq y)$ a (continuous) *resolvent density* on D if the following conditions are satisfied:

(G. 1) $\quad G_\alpha(x, y)\geqq 0, \quad \alpha>0, \quad x, y\in D, \quad x\neq y\,.$

(G. 2) $\quad \alpha\int_D G_\alpha(x, y)dy\leqq 1, \quad \alpha>0, \quad x\in D\,.$[1)]

(G. 3) $\quad G_\alpha(x, y)-G_\beta(x, y)+(\alpha-\beta)\int_D G_\alpha(x, z)G_\beta(z, y)dz=0\,,$

$\alpha,\ \beta>0,\ x,\ y\in D,\ x\neq y\,.$

(G. 4) For fixed $\alpha>0$, $G_\alpha(x, y)$ is continuous in (x, y) on $D\times D$ off the diagonal.

A resolvent density on D is called *conservative* if the equality holds in (G.2) for all $\alpha>0$ and all $x\in D$.

In this paper, we will construct a conservative resolvent density on D and show that it determines a diffusion process (that is, a strong Markov process having continuous trajectories) which takes values in a natural enlarged state space D^*. When the relative boundary ∂D of D is sufficiently smooth, our diffusion process is shown (Theorem 6) to be the well known reflecting barrier Brownian motion on $D\cup\partial D$. For this reason, our process for an arbitrary D may be considered the reflecting barrier Brownian motion in an extended sense.

A function $p(t, x, y)$, $t>0$, x, $y\in D$, will be called a (continuous) *transition density* on D, if it satisfies the following conditions:

(T. 1) $\quad p(t, x, y)\geqq 0,\ t>0,\ x,\ y\in D\,.$

1) dy denotes the Lebesgue measure on D.

(T. 2) $\int_D p(t, x, y)dy \leqq 1,\ t>0,\ x\in D$.

(T. 3) $p(t+s, x, y)=\int_D p(t, x, z)p(s, z, y)dz,\ t, s>0,\ x, y\in D$.

(T. 4) $p(t, x, y)$ is continuous in $(t, x, y)\in(0, +\infty)\times D\times D$.

A transition density for which the equality holds in (T. 2) for all $t>0$ and all $x\in D$ will be called *conservative*.

Let $p^0(t, x, y)$ be the transition density corresponding to the absorbing barrier Brownian motion on D[2]. Set

(1. 1) $$G_\alpha^0(x, y)=\int_0^{+\infty} e^{-\alpha t}p^0(t, x, y)dt, \quad \alpha>0, \quad x, y\in D,$$

then $G_\alpha^0(x, y)$ is a resolvent density on D and can be expressed in the form,

(1. 2) $$G_\alpha^0(x, y)=\Pi_\alpha(x, y)-\tilde{E}_x(e^{-\alpha\tau}\Pi_\alpha(X_\tau, y)) \quad \alpha>0,\ x,y\in D,$$

where,

$$\Pi_\alpha(x, y)=\int_0^{+\infty} e^{-\alpha t}\frac{1}{(2\pi t)^{N/2}}e^{-(|x-y|^2/2t)}dt,\ x, y\in R^N \text{[3]},$$

$\tilde{E}_x$ is the expectation with respect to the standard Brownian measure $\tilde{P}_x$, $x\in D$, and τ is the first exist time from D of the Brownian path X_t.

A function u defined on an open set U of R^N will be called *α-harmonic* on U if

$\left(\alpha-\frac{1}{2}\Delta\right)u(x)=0$, $x\in U$, where Δ is the Laplacian; $\Delta=\sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$. For functions u, v on D, we set

(1. 3) $$(u, v)=\int_D u(x)v(x)dx,$$

$$\boldsymbol{D}(u, v)=\int_D (\text{grad } u, \text{grad } v)(x)dx.$$

For each $\alpha>0$, let $\boldsymbol{H}_\alpha$ be the Hilbert space formed by all α-harmonic functions on D with the following norm:

(1. 4) $$\boldsymbol{D}_\alpha(u, u)=\boldsymbol{D}(u, u)+2\alpha(u, u)<+\infty.$$

In section 2, we shall prove the following.

Theorem 1.
(i) *For each $\alpha>0$ and each $x\in D$, there exists a unique y-function $R_\alpha^x(y)$ $=R_\alpha(x, y)$ in $\boldsymbol{H}_\alpha$ such that the equation*

2) cf. [8].
3) $|x-y|$ denotes the distance between x and y.

$$(1.5) \qquad \boldsymbol{D}(R_\alpha^x, v)+2\alpha(R_\alpha^x, v)=2v(x)$$

holds for all $v \in \boldsymbol{H}_\alpha$.
(ii) *Set*

$$G_\alpha(x, y)=G_\alpha^0(x, y)+R_\alpha(x, y), \quad \alpha>0, \quad x, y \in D.$$

Then $G_\alpha(x, y)$ *is a conservative resolvent density on* D, *symmetric in* $x, y \in D$.
(iii) *Denote by* $\boldsymbol{B}(D)$ (*resp.* $\boldsymbol{C}(D)$) *the collection of all bounded measurable* (*resp. bounded continuous*) *functions on* D. *The operator* G_α *defined by*

$$(1.6) \qquad G_\alpha f(\cdot)=\int_D G_\alpha(\cdot, y) f(y)\, dy, \quad f \in \boldsymbol{B}(D),$$

maps $\boldsymbol{B}(D)$ *into* $\boldsymbol{C}(D)$. *Moreover, if* $f \in \boldsymbol{C}(D)$, *then* $\lim_{\alpha \to +\infty} \alpha G_\alpha f(x)=f(x)$, $x \in D$.
(iv) *Suppose that* K_1 *and* K_2 *are compact,* D_1 *is open and* $K_1 \subset D_1 \subset K_2 \subset D$. *Then,* $\sup_{x \in K_1, y \in D-K_2} G_\alpha(x, y)$ *is finite.*
(v) *There is a unique transition density* $p(t, x, y)$ *on* D *satisfying*

$$(1.7) \qquad G_\alpha(x, y)=\int_0^{+\infty} e^{-\alpha t} p(t, x, y)\, dt, \quad \alpha>0, \quad x, y \in D.$$

$p(t, x, y)$ *is conservative and* $\int_D p(t, x, y) f(y)\, dy$ *is continuous in* $(t, x) \in (0, +\infty) \times D$ *for any* $f \in \boldsymbol{B}(D)$.

When ∂D is suffciently smooth, the transition density in Theorem 1 turns out to be the fundamental solution of the heat equation $\left(\frac{\partial}{\partial t}-\frac{1}{2}\Delta_x\right)u(t, x)=0$, $t>0$, $x \in D$, with the boundary condition $\frac{\partial}{\partial n_x} u(t, x)=0$, $t>0$, $x \in \partial D$, where n_x is the inner normal at the point $x \in \partial D$. Indeed, assuming that ∂D is in class $\boldsymbol{C}^3$, let us denote the latter by $\dot{p}(t, x, y)$, $t>0$, $x, y \in D$. Then, it is a transition density and
$\dot{R}_\alpha(x, y)=\int_0^{+\infty} e^{-\alpha t} \dot{p}(t, x, y)\, dt-G_\alpha^0(x, y)$ is an α-harmonic function in the class $\boldsymbol{C}^1(D \cup \partial D)$ as a function of y[4]. Hence, we have only to show that $\dot{R}_\alpha^x=\dot{R}_\alpha(x, \cdot)$ satisfies equation (1.5). Applying the Green formula to the identity $-\frac{\partial}{\partial n_y} \dot{R}_\alpha^x(y)=\frac{\partial}{\partial n_y} G_\alpha^0(x, y)$, $y \in \partial D$, we see that

$$(1.8) \qquad \frac{1}{2} \boldsymbol{D}(\dot{R}_\alpha^x, v)+\alpha(\dot{R}_\alpha^x, v)=\frac{1}{2} \int_{\partial D} \frac{\partial}{\partial n_y} G_\alpha^0(x, y) v(y) \sigma(dy)$$

4) cf. [7]. $\boldsymbol{C}^1(D \cup \partial D)$ denotes the totality of continuously differentiable functions on $D \cup \partial D$.

holds for every $v \in \boldsymbol{C}^1(D \cup \partial D)$, $\sigma(dy)$ standing for the surface Lebesgue measure of ∂D. The right hand side of (1.8) is the α-harmonic function with the boundary value v. A usual limiting procedure leads us to the validity of (1. 5) for $\dot{R}_\alpha^x$ and for every $v \in \boldsymbol{H}_\alpha$[5].

We call a compact set D^* a *compactification* of D if D^* contains D as an open dense subset and the relative topology of D in D^* is equivalent to the original Euclidean topology there. In Sections 3 and 4, the following theorem will be proved.

Theorem 2.

(i) *There is a compactification D^* of D such that $p(t, x, y)$, $t>0$, of Theorem* 1 *is extended to $(x, y) \in D^* \times D$ uniquely in a certain way and the extended function (denoted again by $p(t, x, y)$) satisfies conditions* (T. 1), (T. 2) *and* (T. 3) *for $x \in D^*$ and $y \in D$.*

(ii) *There exists a Markov process $\boldsymbol{X} = \{X_t, P_x, x \in D^*\}$ possessing the following properties.*

(a) *For each Borel set A of D^*,*

$$P_x(X_t \in A) = \int_{D \cap A} p(t, x, y)\, dy, \quad t>0, \quad x \in D^*\,.$$

(b) *$\boldsymbol{X}$ is continuous;*

$$P_x(X_t \text{ is continuous in } t \text{ for every } t \geqq 0) = 1, \quad x \in D^*\,.$$

(c) *$\boldsymbol{X}$ has the strong Markov property.*

(d) *The part of $\boldsymbol{X}$ on the set D is the absorbing barrier Brownian motion there; for every $x \in D$ and Borel set A of D,*

$$P_x(X_t \in A;\, t<\tau) = \int_A p^0(t, x, y)\, dy, \quad t>0\,,$$

τ being the first exit time from D.

(e) *There exists a Borel subset D_1^* of D^* containing D such that*

$$P_x(X_0 = x) = 1, \quad x \in D_1^*\,,$$
$$P_x(X_0 = x) = 0, \quad x \in D^* - D_1^*\,.$$

Moreover $\boldsymbol{X}$ is conservative on D_1^*; $P_x(X_t \in D_1^*$ for every $t \geqq 0) = 1$, $x \in D_1^*$.

---

5) For $v \in \boldsymbol{H}_\alpha$, we can find a sequence of functions $v_n \in \boldsymbol{C}^1(D \cup \partial D)$ which converges to v with respect to the norm $\sqrt{\boldsymbol{D}(v, v) + 2\alpha(v, v)}$. The boundary function of v_n, then, converges to that of v (which is determined by v, σ-almost everywhere on ∂D) in $\mathbf{L}^2(\sigma)$ sense.

Let D^* be the completion of D of the Martin-Kuramochi type with respect to the resolvent density $G_1(x, y)$ of Theorem 1[6]. In Section 3, we will show that this D^* satisfies condition (i) of Theorem 2 and we will derive a right continuous strong Markov process $\boldsymbol{X}$ on D^* satisfying the condition (ii, a). Moreover, the property (ii, d) will be verified.

We now give some conments on the completion in Theorem 2. The first remark is that the validity of Theorem 2 (i) for our D^* owes essentially to the conservativity of the resolvent density of Theorem 1. The second remark is concerned with the strong Markov property of $\boldsymbol{X}$ in the theorem. D. Ray [20] proved that, under certain hypotheses, to a resolvent on a compact space corresponds a strong Markov process. One of Ray's hypotheses is that the given resolvent makes invariant the space of all continuous functions. This condition, however, is not necessarily satisfied by the resolvent (operator) induced by the density function $G_\alpha(x, y)$ on the extended space D^*. Therefore, Ray's original theorem is not enough to verify the strong Markov property of our $\boldsymbol{X}$. We will reproduce the proof of H. Kunita and H. Nomoto [9]; they treat a wide class of Markov processes including ours. (T. Watanabe pointed out that there is another nice completion for which Ray's original results can be applied in themselves. Under this completion, Theorem 2 is still valid and the conservativity of the resolvent density is irrelevant. See [11].) Third, we note that $D^*-D_1^*$ is the set of all branching points in Ray's sense [20][7]. Finally, statements (b) and (e) imply that almost all trajectories starting from a non-branching point never contact with branching points.

In order to complete the proof of Theorem 2, we must show the continuity of trajectories of $\boldsymbol{X}$. Section 4 will be devoted to the proof of the above feature of $\boldsymbol{X}$ by a potential-theoretic method. First, $G_1(x, y)$ of Theorem 1 will be extended to $(x, y)\in D^*\times D^*$ and every summable 1-excessive function will be expressed as the integral of the kernel $G_1(x, y)$ with a unique measure on D_1^* (Theorem 3). Second, we will introduce the notion of the Dirichlet norm $|||u|||_X$ of the function $u(x)=\int_D G_1(x, y)f(y)\,dy$, $x\in D^*$, $f\in \boldsymbol{B}(D)$, with respect to our process $\boldsymbol{X}$ and we will then show (Theorem 4) that the equality $|||u|||_X^2 =\int_D (\operatorname{grad} u, \operatorname{grad} u)(x)\,dx$ holds for each function of above type. This is a characteristic feature of reflecting barrier Brownian motions. Owing to the result of M. Motoo and S. Watanabe [18], this characteristic property of $\boldsymbol{X}$ permits us to conclude that, for any additive functional A_t of $\boldsymbol{X}$ such as $E_x(A_t)=0$ and $E_x(A_t^2)<+\infty$, $x\in D^*$, $t>0$, the stochastic integral $\int \chi_{D_1^*-D}\,dA_s$ vanishes

6) cf. [12] and [13].

7) For $x\in D^*-D_1^*$, the life time of our path X_t is either infinity or zero P_x-almost every-where (see Lemma 3.4 and 3.5).

identically (Theorem 5). Here, $\chi_{D_1^*-D}$ is the indicator function of D_1^*-D. This property of $\boldsymbol{X}$ will exclude the possibility that the trajectories of $\boldsymbol{X}$ have jumps on D_1^*-D with positive probability.

Acknowledgement. K. Ito and N. Ikeda suggested me the problem treated here and encouraged me throughout the research. The analysis of the continuity of trajectories performed in §3 and §4 is in debt to valuable advices by H. Kunita and S. Watanabe. I wish to thank them all for their kindness. Thanks are due to K. Sato and T. Watanabe for their kind and useful opinion on the manuscript.

2. Construction of resolvent density (proof of Theorem 1)

From now on, we fix an arbitrary bounded domain D of R^N. The following criterion for a function on D to be α-harmonic is easily verified and it will be frequently used in this paper.

Lemma 2.1. *Let α be positive number. A function u on D is α-harmonic, if and only if, for each ball B with closure contained in D, it holds that*

$$u(x)=\int_{\partial B} h_\alpha^B(x, y)u(y)\sigma(dy), \quad x\in B,$$

where $\sigma(dy)$ is the surface Lebesgue measure of ∂B and $h_\alpha^B(x, y)=\frac{1}{2}\frac{\partial}{\partial n_y}\,{}^BG_\alpha^0(x, y)$, $x\in B$, $y\in\partial B$, ${}^BG_\alpha^0(x, y)$ being the resolvent density defined by (1. 1) for the ball B.

For functions u and v on D, define $\boldsymbol{D}(u, v)$ and (u, v) by (1. 3). Put

$$(2.\ 1) \qquad \boldsymbol{D}_\alpha(u, v)=\boldsymbol{D}(u, v)+2\alpha(u, v), \quad \alpha>0\,.$$

Denote by $\boldsymbol{H}_\alpha$ the space of all α-harmonic functions u satisfying $\boldsymbol{D}_\alpha(u, u)<+\infty$.

Lemma 2.2. *For each $\alpha>0$, $\boldsymbol{H}_\alpha$ forms a real Hilbert space with the inner product $\boldsymbol{D}_\alpha(u, v)$. Moreover, any Cauchy sequence of functions in $\boldsymbol{H}_\alpha$ with respect to the norm $\sqrt{\boldsymbol{D}_\alpha(u, u)}$ converges on D uniformly on any compact subset of D.*

Proof. Suppose that $u_n\in\boldsymbol{H}_\alpha$, $n=1, 2, \cdots$, and $\boldsymbol{D}_\alpha(u_n-u_m, u_n-u_m)\xrightarrow[n,m\to+\infty]{} 0$. Let K be any compact subset of D. Choose $\varepsilon>0$ smaller than the distance of K with ∂D. Let $B_\varepsilon(x)$ be the ball with radius ε centered at x in K. Applying Lemma 2. 1 to the α-harmonic function u_n-u_m, we have

$$(2.\ 2) \qquad \begin{aligned} &u_n(x)-u_m(x)\\ &=\frac{1}{V_\varepsilon}\int_{B_\varepsilon(x)}\eta_\alpha(|y-x|)(u_n(y)-u_m(y))\,dy, \quad x\in K, \end{aligned}$$

where V_ε is the volume of $B_\varepsilon(x)$, $|y-x|$ is the distance between x and y, and $\eta_\alpha(r)$ is a function of real $r>0$ which depends only on $\alpha>0$ and satisfies

$0<\eta_\alpha(r)<1$. The Schwarz inequality applied to (2.2) leads to

$$(u_n(x)-u_m(x))^2 \leqq \frac{1}{V_\varepsilon}(u_n-u_m,\, u_n-u_m)$$

$$\leqq \frac{1}{2\alpha V_\varepsilon} \boldsymbol{D}_\alpha(u_n-u_m,\, u_n-u_m), \quad x \in K .$$

Thus, u_n converges to a function u on D uniformly on any compact subset of D. By virtue of Lemma 2. 1, u is also α-harmonic on D and the first derivatives of u_n converge to those of u uniformly on any compact subset of D. On the other hand, since u_n, $n=1, 2, \cdots$, form a Cauchy sequence with respect to the norm $\boldsymbol{D}_\alpha$, one can find, for any $\varepsilon>0$, a compact subset $K\subset D$ such that

$$\int_{D-K} |\text{grad } u_n|^2(x)\,dx + 2\int_{D-K} u_n(x)^2\,dx < \varepsilon$$

uniformly in n. Hence, $u\in \boldsymbol{H}_\alpha$ and $\boldsymbol{D}_\alpha(u_n-u,\, u_n-u) \xrightarrow[n\to+\infty]{} 0$.

Lemma 2.3. *Let $\alpha>0$ be fixed.*
(i) *For each $x\in D$, there exists a function $u^{(x)}\in \boldsymbol{H}_\alpha$ uniquely such that*

(2. 3) $\qquad \boldsymbol{D}_\alpha(u^{(x)}, v)=2v(x)$, *for any $v\in \boldsymbol{H}_\alpha$.*

(ii) *The function $u^{(x)}$ in* (i) *is a unique element of $\boldsymbol{H}_\alpha$ minimizing the value of the functional $\Psi(u)=\boldsymbol{D}_\alpha(u, u)-4u(x)$ on $\boldsymbol{H}_\alpha$.*

Proof. (i). For a fixed $x\in D$, define the linear mapping Φ from $\boldsymbol{H}_\alpha$ to R^1 by $\Phi(v)=2v(x)$, $v\in \boldsymbol{H}_\alpha$. Φ is continuous by the latter half of Lemma 2.2. The Riesz theorem implies (i).

(ii). We have only to notice the equality $\Psi(u)=\Psi(u^{(x)})+\boldsymbol{D}_\alpha(u-u^{(x)}, u-u^{(x)})$, $u\in \boldsymbol{H}_\alpha$.

DEFINITION 1. For $\alpha>0$ and $x, y\in D$, denote by $R_\alpha^x(y)=R_\alpha(x, y)$, $y\in D$, the function $u^{(x)}(y)$ of Lemma 2.3.

DEFINITION 2. Let $G_\alpha^0(x, y)$ by the resolvent density defined by (1.1). Define the function $G_\alpha(x, y)$, $\alpha>0$, $x, y\in D$, by

$$G_\alpha(x, y)=G_\alpha^0(x, y)+R_\alpha(x, y) .$$

Before examining those properties of $G_\alpha(x, y)$ stated in Theorem 1, we prepare three lemmas.

An *exhaustion* of D is a sequence of domains D_n, $n=1, 2, \cdots$, such that the closure of D_n is contained in D_{n+1} and D_n converges monotonically to D. An exhaustion $\{D_n\}$ of D is called *regular* if ∂D_n are of class $\boldsymbol{C}^3$.

Lemma 2.4. *Let $\alpha>0$ be fixed.*
(i) *Any non-negative α-harmonic function on D is either identically zero on D or strictly positive on D.*
(ii) *The function $w=1-\alpha G_{\alpha}^{0}1$ is strictly positive on D. Moreover w is the unique element in $\boldsymbol{H}_{\alpha}$ satisfying*

(2.4) $\boldsymbol{D}_{\alpha}(w, v)=2\alpha(1, v)$ *for all* $v\in\boldsymbol{H}_{\alpha}$.

Proof. (i). Since Lemma 2.1 implies that the value of an α-harmonic function at any point of D is a weighted volume mean on the ball centered at the point, property (i) is verified in the same manner as in the case of harmonic functions.

(ii). It is evident, by expression (1.2) of G_{α}^{0}, that w is α-harmonic and strictly positive on D. In order to show identity (2.4), consider a regular exhaustion $\{D_n\}$ of D.

Put $w_n=\chi_{D_n}-\alpha\, {}^{n}G_{\alpha}^{0}\chi_{D_n}$, where χ_{D_n} is the indicator function of D_n, ${}^{n}G_{\alpha}^{0}\chi_{D_n}(x)=\int_{D_n}{}^{n}G_{\alpha}^{0}(x, y)dy$ and ${}^{n}G_{\alpha}^{0}(x, y)$ is the resolvent density (1. 1) for D_n. The function w_n is α-harmonic in D_n, converges to w monotonically and (consequently) uniformly on any compact subset of D. On account of Lemma 2. 1, the derivatives of w_n converge to those of w on D. Denote by $\boldsymbol{D}_{\alpha}^{n}(\ ,\ )$ the integral (2. 1) on D_n. Since w_n belongs to $C^1(D_n\cup\partial D_n)$, we can apply Green's formula to w_n and $v\in\boldsymbol{H}_{\alpha}$, obtaining $\boldsymbol{D}_{\alpha}^{n}(w_n, v)=2\alpha(\chi_{D_n}, v)$. This equality implies the inequality $\boldsymbol{D}_{\alpha}^{n}(w_n, w_n)-4\alpha(\chi_{D_n}, w_n)\leqq\boldsymbol{D}_{\alpha}^{n}(v, v)-4\alpha(\chi_{D_n}, v)$ for all $v\in\boldsymbol{H}_{\alpha}$. Letting n tend to infinity and using Fatou's lemma, we obtain

$$\boldsymbol{D}_{\alpha}(w, w)-4\alpha(1, w)\leqq\boldsymbol{D}_{\alpha}(v, v)-4\alpha(1, v).$$

Thus, $w\in\boldsymbol{H}_{\alpha}$, and if we put, instead of v, $w+\varepsilon v$ in the inequality above, we arrive at (2.4). The proof of the uniqueness is straightforward.

Lemma 2.5. *Take an exhaustion $\{D_n\}$ of D arbitrarily. Let ${}^{n}R_{\alpha}^{x}(y)$ and ${}^{n}G_{\alpha}(x, y)$, $\alpha>0$, $x, y\in D_n$ be the functions defined by Definition 1 and Definition 2 for the domain D_n. Then, $\lim_{n\to+\infty}{}^{n}G_{\alpha}(x, y)=G_{\alpha}(x, y)$, $\alpha>0$, $x, y\in D$, $x\neq y$. Moreover, for each $x\in D$, the equality*

(2. 5) $\lim_{n\to+\infty}{}^{n}R_{\alpha}^{x}(y)=R_{\alpha}^{x}(y), \quad y\in D,$

holds and the convergence is uniform in y on any compact subset of D.

Proof. Let ${}^{n}G_{\alpha}^{0}(x, y)$ be the resolvent density defined by (1. 1) for the domain D_n. Since ${}^{n}G_{\alpha}^{0}(x, y)$ increases to $G_{\alpha}^{0}(x, y)$ we have only to discuss the convergence of ${}^{n}R_{\alpha}^{x}$ to R_{α}^{x}.

Let us fix $x\in D$. We can assume that x is in D_1. For each D_n, denote its associated α-Dirichlet norm by $\boldsymbol{D}_\alpha^n$ and its associated Hilbert space by $\boldsymbol{H}_\alpha^n$. It is clear that, if $m<n$, the restriction of the function of $\boldsymbol{H}_\alpha^n$ to D_m is an element of $\boldsymbol{H}_\alpha^m$.

If $m<n$, we have

$$\begin{aligned}&\boldsymbol{D}_\alpha^m({}^nR_\alpha^x-{}^mR_\alpha^x,\ {}^nR_\alpha-{}^mR_\alpha^x)\\&=\boldsymbol{D}_\alpha^n({}^nR_\alpha^x,\ {}^nR_\alpha^x)-2\boldsymbol{D}_\alpha^m({}^mR_\alpha^x,\ {}^nR_\alpha^x)+\boldsymbol{D}_\alpha^m({}^mR_\alpha^x,\ {}^mR_\alpha^x)\,.\end{aligned}$$

We will apply Lemma 2. 3 to each term of the last expression. The first term is not greater than $\boldsymbol{D}_\alpha^n({}^nR_\alpha^x,\ {}^nR_\alpha^x)=2\,{}^nR_\alpha^x(x)$. The second and third terms are equal to $-4\,{}^nR_\alpha^x(x)$ and $2\,{}^mR_\alpha^x(x)$, respectively. Therefore, for each N, it holds that

$$(2.\ 6)\qquad 0\leqq\boldsymbol{D}_\alpha^N({}^nR_\alpha^x-{}^mR_\alpha^x,\ {}^nR_\alpha^x-{}^mR_\alpha^x)\leqq2({}^mR_\alpha^x(x)-{}^nR_\alpha^x(x))\,,$$

for any m and n such that $N\leqq m<n$. Inequality (2. 6) implies that ${}^nR_\alpha^x(x)$ is non-increasing in n and since ${}^nR_\alpha^x(x)=\frac{1}{2}\boldsymbol{D}_\alpha^n({}^nR_\alpha^x,\ {}^nR_\alpha^x)$ is non-negative, ${}^nR_\alpha^x(x)$ converges. Thus, inequality (2. 6) and Lemma 2. 1 show that ${}^nR_\alpha^x(y)$ converges to an α-harmonic function $\tilde{R}_\alpha^x(y)$ on D uniformly on any compact subset of D, and for each N, the restriction of ${}^nR_\alpha^x$ to D_N converges to that of $\tilde{R}_\alpha^x$ in the norm $\boldsymbol{D}_\alpha^N$.

Let us prove that $\tilde{R}_\alpha^x(y)=R_\alpha^x(y)$, $y\in D$. Since R_α^x belongs to $\boldsymbol{H}_\alpha^n$, Lemma 2. 3 (ii) implies

$$\boldsymbol{D}_\alpha^n({}^nR_\alpha^x,\ {}^nR_\alpha^x)-4\,{}^nR_\alpha^x(x)\leqq\boldsymbol{D}_\alpha^n(R_\alpha^x,\ R_\alpha^x)-4R_\alpha^x(x)\,.$$

Letting n tend to infinity, we have, for each N,

$$\boldsymbol{D}_\alpha^N(\tilde{R}_\alpha^x,\ \tilde{R}_\alpha^x)-4\tilde{R}_\alpha^x(x)\leqq\boldsymbol{D}_\alpha(R_\alpha^x,\ R_\alpha^x)-4R_\alpha^x(x)\,.$$

Let N tend to infinity, then

$$\boldsymbol{D}_\alpha(\tilde{R}_\alpha^x,\ \tilde{R}_\alpha^x)-4\tilde{R}_\alpha^x(x)\leqq\boldsymbol{D}_\alpha(R_\alpha^x,\ R_\alpha^x)-4R_\alpha^x(x)\,.$$

Thus, we see that $\tilde{R}_\alpha^x\in\boldsymbol{H}_\alpha$ and that, by Lemma 2. 3 (ii), the inequality above is just the equality and $\tilde{R}_\alpha^x(y)=R_\alpha^x(y)$, $y\in D$. The proof of Lemma 2. 5 is complete.

We have seen (in the paragraph following Theorem 1) that, if ∂D_n is of class $\boldsymbol{C}^3$, ${}^nG_\alpha(x,\ y)$ is nothing but the Laplace transform of the fundamental solution of the heat equation on D_n with the boundary condition $\dfrac{\partial}{\partial n_x}u=0$ and this solution is a transition density on D_n. Hence, we have

Lemma 2.6. *Let* $\{D_n\}$, $\{{}^nR_\alpha(x,\ y)\}$ *and* $\{{}^n(G_\alpha(x,\ y)\}$ *be those in Lemma 2.5. If* D_n *is regular, then we have*

(2. 7) $\quad {}^nG_\alpha(x, y)\geqq 0, \quad \alpha>0, \quad x, y\in D_n, \quad x\neq y$.

(2. 8) $\quad {}^nR_\alpha(x, y)\geqq 0, \quad \alpha>0, \quad x, y\in D_n$.

(2. 9) $\quad \alpha\int_{D_n} {}^nG_\alpha(x, y)dy\leqq 1, \quad \alpha>0, \quad x\in D_n$.

(2. 10) $\quad {}^nG_\alpha(x, y)-{}^nG_\beta(x, y)+(\alpha-\beta)\int_{D_n} {}^nG_\alpha(x, z){}^nG_\beta(z, y)\,dz=0,$
$\alpha, \beta>0, \quad x, y\in D_n, \quad x\neq y$.

We note that (2. 8) follows from (2. 7).

Now, let us complete the proof of Theorem 1 by the following series of lemmas.

Lemma 2.7. *$R_\alpha(x, y)$ is non-negative for $\alpha>0$, $x, y\in D$ and $\alpha\int_D G_\alpha(x, y)dy\leqq 1$, for $\alpha>0$, $x\in D$. $G_\alpha(x, y)$ is symmetric in $x, y\in D$ and continuous in (x, y) on $D\times D$ off the diagonal.*

Proof. The first part of Lemma 2. 7 is an immediate consequence of Lemma 2. 5 and Lemma 2. 6. It is well known that $G_\alpha^0(x, y)$ is symmetric in $x, y\in D$ and continuous in $(x, y)\in D\times D$ off the diagonal set. $R_\alpha(x, y)$ is symmetric because $\boldsymbol{D}_\alpha(R_\alpha^x, R_\alpha^y)=2R_\alpha^x(y)=2R_\alpha^y(x)$, $x, y\in D$.

We shall show that $R_\alpha(x, y)$ is continuous in $(x, y)\in D\times D$. Since $R_\alpha(x, y)$ is α-harmonic in x and in y, applying Lemma 2. 1 for any $x, y\in D$ and for sufficiently small balls B_1 and B_2 containing x and y, respectively, we have $R_\alpha(x, y)=\int_{\partial B_1}\int_{\partial B_2} h_\alpha^{B_1}(x, z)R_\alpha(z, z')h_\alpha^{B_2}(y, z')\sigma_1(dz)\sigma_2(dz')$, where $\sigma_1(dz)$ and $\sigma_2(dz')$ are the surface Lebesgue measures of ∂B_1 and ∂B_2, respectively. While, $R_\alpha(z, z')$ being continuous in z' for each z, $\int_{\partial B_2} R_\alpha(z, z')\sigma_2(dz')$ is finite and α-harmonic in z. Thus,

$$\int_{\partial B_1}\int_{\partial B_2} R_\alpha(z, z')\sigma_1(dz)\sigma_2(dz')<+\infty .$$

Since R_α is non-negative, Lebesgue's convergence theorem implies continuity of $R_\alpha(x, y)$. The proof of the latter half of Lemma 2. 7 is complete.

We will show assertion (iv) of Theorem 1.

Lemma 2.8. *Let K_1 and K_2 be compact subsets of D such that K_1 and the closure of D-K_2 are disjoint. Then, $\sup_{x\in K_1, y\in D-K_2} G_\alpha(x, y)$ is finite.*

Proof. Without loss of generality, we can assume that $S=\partial(D-K_2)\cap D$ is sufficiently regular. Consider a regular exhaustion $\{D_n\}$ of D such that $D_1\supset K_2$. Let x be fixed in K_1. For a fixed n, set $D'=D_n-K_2$ and $u(y)$

$={}^nG_\alpha(x, y)$, $y\in D'\cup\partial D'$. Since $\frac{\partial}{\partial n_y}u(y)=0$, $y\in\partial D_n$, we see by Green's formula that $\boldsymbol{D}_\alpha'(u, v-u)=0$ holds if $v\in \boldsymbol{C}^1(D'\cup\partial D')$ and $v=u$ on S[8]. Hence, the equality

$$(2.11)\qquad \boldsymbol{D}_\alpha'(u, u)=\boldsymbol{D}_\alpha'(v, v)-\boldsymbol{D}_\alpha'(u-v, u-v)$$

is valid for each v belonging to $\mathfrak{D}_u=\{v;\ v$ is square summable on D', v has square summable weak-derivatives on D', $v\in \boldsymbol{C}(D'\cup S)$ and $v=u$ on $S\}$[9]. Set $\delta=\sup_{y\in S}u(y)$ and $u_1(y)=\min(u(y), \delta)$, $y\in D'\cup S$. Obviously, $\boldsymbol{D}_\alpha'(u, u)\geqq \boldsymbol{D}_\alpha'(u_1, u_1)$. But, since $u_1\in\mathfrak{D}_u$, (2.11) holds for $v=u_1$ and consequently $u_1(y)=u(y)$ on D'.

We have proved that, if $x\in K_1$ and $y\in D_n-K_2$, then ${}^nG_\alpha(x, y)\leqq \sup_{y\in S}{}^nG_\alpha(x, y)$. Letting n tend to infinity, we see by virtue of Lemma 2. 5, $G_\alpha(x, y)\leqq \sup_{y\in S}G_\alpha(x, y)$, $x\in K_1$, $y\in D-K_2$. Thus,

$$\sup_{x\in K_1,\, y\in D-K_2} G_\alpha(x, y)\leqq \sup_{x\in K_1,\, y\in S} G_\alpha(x, y)\,.$$

The right hand side above is finite by Lemma 2. 7.

Let us show statement (iii) of Theorem 1.

Lemma 2.9. *The operator G_α defined by* (1. 6) *maps $\boldsymbol{B}(D)$ into $\boldsymbol{C}(D)$. Moreover, if $f\in \boldsymbol{C}(D)$, then $\lim_{\alpha\to+\infty}\alpha G_\alpha f(x)=f(x)$, $x\in D$.*

Proof. We note that G_α^0 has those properties in Lemma 2. 9[10]. For $f\in \boldsymbol{B}(D)$, $R_\alpha f(x)=\int_D R_\alpha(x, y)f(y)\,dy$ is α-harmonic and bounded on account of Lemma 2. 1 and Lemma 2. 7. Moreover, we see by Lemma 2. 1 that, for any $x\in D$ and sufficiently small ball B containing x.,

$$|\alpha R_\alpha f(x)|\leqq \int_{\partial B} h_\alpha^B(x, y)|\alpha R_\alpha f(y)|\sigma(dy)$$
$$\leqq \sup_{x\in D}|f(x)|\int_{\partial B} h_\alpha^B(x, y)\sigma(dy)\xrightarrow[\alpha\to+\infty]{}0\,.$$

The proof of Lemma 2. 9 is complete.

The following lemmas are statements (ii) and (v) of Theorem 1.

8) $\boldsymbol{D}_\alpha'$ denotes the integral (2.1) on D'.

9) We call f the weak derivative of v with respect to the coodinate x_i, if $(f, \varphi)_{D'}=-\left(v, \frac{\partial}{\partial x_i}\varphi\right)_{D'}$ holds for every infinitely differentiable function on D' with a compact support, $(\ \ ,\ \ )_{D'}$ being the integral (1. 3) on D'.

10) See (1. 2).

Lemma 2.10. *$G_\alpha(x, y)$ is a conservative resolvent density on D. $R_\alpha(x, y)$ is strictly positive.*

Proof. We must prove that $G_\alpha(x, y)$ satisfies conditions (G. 1)∼(G. 4) stated in the beginning of Section 1 and the conservativity condition. Condition (G. 1), (G. 2) and (G. 4) were already proved in Lemma 2. 7.

Proof of the resolvent equation (G. 3). Take a regular exhaustion $\{D_n\}$ of D. Let f and g be non-negative continuous functions on D with compact supports. Owing to equation (2. 10) of Lemma 2. 6, we have for sufficiently large n,

$$(2.\ 12)\qquad (f, {}^nG_\alpha g)_n-(f, {}^nG_\beta g)_n+(\alpha-\beta)({}^nG_\alpha f, {}^nG_\beta g)_n=0\ ,$$

where $(u, v)_n$ denotes the integral of $u\,v$ on D_n.
Note that $0\leqq {}^nG_\alpha f(x)\,{}^nG_\beta g(x)\leqq \frac{1}{\alpha\beta}\sup_{x\in D} f(x)\cdot\sup_{x\in D} g(x)$ and that ${}^nG_\alpha g$ converges to $G_\alpha g$ on D (since, ${}^nG_\alpha^0 g$ increases to $G_\alpha^0 g$ and ${}^nR_\alpha^x(y)$ converges uniformly on any compact subset).
Hence, we can delete both superscript and subscript n in (2. 12). Owing to Lemma 2. 8 and Lemma 2. 9, the left hand side of (G. 3) is, for each $x\in D$, continuous in $y\in D-\{x\}$, and we can see that the resolvent equation (G. 3) is valid.

Proof of conservativity. If we show that $R_\alpha 1\in \boldsymbol{H}_\alpha$ and that

$$(2.\ 13)\qquad \boldsymbol{D}_\alpha(\alpha R_\alpha 1, v)=2\alpha(1, v)\ ,$$

holds for all $v\in \boldsymbol{H}_\alpha$, then, we have, by (ii) of Lemma 2. 4, $1-\alpha G_\alpha^0 1=\alpha R_\alpha 1$ and $\alpha G_\alpha 1=1$.

Let D_n be an exhaustion of D. Integrating $\boldsymbol{D}_\alpha(R_\alpha^x, R_\alpha^y)=2R_\alpha(x, y)$ on $D_m\times D_n$, we obtain

$$(2.\ 14)\qquad \boldsymbol{D}_\alpha(R_\alpha\chi_{D_m}, R_\alpha\chi_{D_n})=2\int_{D_m}\int_{D_n} R_\alpha(x, y)\,dx\,dy\ .$$

Here, we have used the Fubini theorem, which is valid for the following reason: if $m\leqq n$,

$$\int_{D_m}\int_{D_n} dx\,dy\int_D |(\mathrm{grad}_z\, R_\alpha^x(z),\ \mathrm{grad}_z\, R_\alpha^y(z))|\,dz$$
$$\leqq\int_{D_n}\int_{D_n}\sqrt{\boldsymbol{D}_\alpha(R_\alpha^x, R_\alpha^x)}\sqrt{\boldsymbol{D}_\alpha(R_\alpha^y, R_\alpha^y)}\,dx\,dy$$
$$=\Big(\int_{D_n}\sqrt{2R_\alpha(x, x)}\,dx\Big)^2\leqq 2\int_{D_n} R_\alpha(x, x)\,dx\times \text{Lebesgue measure of } D_n,$$

the integral in the last expression being finite by Lemma 2. 7. In view of

Lemma 2. 7, $R_\alpha(x, y) \geqq 0$ and $\int_D \int_D R_\alpha(x, y)\, dx\, dy \leqq \frac{1}{\alpha} \times$ Lebesgue measure of D. Therefore, $R_\alpha \chi_{D_n}$ forms a Cauchy sequence in $\boldsymbol{H}_\alpha$ and, by Lemma 2. 2, converges to $R_\alpha 1$ in $\boldsymbol{H}_\alpha$. We have $\boldsymbol{D}_\alpha(R_\alpha 1, R_\alpha 1)=2(1, R_\alpha 1)$. In the same way, identity (2. 13) is obtained. Strict positivity of $R_\alpha(x, y)$ follows from Lemma 2. 4.

Lemma 2.11. *There is a unique transition density $p(t, x, y)$ on D satisfying the following conditions.*

(i) $G_\alpha(x, y)=\int_0^{+\infty} e^{-\alpha t} p(t, x, y)\, dt, \quad \alpha>0.$

(ii) *For each $t>0, f \in \boldsymbol{B}(D)$,*

$$\int_D p(t, x, y) f(y)\, dy \text{ is continuous in } (t, x) \in (0, +\infty) \times D.$$

(iii) *$p(t, x, y)$ is symmetric in $x, y \in D$ and it is conservative.*

(iv) *Set $\gamma(t, x, y)=p(t, x, y)-p^0(t, x, y)$, then*

$$\frac{1}{t} \int_D \gamma(t, x, y)\, dy \xrightarrow[t \to 0]{} 0 \text{ uniformly in } x \text{ on any compact subset of } D.$$

Proof. First of all, we will show the existence of a non-negative function $\gamma(t, x, y)$ continuous in $t>0$, satisfying

$$(2.\ 15) \qquad R_\alpha(x, y)=\int_0^{+\infty} e^{-\alpha t} \gamma(t, x, y)\, dt, \quad \alpha>0, \quad x, y \in D.$$

If $x \neq y$, $R_\alpha(x, y)$ is completely monotonic in $\alpha \in (0, +\infty)$. In fact, by the resolvent equation (G. 3) for G_α and G_α^0, we have, if $x \neq y$,

$$(2.\ 16) \qquad (-1)^n \frac{d^n}{d\alpha^n} R_\alpha(x, y)=n!\, [G_\alpha^{[n+1]}(x, y)-(G_\alpha^0)^{[n+1]}(x, y)], \quad n=0, 1, 2, \cdots.$$

Here $G_\alpha^{[1]}(x, y)=G_\alpha(x, y)$ and $G_\alpha^{[n+1]}(x, y)=\int_D G_\alpha^{[n]}(x, z) G_\alpha(z, y)\, dz$, $n=1, 2, \cdots$. $(G_\alpha^0)^{[n]}$ is defined similarly. Evidently, the right hand side of (2. 16) is non-negative and, by Lemma 2. 8, finite. Hence, $R_\alpha(x, y)$ is expressed by a measure on $[0, +\infty)$ as

$$(2.\ 17) \qquad R_\alpha(x, y)=\int_0^{+\infty} e^{-\alpha s} \gamma(ds, x, y), \quad x \neq y, \quad \alpha>0.$$

Take a ball B with closure contained in D. Since $R_\alpha(x, y)$ is α-harmonic in x, we see, by Lemma 2. 1, for any $x \in B$ and any $y \in D$,

$$(2.\ 18) \qquad R_\alpha(x, y)=\int_{\partial B} h_\alpha^B(x, z) R_\alpha(z, y) \sigma(dz).$$

Note that $h_\alpha^B(x, z)$ is written in the form

$$(2.19)\quad h_\alpha^B(x, z)=\int_0^{+\infty} e^{-\alpha t} h^B(t, x, z)\,dt, \quad z\in B, \quad z\in\partial B,$$

where $h^B(t, x, z)=\frac{1}{2}\frac{\partial}{\partial n_z} p_B^0(t,x, z)$, p_B^0 being the transition density p^0 for B. Let us put, for $t>0$, $x\in B$ and $y\in D$,

$$(2.20)\quad \gamma(t, x,y)=\int_{\partial B}\int_0^t h^B(t-s, x, z)\gamma(ds, z, y)\sigma(dz).$$

Owing to equations (2. 17), (2. 18) and (2. 19), the function $\gamma(t, x, y)$ of (2. 20) satisfies the desired equation (2. 15). On the other hand, for any ball B' such as $B'\cup\partial B'\subset B$, the obvious idenity $h^B(t, x, z)=\int_{\partial B'}\int_0^t h^{B'}(t-s, x, z')h^B(s, z', z)\,ds\,\sigma'(dz')$, $x\in B'$, $z\in\partial B$,
leads us to the relation

$$(2.21)\quad \gamma(t, x, y)=\int_{\partial B'}\int_0^t h^{B'}(t-s, x, z')\gamma(s, z', y)\,ds\,\sigma'(dz'),$$
$$t>0, \quad x\in B' \quad y\in D,$$

which implies the continuity of $\gamma(t, x, y)$ in $(t, x)\in(0, +\infty)\times B'$.
Here, we have used the following estimate which is a consequence of (2. 17), (2. 20) and Lemma 2. 8.

$$(2.22)\quad \sup_{0<t\leq T, x\in B', y\in D}\gamma(t, x, y)\leq C\cdot e^T\cdot \sup_{z\in\partial B, y\in D} R_1(z, y)<+\infty,$$

where T is an arbitrary positive number and C is a constant determined by T, B and B'. Hence, we see that, for any x and y in D, $\gamma(t, x, y)$ defined by (2. 20) is independent of ball B such that $x\in B$ and $B\cup\partial B\subset D$, because it satisfies (2. 15) and it is continuous in t. It is symmetric in x, y because of the symmetry of $R_\alpha(x, y)$ (Lemma 2.7). Henceforce, it is continuous in y, and (2. 21) and (2. 22) imply its continuity in $(t, x, y)\in(0, +\infty)\times D\times D$. In view of (2. 22), we see that $\int_D\gamma(t, x, y)f(y)\,dy$ is continuous in $(t, x)\in(0, +\infty)\times D$ for each $f\in \boldsymbol{B}(D)$.

Now put, for $t>0$, x, $y\in D$,

$$(2.23)\quad p(t, x, y)=p^0(t, x, y)+\gamma(t, x, y).$$

Then, $p(t, x, y)$ is continuous in $(t, x, y)\in(0, +\infty)\times D\times D$ and satisfies conditions (i), (ii) and the first half of Lemma 2. 11 (iii). In particular, $\int_D p(t, x, y)\,dy$ is continuous in t, so that, the conservativity of $p(t, x, y)$ follows from that of $G_\alpha(x, y)$. For each $x, y \in D$, $p(t+s, x, y)$ and $\int_D p(t, x, z)$ $p(s, z, y)dz$ are continuous in $(t, s) \in (0, +\infty) \times (0, +\infty)$, and so, they are identical by virtue of (G. 3) for $G_\alpha(x, y)$. Thus, $p(t, x, y)$ is a transition density. Assersion (iv) of Lemma 2. 11 follows from (2. 21) and the inequality $\int_D \gamma(t, x, y)dy \leqq 1$, $t>0$, $x \in D$.

3. Compactification of D. Construction of a strong Markov process on the compactified space

Consider the resolvent density $G_\alpha(x, y)$, $\alpha>0$, $x, y \in D$, in Theorem 1. Let $x_n \in D$, $n=1, 2, \cdots$, be a sequence having no accumulation point in D and $\{D_l, l=1, 2, \cdots\}$ be an exhaustion of D. For each l, there exists N such that $x_n \in D-D_{l+2}$, $n \geqq N$. By Theorem 1 (iv), the family of functions $\{G_1(x_n, y), n \geqq N\}$ of y is uniformly bounded in $y \in D_{l+1}$. Moreover, Lemma 2. 1 implies that, for $n \geqq N$, the first derivatives of $G_1(x_n, y)$, $n \geqq N$, are also uniformly bounded in $y \in D_l$ and that functions $G_1(x_n, y)$, $n \geqq N$, are equi-continuous there. Hence, a subsequence of $G_1(x_n, y)$ converges uniformly on each D_l and consequently, by Lemma 2. 1, the limit function is 1-harmonic in D.

A sequence $x_n \in D$, $n=1, 2, \cdots$ having no accumulation point in D is called *fundamental*, if $\lim_{n \to +\infty} G_1(x_n, y)$ exists for each $y \in D$.

Two fundamental sequences $\{x_n\}$ and $\{x_n'\}$ are called *equivalent*, if $\lim_{n \to +\infty} G_1(x_n, y) = \lim_{n \to +\infty} G_1(x_n', y)$, $y \in D$. This defines a usual equivalence relation among fundamental sequences.

DEFINITION 3.

(i) Denote by $\triangle$ the collection of equivalent classes of fundamental sequences.

(ii) For $x \in \triangle$, define $G_1(x, y)$ by $G_1(x, y) = \lim_{n \to +\infty} G_1(x_n, y)$, $y \in D$, where, $\{x_n\}$ is a fundamental sequence belonging to x.

(iii) Set $D^* = D \cup \triangle$. For $x_1, x_2 \in D^*$, set

$$(3.1) \qquad \rho(x_1, x_2) = \int_D \frac{|G_1(x_1, y) - G_1(x_2, y)|}{1 + |G_1(x_1, y) - G_1(x_2, y)|} dy .$$

Evidently, ρ defines a metric on D^*.

Lemma 3.1.

(i) *(D^*, ρ) is a compactification of D.*

(ii) *For each y in D, the extended function $G_1(x, y)$ is ρ-continuous in x on $D^* - \{y\}$ and the class of functions (of x), $\{G_1(x, y), y \in D\}$, separates points of D^*.*

(iii) *If K is a compact subset of D and F is a closed subset of D^*-K, then* $\sup_{x\in F, y\in K} G_1(x, y)$ *is finite.*

(iv) *When the relative boundary ∂D of D in R^N is of class C^3, $D\cup\partial D$ coincides with D^* up to a homeomorphism which is the identity on D.*

Proof. Martin's original proof (cf. [13], §2, Theorem I and II) can be applied with no change to obtain the statements (i) and (ii). Third assertion is a consequence of Theorem 1 (iv). Suppose that ∂D is of class C^3. As we have seen in Section 1, $G_\alpha(x, y)$ of Theorem 1 is the Laplace transform of a fundamental solution $\mathring{p}(t, x, y)$ of a boundary problem of the heat equation. $\mathring{p}(t, x, y)$ and $G_\alpha(x, y)$ can be continuously extended to $D\cup\partial D$ as functions of x and it holds that, for each $x\in D\cup\partial D$, $f\in C(D\cup\partial D)$, $\lim_{t\to 0}\int_D \mathring{p}(t, x, y) f(y)\,dy=f(x)$[11], which implies $\lim_{\alpha\to+\infty}\alpha\int_D G_\alpha(x, y)f(y)\,dy=f(x)$. Hence, $\{G_1(x, y), y\in D\}$ separate points of $D\cup\partial D$. Therefore, $D\cup\partial D$ is homeomorphic to D^* (cf. [1], §9).

Denote by $\mathfrak{B}(D^*)$ the σ-field of all Borel subsets of D^*. $B(D^*)$, $C(D^*)$ and $C_0(D)$ will stand for the classes of all bounded Borel measurable functions on D^*, ρ-continuous functions on D^* and continuous functions on D with compact supports in D, respectively. Each $f\in C_0(D)$ will be considered as a function on D^* by setting $f(x)=0$, $x\in\triangle$.

As an immediate consequence of Lemma 3. 1 and Theorem 1 (iii), we have

Corollary. *The operator G_1, defined by $G_1f(x)=\int_D G_1(x, y)f(y)\,dy$, $x\in D^*$, maps $C_0(D)$ into $C(D^*)$ and the collection of functions G_1f, $f\in C_0(D)$, separates points of D^*.*

Now, let us extend every function $G_\alpha(x, y)$, $\alpha>0$, as follows.

DEFINITION 4. For $\alpha>0$, $x\in\triangle$, $y\in D$, define $G_\alpha(x, y)$ by

$$G_\alpha(x, y)=G_1(x, y)-(\alpha-1)\int_D G_1(x, z)G_\alpha(z, y)\,dz\,. \tag{3. 2}$$

Lemma 3.2. *For each $x\in\triangle$, $G_\alpha(x, y)$ has the following properties:*

(G. 1)′ *$G_\alpha(x, y)$, $\alpha>0$, $y\in D$, is non-negative, finite and α-harmonic in $y\in D$,*

(G. 2)′ $\alpha G_\alpha 1(x)=G_1 1(x)\leqq 1, \quad \alpha>0,$

where $G_\alpha 1(x)=\int_D G_\alpha(x, y)\,dy.$

(G. 3)′ $G_\alpha(x, y)-G_\beta(x, y)+(\alpha-\beta)\int_D G_\alpha(x, z)G_\beta(z, y)\,dz=0,\ \alpha,\ \beta>0,\ y\in D.$

11) cf. [7].

Proof. Let us fix $x \in \triangle$. By Fatou's lemma,

(3. 3) $\quad G_1 1(x) \leqq 1$.

By virtue of (3. 3), assertion (iii) of Lemma 3. 1 and assertion (iv) of Theorem 1, the integral appering in (3. 2) turns out to be finite for $\alpha > 0$ and $y \in D$. When $\alpha < 1$, $G_\alpha(x, y)$ is clearly non-negative. By Fatou's lemma, $G_\alpha(x, y) \geqq 0$ for $\alpha > 1$. We can easily verify

$$\left(\alpha - \frac{1}{2} \triangle_y\right) G_\alpha(x, y) = 0, \quad \alpha > 0, \quad y \in D .$$

Integrating both sides of (3. 2) in y and noting the conservativity of G_α of Theorem 1, we get $\alpha G_\alpha 1(x) = G_1 1(x)$, $\alpha > 0$. The equation (G. 3)′ is obtained from (3. 2) by a simple calculation.

We now extend $p(t, x, y)$ of Thoerem 1 (v) from D to D^* with respect to x.

Lemma 3.3. *For each $x \in \triangle$, there is one and only one function $p(t, x, y)$, $t > 0$, $y \in D$, which is continuous in t and satisfies*

(3. 4) $\quad G_\alpha(x, y) = \int_0^{+\infty} e^{-\alpha t} p(t, x, y) dt, \quad \alpha > 0, \quad y \in D .$

Moreover the function $p(t, x, y)$ has the following properties:

(T. 1)′ *It is non negative.*

(T. 2)′ $\int_D p(t, x, y) dy = G_1 1(x) \leqq 1, \quad t > 0 .$

(T. 3)′ $\int_D p(t, x, z) p(s, z, y) dz = p(t+s, x, y), \quad t, s > 0, \quad y \in D .$

(T. 4)′ *For each $x \in \triangle$, it is continuous in $(t, y) \in (0, +\infty) \times D$ and, for each $t > 0$ and $y \in D$, it is measurable in x on $\triangle$. Moreover, for any $f \in \boldsymbol{B}(D^*)$ and $x \in \triangle$, $\int_D p(t, x, y) f(y) dy$ is continuous in $t > 0$.*

Proof. In view of (G. 3)′ of Lemma 3. 2, we see that $G_\alpha(x, y)$, $x \in \triangle$, $y \in D$ is completely monotonic in $\alpha \in (0, +\infty)$. By (G. 1)′ of Lemma 3. 2, it is α-harmonic in $y \in D$. Hence, we can construct $p(t, x, y)$, $t > 0$, $x \in \triangle$, $y \in D$, satisfying (3. 4), (T. 1)′ and the first half of (T. 4)′ in the same manner as the construction of $\gamma(t, x, y)$ of Lemma 2. 11.

As consequences of properties (G. 2)′ and (G. 3)′ of Lemma 3. 2, the equation in (T. 2)′ holds for almost all $t > 0$ and relation (T. 3)′ holds for almost all $t, s > 0$. By virtue of (2. 22), the left hand side of (T. 3)′ is continuous in $s > 0$ for each t satisfying (T. 2)′. So the equation (T. 3)′ holds for almost all $t > 0$ and for all $s > 0$. In view of property (T. 3) of the transition density $p(t, x, y)$,

$t>0$, $x, y\in D$, (T. 3)′ holds for all $t, s>0$. (T. 3)′ implies that the left hand side of (T. 2)′ is a constant in t. Hence (T. 2)′ holds for all $t>0$. It follow from the first half of (T. 4)′ that $\int_D p(t, x, y)f(y)dy$ is lower semi-continuous in t for each non-negative bounded function f on D. Moreover, on account of (T. 2)′, it is continuous in t. Thus, $\int_D p(t, x, y)f(y)dy$ is continuous in $t>0$ for each $f\in \boldsymbol{B}(D^*)$ and $x\in \triangle$.

Now, we are in a position to construct the Markov process (on D^*) associated with $p(t, x, y)$, $x\in D^*$, $y\in D$, and investigate its properties.

Add a point ∂ to D^* as an isolated point. $\mathfrak{B}(D^*\cup\partial)$ will stand for the collection of sets whose restrictions to D^* are the elements of $\mathfrak{B}(D^*)$. Denote by $\boldsymbol{B}(D^*\cup\partial)$ ($\boldsymbol{C}(D^*\cup\partial)$) the aggregate of all the functions on $D^*\cup\partial$ whose restrictions to D^* are the elements of $\boldsymbol{B}(D^*)$ (resp. $\boldsymbol{C}(D^*)$). Each element f of $\boldsymbol{B}(D^*)$ will always be considered as the one of $\boldsymbol{B}(D^*\cup\partial)$ by setting $f(\partial)=0$, unless particularly mentioned. Let $p(t, x, y)$ be the function defined for $t>0$, $x\in D^*$ and $y\in D$ by Theorem 1 (v) and Lemma 3.3. For $E\in \boldsymbol{B}(D^*\cup\partial)$, define

$$\begin{aligned} &p(t, x, E)=\int_{E\cup D} p(t, x, y)dy+(1-q(x))\chi_E(\partial), \quad x\in D^*, \\ &p(t, \partial, E)=\chi_E(\partial), \end{aligned} \tag{3. 5}$$

where χ_E is the indicator function of the set E, and

$$q(x)=\int_D G_1(x, y)dy, \quad x\in D^*. \tag{3. 6}$$

We put for $f\in \boldsymbol{B}(D^*\cup\partial)$,

$$\begin{aligned} &T_t f(x)=\int_{D^*\cup\partial} p(t, x, dy)f(y), \\ &G_\alpha f(x)=\int_0^{+\infty} e^{-\alpha t}T_t f(x)dt, \quad x\in D^*\cup\partial, \quad t>0, \quad \alpha>0. \end{aligned} \tag{3. 7}$$

$G_\alpha f$ is expressed in the form

$$\begin{aligned} &G_\alpha f(x)=\int_D G_\alpha(x, y)f(y)dy+\frac{1-q(x)}{\alpha}f(\partial), \quad x\in D^*, \\ &G_\alpha f(\partial)=\frac{f(\partial)}{\alpha}. \end{aligned} \tag{3. 8}$$

By virtue of Theorem 1 (v) and Lemma 3. 3, $p(t, x, E)$ defined by (3. 5) is a transition function on $D^*\cup\partial$; $p(t, x, \cdot)$ is a probability measure on $D^*\cup\partial$, $p(\cdot, \cdot, E)$ is, for each $E\in\mathfrak{B}(D^*\cup\partial)$, measureable in $(t, x)\in(0, +\infty)\times\{D^*\cup\partial\}$

and it satisfies the Chapmann-Kolmogorov equation.

Let Ω be the product compact space $\{D^* \cup \partial\}^{(0,+\infty)}$. Denote by $\tilde{X}_t(\omega)$ the t-th coodinate of $\omega \in \Omega$. Let $\mathfrak{F}(\mathfrak{F}_t)$ be the σ-field of subsets of Ω generated by the cylindrical open sets of Ω (resp. cylindrical open sets depending on the coodinates up to and including t). Denote by $\mathfrak{A}$ the σ-field of subsets of Ω generated by all open set of Ω. For each $x \in D^* \cup \partial$, there is a unique Radon measure[12] P_x over $(\Omega, \mathfrak{A})$ which is a probability measure and satisfies the following conditions.

(3. 9) $P_x(\tilde{X}_t \in E) = p(t, x, E)$,
$t > 0, \quad x \in D^* \cup \partial, \quad E \in \mathfrak{B}(D^* \cup \partial)$,

(3. 10) For each $\Lambda \in \mathfrak{F}_t$ and bounded $\mathfrak{F}$-measurable function F on Ω, $E_x(F_x(\theta_t\omega); \Lambda) = E_x(E_{\tilde{X}_t}(F); \Lambda)$, $x \in D^* \cup \partial$,
where E_x denotes the integration with respect to P_x-measure and θ_t; $t > 0$, is the shift from Ω to Ω defined by $\tilde{X}_s(\theta_t\omega) = \tilde{X}_{t+s}(\omega)$, $s > 0$.

Lemma 3.4.
(i) *Set* $\Omega_1 = \{\omega; \tilde{X}_t(\omega) \in D^*$ *for every* $t > 0\}$ *and* $\Omega_2 = \{\omega: \tilde{X}_t(\omega) \in \{\partial\}$ *for every* $t > 0\}$. *Then,* $P_x(\Omega_1) = q(x)$, $P_x(\Omega_2) = 1 - q(x)$, $x \in D^*$ *and* $P_{\{\partial\}}(\Omega_2) = 1$.
(ii) *For each* $x \in D^* \cup \partial$, *we have* $P_x(\tilde{X}_t$ *has the right limits for all* $t \geqq 0$ *and the left limits for all* $t > 0) = 1$.

Proof. (i). Relations (3. 5), (3. 9) and (3. 10) imply $P_x(\tilde{X}_t \in D^*, \tilde{X}_s \in \{\partial\}) = 0$ for every t, s such as $t > s > 0$ and for every $x \in D^*$. Since $\{\tilde{X}_t, P_x\}$, $x \in D^*$, is separable,[13] we see $P_x(\Omega_1) = \lim_{t \to +\infty} P_x(\tilde{X}_t \in D^*) = q(x)$ and $P_x(\Omega_2) = \lim_{t \to 0} P_x(\tilde{X}_t \in \{\partial\}) = 1 - q(x)$.
(ii). Denote by $C_0^+(D)$ the collection of all non-negative functions in $C_0(D)$ and by $S_0(D)$ a countable dense subset of $C_0^+(D)$ in uniform norm. By virtue of Corollary to Lemma 3. 1, functions $G_1 f$, $f \in S_0(D)$, are continuous on D^* and separate points of D^*. Moreover, $\{Z_t = e^{-t} G_1 f(\tilde{X}_t), \mathfrak{F}_t, P_x\}$, $f \in S_0(D)$, $x \in D^*$, is a bounded supermartingale. Hence, we have assertion (ii) by a standard argument[14].

It follows from Lemma 3. 5 that there is well defined $X_t(\omega) = \lim_{t' \downarrow t} \tilde{X}_{t'}(\omega)$ for every $t \geqq 0$ almost everywhere (P_x), $x \in D^* \cup \partial$. X_t is right continuous in $t \geqq 0$ and has the left limit in $t > 0$ almost everywhere (P_x), $x \in D^* \cup \partial$. On account of Theorem 1 (v) and Lemma 3. 3 (T. 4)′, X_t is a modification of $\tilde{X}_t$; $P_x(X_t = \tilde{X}_t) = 1$, for each $t > 0$ and $x \in D^* \cup \partial$.

12) cf. [15].
13) cf. [15].
14) cf. [10] and [20].

Let us examine the distribution of X_0.

DEFINITION 5.

(i) For each $x\in D^*\cup\partial$, define a probability measure $\mu(x, E)$ on $\mathfrak{B}(D^*\cup\partial)$ by

$$\mu(x, E)=P_x(X_0\in E), \quad E\in\mathfrak{B}(D^*\cup\partial).$$

This $\mu(x, \cdot)$ is called the branching measure at x.

(ii) A point x in $D^*\cup\partial$ is called a branching point if $\mu(x, \{x\})<1$.

The notion of branching measure was introduced by *D.* Ray [20]. The above definition, slightly different from Ray's original one, is due to H. Kunita and T. Watanabe [10]. We shall use the general results obtained by these authors, whenever their methods of the proof are applicable to our situation without essential change.

Denote by $\triangle_0$ the totality of branching points. Then, we have

Lemma 3.5.

(i) $\triangle_0\subset\triangle$.

(ii) *$\triangle_0$ is an F_σ-set and $\mu(x, \triangle_0)=0$, $x\in\triangle_0$.*

(iii) *Put $\triangle_0'=\{x: q(x)<1\}$, where $q(x)=\int_D G_1(x, y)dy$. Then, $\triangle_0'\subset\triangle_0$ and $\mu(x, \{\partial\})=1-q(x)$, $x\in\triangle_0$.*

Proof. If $f\in \boldsymbol{C}(D^*\cup\partial)$, then

$$(3.11)\qquad \lim_{\alpha\to+\infty}\alpha G_\alpha f(x)=\lim_{\alpha\to+\infty}E_x\left(\int_0^{+\infty}e^{-t}f(X_{t/\alpha})\,dt\right)$$

$$=E_x(f(X_0))=\int_{D^*\cup\partial}\mu(x, dy)f(y), \quad x\in D^*\cup\partial.$$

On the other hand, because of Theorem 1 (ii) and formula (3. 8), $\lim_{\alpha\to+\infty}\alpha G_\alpha f(x)=f(x)$, for $x\in D\cup\partial$, $f\in\boldsymbol{C}(D^*\cup\partial)$. Hence, $D\cup\partial$ contains no branching point.

For the proof of (ii), let us cite a criterion of D. Ray [20] in a modified form fitted to our situation: $x\in\triangle_0$, if and only if $f(x)>\lim_{\alpha\to+\infty}\alpha G_\alpha f(x)$, for some $f\in\boldsymbol{C}_1=\{f=G_1h\wedge c;\ h\in{}_0\boldsymbol{S}(D),\ c$ is non-negative rational$\}$. Since, for $f\in\boldsymbol{C}_1$, $\alpha G_{\alpha+1}f\leqq f$ and $G_{\alpha+1}f=G_1(f-\alpha G_{\alpha+1}f)$ is lower semi-continuous on D^*, $\triangle_0=\bigcup_{f\in C_1}\bigcup_{n=1}^{+\infty}\bigcap_{\alpha>0,\ rational}\{f(x)\geqq\alpha G_{\alpha+1}f(x)+1/n\}$ is an F_σ-set. By (3. 11), we *have* for $f=G_1h$, $h\in\boldsymbol{C}_0(D)$, and conseqeuntly, for $f=G_\alpha h$, $h\in\boldsymbol{B}(D^*)$, $\alpha>0$, the equality $f(x)=\int_{D^*\cup\partial}\mu(x, dy)f(y)$. Therefore,

$$\int_{D^*\cup\partial}\mu(x, dy)\lim_{\alpha\to+\infty}(\alpha G_\alpha f)(y)=\lim_{\alpha\to+\infty}\alpha G_\alpha f(x)$$

$$=\int_{D^*\cup\partial}\mu(x, dy)f(y), \quad f\in\boldsymbol{C}_1.$$

Using the inequality $\lim_{\alpha\to+\infty} \alpha G_\alpha f \leqq f, f\in \boldsymbol{C}_1$ and the criterion above, we can see that $\mu(x, \triangle_0)=0$.

Assertion (iii) is immediate from (3. 8) and (3. 11).

In the next section, we shall see that $\mu(x, D)=0$, $x\in\triangle_0$.

Let us set $D_1^*=D^*-\triangle_0$. By Lemma 3. 5 (i), we see $D\subset D_1^*$. By Lemma 3, 4 (i) and Lemma 3. 5 (iii), we have $P_x(X_t\in D^*$ for every $t\geqq 0)=1$, $x\in D_1^*$. The following two lemmas will assure that the properties stated in Theorem 2 (ii) are valid for $\boldsymbol{X}=\{X_t, P_x, x\in D^*\}$ except the continuity of the trajectory X_t at the boundary $\triangle$.

We call a random time $\sigma\geqq 0$ a Markov time (relative to $\mathfrak{F}_t$) if, for each $t>0$ and each probability measure ν on D^*, the set $\{\sigma<t\}$ is in $\mathfrak{F}_t$ up to a set of P_ν-measure zero ($P_\nu(\cdot)=\int_{D^*}\nu(dx)P_x(\cdot)$). For a Markov time σ, let $\mathfrak{F}_{\sigma+}$ denote the σ-field of subsets Λ of Ω such that, for each $t>0$ and each probability measure ν on D^*, $\Lambda\cap\{\sigma<t\}$ is in $\mathfrak{F}_t$ up to a set of P_ν-measure zero.

Lemma 3.6.

(i) *$\boldsymbol{X}=\{X_t, P_x, x\in D^*\}$ is a strong Markov process; for each Markov time σ, $\Lambda\in\mathfrak{F}_{\sigma+}$ and $f\in\boldsymbol{B}(D^*)$,*

$$E_x(f(X_{\sigma+t});\ \Lambda)=E_x(E_{X_\sigma}(f(X_t));\ \Lambda),\quad x\in D^*.$$

(ii) *For each $x\in D^*$, $P_x(X_t\notin\triangle_0$ for every $t\geqq 0)=1$.*

Lemma 3.7.

(i) *Let $\{D_n\}$ be an exhaustion of D. Set*

$$\tau_n=inf\ \{t\colon X_t\in D^*-D_n\}\quad and\quad \tau=\lim_{n\to+\infty}\tau_n.$$

Then, $P_x(X_t$ is continuous in $0\leqq t<\tau)=1$, $x\in D^*$.

(ii) *For each $x\in D$ and Borel set E of D,*

$$P_x(X_t\in E,\ t<\tau)=\int_E p^0(t, x, y)\,dy.$$

(iii) *For each $x\in D^*$,*

$$P_x(X_t \text{ is continous for any } t\geqq 0 \text{ such that } X_t \text{ or } X_{t-} \text{ is in } D)=1.$$

(iv) *For each $x\in D^*$,*

$$P_x(X_t,\ X_{t-}\notin\triangle_0 \text{ for every } t\geqq 0)=1.$$

(v) *$\boldsymbol{X}$ is quasi-left continuous; for any sequence of Markov times σ_n increasing to σ,*

$$P_x(\lim_{n\to+\infty} X_{\sigma_n}=X_\sigma;\ \sigma<+\infty)=P_x(\sigma<+\infty), \quad x\in D^*.$$

Proof of Lemma 3. 6 (i). Since X_t is a modification of $\tilde{X}_t$, relations (3. 9) and (3. 10) hold for $\boldsymbol{X}$, if we replace $\tilde{X}_t$ there with X_t.

Take a Markov time σ and a set $\Lambda\in\mathfrak{F}_{\sigma+}$. The Markov property (3. 10) for X_t and a usual limiting procedure lead us to

$$\text{(3. 12)}\quad E_x(G_1f(X_\sigma);\ \Lambda)$$
$$=E_x(\int_\sigma^{+\infty} e^{-\alpha(t-\sigma)}(f(X_t)+(\alpha-1)G_1f(X_t)dt;\ \Lambda),$$

for $f\in\boldsymbol{C}_0(D)$, $x\in D^*$. Here, we have used the resolvent equation, the right continuity of X_t in $t\geqq 0$ and the continuity of $G_1f(x)$, $f\in\boldsymbol{C}_0(D)$ in $x\in D^*$. Since $P_x(X_t\in\triangle)=0$, $x\in D^*$, $t>0$, we can see that equation (3. 12) holds also for $f\in\boldsymbol{B}(D^*)$. By setting $f=G_\alpha h$. $h\in\boldsymbol{B}(D)$, $\alpha>0$, in equation (3. 12), we have $E_x(G_\alpha G_1h(X_\sigma);\ \Lambda)=E_x(\int_\sigma^{+\infty} e^{-\alpha(t-\sigma)}G_1h(X_t)dt;\ \Lambda)$. By the resolvent equation (G. 3) and (G. 3)′ (Lemma 3. 2), we have, for $\beta>0$ and $f\in\boldsymbol{C}(D^*)$,

$$E_x(G_\alpha(\beta G_\beta f)(X_\sigma);\ \Lambda)=E_x(\int_\sigma^{+\infty} e^{-\alpha(t-\sigma)}(\beta G_\beta f)(X_t)dt;\ \Lambda)$$
$$=E_x(\int_\sigma^{+\infty} e^{-\alpha(t-\sigma)}(\beta G_\beta f)(X_t)\chi_D(X_t)dt;\ \Lambda).$$

Letting β tend to infinity, we have, by Theorem 1 (iii),

$$E_x(G_\alpha f(X_\sigma);\ \Lambda)$$
$$=E_x(\int_0^{+\infty} e^{-\alpha t}f(X_{\sigma+t})dt;\ \Lambda),\quad \alpha>0,\quad f\in\boldsymbol{C}(D^*),\quad x\in D^*,$$

which proves conclusion (i) of Lemma 3. 6.

Proof of Lemma 3. 6 (ii).

Here, we can go along the same line as in H. Kunita and T. Watanabe [11], Section 2, (j). Set, for $A\subset D^*$,

$$\text{(3. 13)}\quad \sigma_A=\inf\{t>0;\ X_t\in A\},$$
$$=+\infty,\ \text{if there is no such } t.$$

σ_A is a Markov time if A is open or closed. Since $\triangle_0$ is an F_σ-set (Lemma 3. 5 (ii)), Lemma 3. 5 (ii) and the strong Markov property will imply the second assertion of Lemma 3. 6.

Proof of Lemma 3. 7 (i), (ii).

It follows from Lemma 2. 11 (iv), that, for each compact set $K\subset D$ and $\varepsilon>0$,

(3. 14) $\lim_{t\uparrow 0} \frac{1}{t} \sup_{x\in K} p(t, x, D- U_\varepsilon(x))=0$.

where $U_\varepsilon(x)=\{y\in D, \rho(x, y)<\varepsilon\}$.

(3. 14) implies

(3. 15) $P_x(X_t$ is continuous for every $t<\tau_n)=1$,

$x\in D^*$, (see E.B. Dynkin [3], Lemma 6.6). Letting n tend to infinity, we have the first statement of Lemma 3.7.

Next, take a regular exhaustion $\{D_n\}$. Then, we have

(3. 16) $P_x(\tau_n=0)=1, \quad x\in\partial D_n, \quad n=1, 2, \cdots,$

(3. 17) for each n and compact set $K\subset D_n$,

$$\lim_{u\downarrow 0} \sup_{x\in K} P_n(\tau_n\leqq u)=0 ,$$

(3. 18) for each twice continuously differentiable functions f on D,

$$\lim_{t\downarrow 0} \frac{1}{t}(T_t f(x)-f(x))=\frac{1}{2}\triangle f(x), \quad x\in D .$$

Indeed, (3. 18) is immediate. Property (3. 16) follows from $P_x(\tau_n>t)\leqq 1-P_x(X_t\in D-D_n)$ and $P_x(X_t\in D^*-D_n)\geqq\int_{D-D_n} p^0(t, x, y)\,dy$. Property (3. 17) follows from the following estimate ([3], Lemma 6.1): for any Borel subset G of D, $P_x(X_t\in D_n\cup\partial D_n$ for every $t\leqq u)\geqq p(u, x, G)-\sup_{y\in D-D_n, 0<t\leqq u} p(t, y, G)$. Since T_t maps $\boldsymbol{B}(D)$ into $\boldsymbol{C}(D)$ (Theorem 1 (v)), it follows from (3. 16) and (3. 17) that the operator T_t^n, defined by $T_t^n f(x)=E_x(f(X_t); t<\tau_n)$, $x\in D_n$, makes invariant the space of all continuous functions which vanish on ∂D_n (see E.B. Dynkin [4]. Theorem 13.1 and Theorem 13.8). Let $p^{(n)}(t, x, y)$ denote the transition density of the absorbing barrier Brownian motion on D_n. Then, combining the above property of T_t^n, the continuity of trajectory X_t, $t<\tau$, and formula (3. 18), we can conclude ([4], chap. V, §6) that, for any Borel subset E of D_n,

$$P_x(X_t\in E, t<\tau_n)=\int_E p^{(n)}(t, x, y)\,dy, \ t>0, \quad x\in D_n .$$

Let n tend to infinity to obtain conclusion (ii) of our lemma.

Proof of Lemma 3.7 (iii), (iv).

Let us fix $c>0$. Denote by $\mathfrak{L}$ the class of all D^*-valued functions defined on $[0, c]$. Define the operator $\mathfrak{q}$ from $\mathfrak{L}$ to $\mathfrak{L}$ by $\mathfrak{q}\varphi(t)=\varphi(c-t)$, $0\leqq t\leqq c$, $\varphi\in\mathfrak{L}$. For $\omega\in\Omega$, we define $\nu(\omega)=\{X_t(\omega); 0\leqq t\leqq c\}$.

$\nu(\omega)\in\mathfrak{L}$ for almost all $\omega(P_x)$. We set for $A\in\mathfrak{F}_c$ $\gamma A=\nu^{-1}\mathfrak{q}\nu A$. According to the symmetry and the conservativity of $p(t, x, y)$, it is easy to see that

$$(3.19)\qquad \int_D P_x(\gamma A)\,dx=\int_D P_x(A)\,dx,\quad A\in\mathfrak{F}_c.$$

We shall first prove assertion (iv).
Put $A_h^{c+h}=\{\omega;\ X_{t-}\in\triangle_0$ for some $t\in(h, c+h)\}$
and $B_0^c=\{\omega;\ X_t\in\triangle_0$ for some $t\in(0, c)\}$, $h\geqq 0$.
Obviously, $A_0^c=\gamma B_0^c$, and by Lemma 3.6 (ii), and (3. 19), we have $\int_D P_x(A_0^c)\,dx=\int_D P_x(B_0^c)\,dx=0$. Hence, $P_x(A_0^c)=0$ for almost all $x\in D$. By (3. 10), we see, for each $x\in D^*$, $P_x(A_h^{c+h})=\int_D p(h, x, y)P_y(A_0^c)\,dy=0$. Letting c tend to infinity and then h tend to zero, we obtain conclusion (iv) of the present lemma.

Coming to the proof of assertion (iii), consider the set $\tilde{A}_0^c=\{\omega;\ X_{t-}\in D, X_{t-}\neq X_t$ for some $t\in(0, c)\}$. Then, $\tilde{A}_0^c=A_1\cup A_2$, where, $A_1=\{\omega;\ X_{t-}\in D, X_t\in D, X_t\neq X_{t-}$ for some $t\in(0, c)\}$ and $A_2=\{\omega;\ X_{t-}\in D, X_t\in\triangle$ for some $t\in(0, c)\}$. Denote by S a countable dense subset of $(0, c)$. Obviously, $A_1\subset\bigcup_{s\in S}\{\omega;\ X_s\in D, X_t$ has a discontinuity for some $t\in(s, (s+\tau(\theta_s\omega))\wedge c)\}$ and $A_2\subset\bigcup_{s\in S}\{\omega;\ X_s\in D, X_{\tau_n(\theta_s\omega)}\notin\partial D_n$ for some n such as $s+\tau_n(\theta_s\omega)<c\}$. By virtue of (i) and (ii) of Lemma 3. 7, one has $P_x(A_1\cup A_2)=0$ for $x\in D$, and consequently (see the proof of (iv)) for all $x\in D^*$. Set $\tilde{B}_0^c=\gamma\tilde{A}_c^0$, then the same argument as in the proof of (iv) leads to $P_x(\tilde{B}_0^c)=0$, $x\in D^*$.

The final statement of Lemma 3. 7 follows from assertion (iv) of the lemma and assertion (i) of Lemma 3. 6. (see [11], Section 2, (i)).

4. The Dirichlet norm related to the process and the continuity of trajectories at the boundary

The main purpose of this section is to show in Lemma 4. 5 that, for almost all ω, the entire trajectory $X_t(\omega)$, $0\leqq t<+\infty$, is continuous. Since we already proved that $X_t(\omega)$ is continuous for all $t>0$ such that $X_t(\omega)$ or $X_{t-}(\omega)\in D$, it remains to prove that $X_t(\omega)$ has no jumps at the boundary $\triangle$.

First, we will give an integral representation of 1-excessive functions.

DEFINITION 6. A non-negative function u on D^* is called *α-excessive* if

$$(4.1)\qquad e^{-\alpha t}T_t u(x)\uparrow u(x)\ \text{as}\ t\downarrow 0\ \text{for each}\ x\in D^*.$$

Lemma 4.1.

(i) *If a non-negative function u defined on D satisfies* (4. 1) *for every $x\in D$, then u is uniquely extended to an α-excessive function on D^*.*

(ii) *If u_1 and u_2 are α-excessive and $u_1(x)=u_2(x)$ almost everywhere on D, then u_1 and u_2 coincide on D^*.*

Proof. (i). For $x\in D^*$, $e^{-\alpha t}T_t u(x)=e^{-\alpha t}\int_D p(t, x, y)u(y)dy$ is monotone increasing as $t\downarrow 0$, and we have only to set $\tilde{u}(x)=\lim_{t\downarrow 0} T_t u(x)$. The uniqueness of $\tilde{u}$ and assertion (ii) are easily verified.

Set $\triangle_1=\triangle-\triangle_0$.

Lemma 4.2.

(i) *$G_\alpha(x, y)$, $(x, y)\in D^*\times D$, can be extended to $(x, y)\in D^*\times D^*$ in such a way that the extended function $G_\alpha(x, y)$ is symmetric in $x, y\in D^*$ and, for each x(resp. y)$\in D^*$, it is α-excessive in y(resp. x).*

(ii) *For each branching point $x\in\triangle_0$, the branching measure $\mu(x, \cdot)$ is concentrated on $\triangle_1\cup\partial$.*

Proof. (i). By Theorem 1 (v) and Lemma 3.3, $G_\alpha(x, y)$ is, for each $y\in D$, α-excessive in $x\in D^*$ and it satisfies (4.1) as a function of $y\in D$, for each $x\in D^*$. By virtue of Lemma 4.1, $G_\alpha(x, y)$, $x\in D^*$, has an α-excessive extension with respect to y. The symmetry of the extended kernel follows from Theorem 1 (ii). (ii). As we have seen in Section 3, (see the proof of Lemma 3.5),

$$f(x)=\int_{D\cup\triangle_1}\mu(x, dy)f(y), \text{ for } f=G_\alpha h,\ h\in \boldsymbol{B}(D^*).$$

Hence, by Lemma 4.1 (ii),

$$G_\alpha(x, y)=\int_{D\cup\triangle_1}\mu(x, dz)G_\alpha(z, y), \quad y\in D. \tag{4.2}$$

When $x\in\triangle_0$, $G_\alpha(x, y)$ is α-harmonic in y and equation (4.2) implies that $\mu(x, \cdot)$ has no mass on D (see Lemma 2.1).

Theorem 3.

If u is 1-excessive and $\int_D u(x)dx<+\infty$, then there exists a unique measure ν concentrated on $D\cup\triangle_1$ such as

$$u(x)=\int_{D\cup\triangle_1}G_1(x, y)\nu(dy), \quad x\in D^*. \tag{4.3}$$

We call ν the canonical measure corresponding to u.

Proof. Since u is 1-excessive, there is an increasing sequence of non-negative functions f_n, $n=1, 2, \cdots$, such that

$$G_1 f_n(x) \underset{n\to+\infty}{\uparrow} u(x), \quad x \in D^* .$$

Because of Theorem 1 (ii), $\int_D f_n(x)dx = (f_n, G_1 1) = (G_1 f_n, 1) \leqq \int_D u(x)dx < +\infty$. Hence, extracting a subsequence if necessary, the sequence of measures $f_n(x)dx$ converges weakly to a measure $\nu_0(dx)$ on D^*. By Corollary to Lemma 3. 1, $G_1\varphi$ is continuous if $\varphi \in C_0(D)$, so that $(\varphi, u) = \lim_{n\to+\infty}(\varphi, G_1 f_n) = \lim_{n\to+\infty}(G_1\varphi, f_n)$ $= \int_{D\cup\triangle} G_1\varphi(x)\nu_0(dx)$, $\varphi \in C_0(D)$. Thus, it holds that

$$(4.\ 4) \qquad u(x) = \int_{D\cup\triangle} G_1(x, y)\nu_0(dy),$$

for almost all $x \in D$, and consequently (Lemma 4. 1 (ii)) for every $x \in D^*$. Using (4. 2) and Lemma 4. 2 (ii), we can rewrite (4. 4) in the form (4. 3) with ν defined by $\nu(dy) = \nu_0(dy) + \int_{\triangle_0} \nu_0(dz)\mu(z, dy)$. The measure ν of (4. 3) is uniquely determined by u. In fact, for any $f \in C(D^*)$, $\int_{D^*} f(x)\nu(dx) = \lim_{\alpha\to+\infty} \alpha \int_{D\cup\triangle_1} G_\alpha f(x)\nu(dx)$ $= \lim_{\alpha\to+\infty} \alpha \int_{D\cup\triangle_1} (G_1 f(x) - (\alpha-1)G_1 G_\alpha f(x))\nu(dx) = \lim_{\alpha\to+\infty} \alpha(u, f-(\alpha-1)G_\alpha f)$. The proof of Theorem 3 is complete.

Our next task is about the canonical measures corresponding to a special class of excessive functions.

DEFINITION 7. The $(-\infty, +\infty]$-valued function $A_t(\omega)$ on $[0, +\infty]\times\Omega$ is called an *α-additive functional* of $\boldsymbol{X}$, if

(A. 1) for fixed t, $A_t(\omega)$ is $\mathfrak{F}_{t+}$-measurable in ω,

and if there is $\mathfrak{A}$-measurable set Ω_A closed under the operation θ_t, $t>0$, such that $P_x(\Omega_A) = 1$, $x \in D^*$, and for each fixed $\omega \in \Omega_A$,

(A. 2) $A_t(\omega)$ is right continuous and has the left limit in t,

(A. 3) $\zeta(\omega) = 0$ implies $A_t(\omega) = 0$ for $t \geqq 0$,

where $\zeta(\omega)$ is a hitting time to ∂, and

(A. 4) $A_{t+s}(\omega) = A_t(\omega) + e^{-\alpha t}A_s(\theta_t\omega)$, for $t, s \geqq 0$.

Two α-additive functionals A and B are called *equivalent* and denoted by $A \approx B$, when $A_t = B_t$ holds almost everywhere (P_x) for each $t \geqq 0$ and $x \in D^*$. A 0-additive functional will be called an *additive functional* simply.

Put $\mathfrak{R} = \{u; u = G_\alpha f, f \in \boldsymbol{B}(D^*)\}$. $\mathfrak{R}$ is contained in $\boldsymbol{B}(D^*)$ and independent of $\alpha > 0$. If $G_\alpha f_1(x) = G_\alpha f_2(x)$, $x \in D^*$, $f_1, f_2 \in \boldsymbol{B}(D^*)$, then, as one easily sees,

$f_1=f_2$ almost every-where on D.

Take $u\in\Re$. If $u=G_{1/2}f$, $f\in \boldsymbol{B}(D^*)$, we set

$$(4.5)\qquad A_t^u=e^{-t/2}u(X_t)-u(X_0)+\int_0^t e^{-s/2}f(X_s)\,ds,\quad t\geqq 0\,.$$

It is easy to see that A_t^u is a 1/2-additive functional and it is uniquely determined by u up to equivalence. Clearly $E_x(A_t^u)=0$, $x\in D^*$, $t\geqq 0$. We see that

$$(4.6)\qquad v_u(x)=E_x((A_{+\infty}^u)^2)$$

is a 1-excessive function. In fact, $A_{+\infty}^u(\omega)=A_t^u(\omega)+e^{-t/2}A_{+\infty}^u(\theta_t\omega)$ implies $v_u(x) =E_x((A_t^u)^2)+2E_x(e^{-t/2}A_t^uE_{X_t}(A_{+\infty}^u))+E_x(e^{-t}E_{X_t}((A_{+\infty}^u)^2)=E_x((A_t^u)^2)+e^{-t}T_tv_u(x)$, and $e^{-t}T_tv_u(x)\uparrow v_u(x)$ as $t\downarrow 0$, $x\in D^*$. Moreover, $\int_D v_u(x)\,dx<+\infty$, and so, v_u is expressed as the G_1-potential of a measure on $D_1^*=D\cup\triangle_1$ according to Theorem 3.

DEFINITION 8. For $u\in\Re$, define A_t^u and v_u by (4.5) and (4.6), respectively. Denote by ν_u the canonical measure on $D\cup\triangle_1$ corresponding to v_u. Set $|||u|||_X=\sqrt{\nu_u(D\cup\triangle_1)}$ and call this the *Dirichlet norm* of $u\in\Re$ with respect to the process $\boldsymbol{X}$.

We will show

Theorem 4. *Let u be in $\Re$. Then,*

(i) $|||u|||_X^2=\int_D(\text{grad } u, \text{grad } u)\,(x)\,dx\,,$

(ii) $\nu_u(\triangle_1)=0\,.$

Let us prepare two lemmas.

Lemma 4.3.
$|||u|||_X^2=2(u,f)-(u,u),\quad u\in\Re.$

Proof. Since $\int_D G_1(x,y)\,dx=\int_D G_1(y,x)\,dx=q(y)=1$ for $y\in D\cup\triangle_1$ (Lemma 3.5 (iii)), we have $|||u|||_X^2=\nu_u(D\cup\triangle_1)=\int_D v_u(x)\,dx$. On the other hand,

$$\begin{aligned}v_u(x)&=E_x((\int_0^{+\infty}e^{-s/2}f(X_s)\,ds)^2)-u(x)^2\\&=2E_x(\int_0^{+\infty}e^{-t/2}f(X_t)\,dt\int_t^{+\infty}e^{-s/2}f(X_s)\,ds)-u(x)^2\\&=2E_x(\int_0^{+\infty}e^{-t}f(X_t)\,dt\,E_{X_t}(\int_0^{+\infty}e^{-s/2}f(X_s)\,ds))-u(x)^2\\&=2\int_D G_1(x,y)f(y)u(y)\,dy-u(x)^2\,.\end{aligned}$$

Hence, Lemma 4.3 is valid.

Lemma 4.4. *Let τ be the first exit time from D defined in Lemma* 3.7 (i). *Then we have, for $u \in \mathfrak{R}$,*

$$E_x((A^u_{\tau-})^2) = \int_D G^0_1(x, y)(\text{grad } u, \text{grad } u)(y)\,dy, \quad x \in D, \tag{4.7}$$

$$E_x((A^u_{\tau-})^2) = \int_D G^0_1(x, y)\,\nu_u(dy), \quad x \in D, \tag{4.8}$$

$$\nu_u(D) = \int_D (\text{grad } u, \text{grad } u)(y)\,dy. \tag{4.9}$$

Proof. Let $\{\tau_n\}$ be the first exit times from an exhaustion $\{D_n\}$ of D. By definition, $\tau_n \uparrow \tau$. In view of Lemma 3.7 (ii), $\{X_t, t<\tau_n\}$ is equivalent to the absorbing barrier standard Brownian motion on D_n. Now, suppose that f belongs to $C^1(D)$. Then, $u = G_{1/2}f = G^0_{1/2}f + R_{1/2}f$ belongs to $C^2(D)$ and $\left(\frac{1}{2} - \frac{1}{2}\triangle\right)u(x) = f(x)$, $x \in D$[15]. Applying the formula concerning stochastic integrals[16] to the function $F(t,x) = e^{-t/2}u(x)$, we obtain $A^u_{\tau_n} = \int_0^{\tau_n} e^{-s/2}\,\text{grad } u(X_s)\,dX_s$, and consequently

$$E_x((A^u_{\tau_n})^2) = E_x\left(\int_0^{\tau_n} e^{-s}(\text{grad } u, \text{grad } u)(X_s)\,ds\right), \quad x \in D. \tag{4.10}$$

Consider the collection $\mathfrak{H}$ of all bounded functions f on D such that $u = G_{1/2}f$ satisfies equation (4.10) for a fixed n. Obviously $\mathfrak{H}$ is a linear space and $C^1(D) \subset \mathfrak{H}$. It is easy to see that, if $f_k \in \mathfrak{H}$ converges boundedly to a bounded function f, then $f \in \mathfrak{H}$. Hence, $\mathfrak{H} = B(D)$. We get formula (4.7) by letting n tend to infinity in (4.10). In order to show identity (4.8), we have only to let n tend to infinity in the first and last term of the following identity.

$$\begin{aligned} E_x((A^u_{\tau_n})^2) &= v_u(x) - E_x(e^{-\tau_n} v_u(X_{\tau_n})) \\ &= \int_{D \cap \triangle_1} G_1(x, y)\,\nu_u(dy) - \int_{D \cup \triangle_1} E_x(e^{-\tau_n} G_1(X_{\tau_n}, y))\,\nu_u(dy) \\ &= \int_D (G_1(x, y) - E_x(e^{-\tau_n} G_1(X_{\tau_n}, y)))\,\nu_u(dy). \end{aligned}$$

The formulae (4.7) and (4.8) imply identity (4.9).

Proof of Theorem 4. It follows from the definition of $R_{1/2}(x, y)$ that, when $u \in \mathfrak{R}$ and $u = G_{1/2}f$, $f \in B(D)$,

15) $C^1(D)$ ($C^2(D)$) is the aggregate of all bounded, continuously (resp. twice continuously) differentiable functions on D.

16) cf. [4], (7. 77).

(4.11) $\quad \boldsymbol{D}_{1/2}(u, u)=2(u, f)$.

Indeed, the same procedure as in the proof of Lemma 2.10 is applicable to get $\boldsymbol{D}_{1/2}(R_{1/2}f, R_{1/2}f)=2(R_{1/2}f, f)$. It is easy to see that $\boldsymbol{D}_{1/2}(G^0_{1/2}f, G^0_{1/2}f)=2(G^0_{1/2}f, f)$ and $\boldsymbol{D}_{1/2}(G^0_{1/2}f, R_{1/2}f)=0$. Rewrite (4.11) in the form, $2(u, f)-(u, u)=\int_D(\text{grad } u, \text{grad } u)(y)\,dy$. Now, assertions (i) and (ii) of Theorem 4 follow from Lemma 4.3 and Lemma 4.4, respectively.

Coming to our main task about the continuity of trajectories of $\boldsymbol{X}$, we shall introduce several notations and concepts given by M. Motoo and S. Watanabe [18]. In [18], Hunt processes are treated. Our process $\boldsymbol{X}$ is not a Hunt process in general: It may include branching points. However, owing to Lemmas 3.6, 3.7 and 4.1, all the results in [18] can be applied to our process.

Set

$\mathfrak{C}_1^+=\{A$; A is an additive functional of $\boldsymbol{X}$ such that $A_t(\omega)$, $t\geqq 0$, $\omega\in\Omega_A$, is non-negative, continuous in t and $E_x(A_t)<+\infty$ for $t\geqq 0$, $x\in D^*\}$[17]

$\mathfrak{C}_1=\{A;\ A=A_1-A_2,\ A_i\in\mathfrak{C}_1^+,\ i=1, 2\}$,

$\mathfrak{M}=\{A$; A is an additive functional of $\boldsymbol{X}$ such that $E_x(A_t^2)<+\infty$ and $E_x(A_t)=0$ for $t\geqq 0$, $x\in D^*\}$.

Let $A, B\in\mathfrak{M}$. Then there exists a unique element of $\mathfrak{C}_1$, denoted by $\langle A, B\rangle$, satisfying the following condition: $E_x(\langle A, B\rangle_t)=E_x(A_tB_t)$ holds for every $t\geqq 0$ and $x\in D^*$. For $A\in\mathfrak{M}$, $\langle A, A\rangle$ will be denoted by $\langle A\rangle$. It is an element of. $\mathfrak{C}_1^+$.

We set, for $A\in\mathfrak{M}$,

$\boldsymbol{L}^2(A)=\{f$; f is a measurable function on D^* such that $E_x(\int_0^t f(X_s)^2 d\langle A\rangle_s)<+\infty$ for every $t>0$, $x\in D^*\}$.

DEFINITION 9.

Let $A\in\mathfrak{M}$ and $f\in\boldsymbol{L}^2(A)$. $B\in\mathfrak{M}$ is called the stochastic integral of f by A and is denoted by $B=\int f dA$ if $E_x(B_tC_t)=E_x(\int_0^t f(X_s)d\langle A, C\rangle_s)$, $t\geqq 0$, holds for every $C\in\mathfrak{M}$.

The stochastic integral exists uniquely for $A\in\mathfrak{M}$ and $f\in\boldsymbol{L}^2(A)$ (Theorem 10.4 of [18]). As a consequence of Theorem 4, we have

Theorem 5. *Denote by $\chi_{\triangle_1}$ the indicator function of the set $\triangle_1$. It holds that $\int\chi_{\triangle_1}dA\approx 0$ for any $A\in\mathfrak{M}$.*

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17) Ω_A is a suitable defining set of A (see Definition 7).

Proof. (i). Set, for $u \in \mathfrak{R}$ and $u = G_{1/2} f$ with $f \in \boldsymbol{B}(D^*)$,

$$\tilde{A}_t^u = u(X_t) - u(X_0) + \int_0^t (f(X_s) - \frac{1}{2} u(X_s))\, ds, \quad t \geqq 0. \tag{4.12}$$

Obviously, $\tilde{A}^u \in \mathfrak{M}$. Let us show, for $u \in \mathfrak{R}$,

$$\int \chi_{\triangle_1} d\tilde{A}^u \approx 0, \text{ or equivalently} \tag{4.13}$$

$$\int_0^t \chi_{\triangle_1}(X_s)\, d\langle \tilde{A}^u \rangle_s = 0, \ t \geqq 0, \text{ almost everywhere } (P_x),\ x \in D^*. \tag{4.14}$$

Since A^u defined by (4.5) is related to $\tilde{A}^u$ by $A_t^u = e^{-t/2} \tilde{A}_t^u + \frac{1}{2} \int_0^t e^{-s/2} \tilde{A}_s^u\, ds$, v_u defined by (4.6) is expressed as

$$v_u(x) = E_x\Big(\int_0^{+\infty} e^{-s}\, d\langle \tilde{A}^u \rangle_s\Big), \quad x \in D^*. \tag{4.15}$$

On the other hand, $v_v(x) = \int_{D \cup \triangle_1} G_1(x, y)\, \nu_u(dy)$, and by virtue of Theorem 4 (which states $\nu_u(\triangle_1) = 0$), $\langle \tilde{A}^u \rangle_t$ can never increase when $X_t \in \triangle_1$ (see [6] or [14]), that is, $\int \chi_{\triangle_1}(X_s)\, d\langle \tilde{A}^u \rangle_s \approx 0$.

(ii). In order to derive Theorem 5 from (4.13), we introduce several notations. We write l.i.m $A^n = A$, for A^n and $A \in \mathfrak{M}$, if and only if $E_x((A_t^u - A_t)^2) \xrightarrow[n \to +\infty]{} 0$, $x \in D^*$, $t \geqq 0$. A subset $\boldsymbol{L}$ of $\mathfrak{M}$ is called a subspace, if $\boldsymbol{L}$ satisfies the following conditions.

(a) If A, $B \in \boldsymbol{L}$, then $A + B \in \boldsymbol{L}$.
(b) If $A^n \in \boldsymbol{L}$ and $A = \text{l.i.m } A^n$, then $A \in \boldsymbol{L}$.
(c) If $A \in \boldsymbol{L}$ and $f \in \boldsymbol{L}^2(A)$, then $\int f\, dA \in \boldsymbol{L}$.

For a subset $\boldsymbol{M}$ of $\mathfrak{M}$, $\boldsymbol{L}(\boldsymbol{M})$ will stand for the minimum subspace which contains $\boldsymbol{M}$. We note that, Theorem 12.2 of [18] states $\mathfrak{M} = \boldsymbol{L}(\tilde{A}^u;\ u \in \boldsymbol{R})$, where $\tilde{A}^u$ is defined by (4.12). If we set $\mathfrak{M}' = \{A;\ A \in \mathfrak{M},\ \int \chi_{\triangle_1} dA \approx 0\}$, then $\mathfrak{M}'$ is a subspace of $\mathfrak{M}$ and contains $\tilde{A}^u$, $u \in \mathfrak{R}$, by (4.13). Hence $\mathfrak{M}' = \mathfrak{M}$, completing the proof of Theorem 5.

By the following lemma, we will complete the proof of Theorem 2 stated in Section 1.

Lemma 4.5. *The strong Markov process* $\boldsymbol{X} = \{X_t, \mathfrak{F}_{t+}, P_x, x \in D^*\}$ *is a diffusion, that is,* $\boldsymbol{X}$ *satisfies the condition*
(b) $P_x(X_t$ *is continuous for every* $t \geqq 0) = 1$, $x \in D^*$.

Proof. Let $\rho(x, y)$ be the metric on D^* defined by (3.1). We shall set, for convenience, $\rho(x, \partial)=+\infty$, $x\in D^*$ and $\rho(\partial, \partial)=0$. For $\varepsilon>0$, define σ^ε by

$$\begin{aligned}\sigma^\varepsilon &= \inf\{t;\ \rho(X_{t-}, X_t)>\varepsilon\}\ , \\ &= +\infty \qquad \text{if there is no such } t\,,\end{aligned}$$

and $\sigma_1^\varepsilon, \sigma_2^\varepsilon, \cdots$, by $\sigma_1^\varepsilon=\sigma^\varepsilon$, $\sigma_n^\varepsilon=\sigma_{n-1}^\varepsilon(\omega)+\sigma^\varepsilon(\theta_{\sigma^\varepsilon_{n-1}}\omega)$. Set $\mathfrak{p}_t^{\varepsilon,E}=\sum_{\sigma_n^\varepsilon\leqq t}\chi_E(X_{\sigma_n^\varepsilon})$, for $E\in\mathfrak{B}(D^*\cup\partial)$ and $t\geqq 0$. Obviously, $\mathfrak{p}_t^{\varepsilon,E}$ is an additive functional. We shall denote $\mathfrak{p}_t^{\varepsilon,D^*\cup\partial}$ by $\mathfrak{p}_t^\varepsilon$. Statement (b) is equivalent to

(4. 16) $\quad \mathfrak{p}_t^\varepsilon\approx 0$, for any $t\geqq 0$ and $\varepsilon>0$.

Let us show (4.16). We can find $B_m\in\mathfrak{B}(D^*\cup\partial)$ such that $B_m\uparrow D^*\cup\partial$ and $E_x(\mathfrak{p}_t^{\varepsilon,B_m})<+\infty$, $x\in D^*$, $t\geqq 0$ (Lemma 3.1 of [22]). For B_m, there is $\tilde{\mathfrak{p}}_t^{\varepsilon,m}\in\mathfrak{C}_1^+$ such as

(4. 17) $\quad E_x(\mathfrak{p}_t^{\varepsilon,B_m})=E_x(\tilde{\mathfrak{p}}_t^{\varepsilon,m}),\quad t\geqq 0,\quad x\in D^*$.

If we put $\mathfrak{q}_t^{\varepsilon,m}=\mathfrak{p}_t^{\varepsilon,B_m}-\tilde{\mathfrak{p}}_t^{\varepsilon,m}$, then $\mathfrak{q}^{\varepsilon,m}\in\mathfrak{M}$ and

(4. 18) $\quad \langle\mathfrak{q}^{\varepsilon,m}\rangle\approx\tilde{\mathfrak{p}}^{\varepsilon,m}$ (Theorem 2.2 of [22]).

Now Theorem 5 implies

(4. 19) $\quad E_x\left(\int_0^t\chi_{\triangle_1}(X_s)\,d\tilde{\mathfrak{p}}_s^{\varepsilon,m}\right)=0,\quad t\geqq 0,\quad x\in D^*$.

On the other hand, we have from identity (4.17),

(4. 20) $\quad E_x\left(\sum_{\sigma_n^\varepsilon\leqq t}\chi_{\triangle_1}(X_{\sigma_n^\varepsilon-})\chi_{B_m}(X_{\sigma_n^\varepsilon})\right)=E_x\left(\int_0^t\chi_{\triangle_1}(X_s)\,d\tilde{\mathfrak{p}}_s^{\delta,m}\right)$,

$x\in D^*$ (Lemma 3.2 of [22]). The left hand side of equation (4.20) is, owing to assertions (iii) and (iv) of Lemma 3.7, no other than $E_x(\mathfrak{p}_t^{\varepsilon,B_m})$. Therefore, the formulae (4.19) and (4.20) imply $\mathfrak{p}_t^{\varepsilon,B_m}\approx 0$, and consequently assertion (4.16).

We call the conservative diffusion process $\{X_t, \mathfrak{F}_{t+}, P_x, x\in D_1^*\}$ the *reflecting barrier Brownian motion* on $D_1^*=D\cup\triangle_1$.

Consider the case when ∂D is of class C^3. By virtue of Lemma 3.1 (iv), we can find a homeomorphism Ψ from $D\cup\partial D$ onto D^* such as $\Psi(x)=x$, $x\in D$. In this case, $\triangle_0$ is empty and so, $D^*=D_1^*$ (see the identity (3.11) and the proof of Lemma 3.1). Set $\dot{X}_t=\Psi^{-1}(X_t)$, $t\geqq 0$ and $\dot{P}_x=P_{\Psi(x)}$, $x\in D\cup\partial D$. Theorem 2 and the argument in the paragraph following Theorem 1 now imply

Theorem 6. *Suppose that ∂D is of class C^3. Then, $\dot{X}=(\dot{X}_t, \dot{P}_x, x\in D\cup\partial D)$ is a conservative diffusion process on $D\cup\partial D$ satisfying $\dot{P}_x(\dot{X}_t\in E)$*

$=\int_{E\cap D}\dot{p}(t, x, y)dy$, $t>0$, $x\in D\cup\partial D$, *for any Borel set* E *of* $D\cup\partial D$. *Here,* $\dot{p}(t, x, y)$, $t>0$, $x\in D^*$, $y\in D$ *is the fundamental solution of the heat equation* $\left(\frac{\partial}{\partial t}-\frac{1}{2}\triangle\right)u(t, x)=0$ *with the condition* $\frac{\partial}{\partial n_x}u(t, x)=0$, $x\in\partial D$. *We call* $\dot{X}$ *the reflecting barrier Brownian motion on* $D\cup\partial D$.

See K. Sato and T. Ueno [21] for another version of $\dot{X}$.

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On boundary conditions for multi-dimensional Brownian motions with symmetric resolvent densities

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§ 1. Introduction.

Let D be an arbitrary bounded domain of the N-dimensional Euclidean space $R^N(N \geqq 1)$. A function $G_\alpha(x, y)$ defined for $\alpha > 0$, $x, y \in D$, $x \neq y$ will be called a *resolvent density* on D, if it satisfies that, $G_\alpha(x, y) \geqq 0$, $\alpha \int_D G_\alpha(x, z)dz \leqq 1$ and $G_\alpha(x, y) - G_\beta(x, y) + (\alpha - \beta)\int_D G_\alpha(x, z)G_\beta(z, y)dz = 0$ for all $\alpha > 0$, $\beta > 0$ and $x, y \in D$, $x \neq y$. Denote by $G_\alpha^0(x, y)$ the resolvent density corresponding to the *absorbing barrier Brownian motion* on D[1].

Consider the family $\boldsymbol{G}$ of all *conservative symmetric* resolvent densities[2] on D possessing the following properties:

(G. a) *$G_\alpha(x, y)$ is written in the form*

$$G_\alpha(x, y) = G_\alpha^0(x, y) + R_\alpha(x, y).$$

$R_\alpha(x, y)$ is a non-negative function of $\alpha > 0$, $x, y \in D$, and α-harmonic[3] *in $x \in D$ for each $\alpha > 0$ and $y \in D$.*

(G. b) *For any compact subset K of D, $\sup_{x \in K, y \in D} R_\alpha(x, y)$ is finite.*

In [15], we constructed a particular element of $\boldsymbol{G}$ and showed that it determines a continuous strong Markov process (called the *reflecting barrier Brownian motion*) on an extended state space D^*.

In the present paper, by studying the structure of Dirichlet spaces associated with elements of $\boldsymbol{G}$, we will answer the questions:

(i) How many elements are there in $\boldsymbol{G}$?

(ii) In what sense is the resolvent density of [15] typical among $\boldsymbol{G}$?

1) Cf. [5].

2) We will say that a resolvent density $G_\alpha(x, y)$ is conservative (resp. symmetric) when $\alpha \int_D G_\alpha(x, z)dz = 1$, $\alpha > 0$, $x \in D$ (resp. $G_\alpha(x, y) = G_\alpha(y, x)$, $\alpha > 0$, $x, y \in D$).

3) We call a function on D α-harmonic when

$$\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2 u(x)}{\partial x_i^2} = \alpha u(x), \qquad x \in D.$$

Our goal is to establish in section 5 and section 7 a one-to-one correspondence between $\boldsymbol{G}$ and a class of Dirichlet spaces formed by functions on the Martin boundary of the domain D.

The present paper consists of nine sections.

Sections 2 and 3 will serve as preparations for later discussions. In section 2 we will introduce the notion of the Dirichlet space (relative to an L^2-space), in a slightly modified sense, due to Beurling and Deny [2]. In section 3, the Dirichlet space formed by every square integrable BLD function (denoted by $\widehat{\mathrm{BLD}}$) will be studied by making use of the Feller kernels on the Martin boundary.

With a given element $G_\alpha(x, y) = G_\alpha^0(x, y) + R_\alpha(x, y)$ of the class $\boldsymbol{G}$, we associate a Dirichlet space $(\mathcal{F}_D, \mathcal{E})$ relative to $L^2(D)$ by

$$\mathcal{F}_D = \{u \in L^2(D)\,;\ \mathcal{E}(u, u) = \lim_{\beta \to +\infty} \beta(u - \beta G_\beta u, u)_{L^2(D)} < +\infty\}\,.$$

In sections 4, 5 and 6, the space $(\mathcal{F}_D, \mathcal{E})$ will be analized in details as outlined in the following.

Let $\mathcal{F}_D^{(0)}$ (actually independent of $\alpha > 0$) be the space spanned by $\{G_\alpha^0 f, f \in \boldsymbol{B}(D)\}$ with respect to the norm $\sqrt{\mathcal{E}^\alpha(u, u)} = \sqrt{\mathcal{E}(u, u) + \alpha(u, u)_{L^2(D)}}$ and $\mathcal{H}_\alpha$, the space spanned by $\{R_\alpha f, f \in \boldsymbol{B}(D)\}$. For each $\alpha > 0$, spaces $\mathcal{F}_D^{(0)}$ and $\mathcal{H}_\alpha$ are orthogonal with respect to $\mathcal{E}^\alpha$ and $\mathcal{F}_D = \mathcal{F}_D^{(0)} \oplus \mathcal{H}_\alpha$. Further the space $(\mathcal{F}_D^{(0)}, \mathcal{E})$ is identical with the space BLD_0 of BLD functions of potential type. The proof of these facts will be carried out in section 4 by making use of a Feller type expression of $R_\alpha f$: $R_\alpha f(x) = H_\alpha^x \tilde{R}^\alpha \hat{H}_\alpha f$.

Denote by M the Martin boundary of the domain D. Using the Feller kernels, we introduce by (3.14) and (3.15) respectively a bilinear form $\boldsymbol{D}(\ ,\ )$ for functions on M and a space $\boldsymbol{H}_M$ of functions on M. Theorem 5.2 and 5.3 will characterize the above-mentioned Hilbert spaces $\{(\mathcal{H}_\alpha, \mathcal{E}^\alpha), \alpha > 0\}$ by means of a Dirichlet space $(\mathcal{F}_M, \mathcal{E}_M(\ ,\ ))$ satisfying the following conditions[4].

(B. 1) *$\mathcal{F}_M$ is a linear subspace of $\boldsymbol{H}_M$. $\mathcal{F}_M$ contains every constant function on M.*

(B. 2) *$\mathcal{E}_M$ is a bilinear form on $\mathcal{F}_M$ which is written as $\mathcal{E}_M(\varphi, \psi) = \boldsymbol{D}(\varphi \cdot \psi) + \boldsymbol{N}(\varphi, \psi)$, $\varphi, \psi \in \mathcal{F}_M$, where $\boldsymbol{N}$ is a non-negative symmetric bilinear form on $\mathcal{F}_M$ satisfying $\boldsymbol{N}(1 \cdot 1) = 0$. The space $\mathcal{F}_M$ is complete with metric $\mathcal{E}_M(\ ,\ ) + \lambda(\ ,\ )_{L^2(M)'}$ for a $\lambda > 0$.*

(B. 3) *If $\varphi \in \mathcal{F}_M$ and if ψ is a normal contraction of φ in the sense of* [4], *then $\psi \in \mathcal{F}_M$ and $\boldsymbol{N}(\psi, \psi) \leqq \boldsymbol{N}(\varphi, \varphi)$.*

Conversely, for any pair $(\mathcal{F}_M, \boldsymbol{N})$ satisfying the conditions (B. 1), (B. 2)

4) Conditions (B.1), (B.2) and (B.3) implies that $(\mathcal{F}_M, \mathcal{E}_M)$ is a Dirichlet space relative to $L^2(M)'$, the space $L^2(M)'$ being defined in section 3.

and (B. 3), we will construct in section 7 an element $G_\alpha(x, y)$ of the class $\boldsymbol{G}$ which corresponds to this pair $(\mathcal{F}_M, \boldsymbol{N})$ in the manner of Theorem 5.2. In this way, we will establish a one-to-one correspondence between the class $\boldsymbol{G}$ and the class of the pairs $(\mathcal{F}_M, \boldsymbol{N})$.

Section 6 will be concerned with the boundary condition. Consider again the Dirichlet space $(\mathcal{F}_D, \mathcal{E})$ associated with a given element $G_\alpha(x, y)$ of $\boldsymbol{G}$. Since $2\boldsymbol{D}(\varphi, \varphi)$ for $\varphi \in \boldsymbol{H}_M$ is nothing but an expression of the Dirichlet integral of the harmonic function with fine boundary function φ (see Doob [7] and Fukushima [**13**]), our results of sections 4 and 5 enable us in Theorem 6.1 to conclude that $\mathrm{BLD}_0 \subset \mathcal{F}$[5] $\subset \widehat{\mathrm{BLD}}$ and, for every $u \in \mathcal{F}$, $\mathcal{E}(u, u) \geqq \frac{1}{2}\int_D (\mathrm{grad}\, u, \mathrm{grad}\, u)(x)dx$. Furthermore, we can see that the space $\mathcal{D} = G_\alpha(L^2(D))$ is a restriction of the domain $\mathcal{D}(\Delta)$ of the generalized Laplacian Δ (denoted by the same symbol Δ as the usual Laplacian), which is defined in terms of the space $\widehat{\mathrm{BLD}}$ (Definition 6.1). This restriction will be decided in terms of $(\mathcal{F}_M, \boldsymbol{N})$ by the boundary condition (6.8). Formula (6.8) includes implicitly the notion of the (generalized) normal derivative in Doob's sense [7]. Moreover, (6.8) is analogous to a boundary condition by Feller [**11**; p. 560], where the Markov chains with a finite number of exit boundary points are treated.

The final two sections will be devoted to the study of several special cases. In section 8, we will be concerned with the subclass $\boldsymbol{G}_1$ formed by those elements of $\boldsymbol{G}$ for which the corresponding forms $\boldsymbol{N}(\,,\,)$ vanish identically on the corresponding spaces $\mathcal{F}_M$. We will see that a diffusion process on an extended state space corresponds to each element of $\boldsymbol{G}_1$. There are two extreme elements of $\boldsymbol{G}_1$: the cases when $\mathcal{F}_M = \boldsymbol{H}_M$ and when $\mathcal{F}_M$ contains only constant functions. We will see that the former case turns out to reconstruct the resolvent density of [**15**]. In section 9, we will examine the cases that the domain D is a disk and an interval[6].

Here are two remarks about our class $\boldsymbol{G}$ of resolvent densities.

First, we note that there is a one-to-one correspondence between $\boldsymbol{G}$ and a family of (equivalent classes of) Markov processes dominating the absorbing Brownian motion on D. Indeed, with each element $G.(\cdot, \cdot)$ of $\boldsymbol{G}$, we can associate, exactly in the same manner as in [**15**; section 3], a right continuous strong Markov process $\boldsymbol{X} = (X_t, P_x, x \in D^*)$ whose state space D^* is the Martin-Kuramochi type completion of D with respect to the class of functions $\{G_1(\cdot, y), y \in D\}$. $\boldsymbol{X}$ has the following properties:

($\boldsymbol{X}$. 1) *$\boldsymbol{X}$ is conservative on D:*

5) $\mathcal{F}$ is the refinement of the space $\mathcal{F}_D$ (see (4.18)).

6) There, we can compare our boundary condition (6.8) with those of Wentzell [**23**] and Feller [**12**].

$$P_x(X_t \in D)=1. \qquad t>0, \qquad x \in D\,.$$

($\boldsymbol{X}$. 2) *Let τ be the first exit time from D of the path X_t, then $(X_t,\ t<\tau,$ $P_x,\ x \in D)$ is the absorbing Brownian motion on D.*

($\boldsymbol{X}$. 3) *For any Borel set E of D^*,*

$$\int_0^{+\infty} e^{-\alpha t} P_x(X_t \in E)dt = \int_{E \cap D} G_\alpha(x, y)dy, \qquad \alpha>0, \qquad x \in D\,.$$

Conversely, suppose that a right continuous strong Markov process $\boldsymbol{X}$ on an enlarged state space D^* satisfies the conditions ($\boldsymbol{X}$. 1) and ($\boldsymbol{X}$. 2). Further we assume the existence of a symmetric, jointly continuous function $G_\alpha(x, y)$, $\alpha>0$, $x,\ y \in D$, $x \neq y$ satisfying the condition ($\boldsymbol{X}$. 3). Then, as one easily verifies, this function is an element of $\boldsymbol{G}$.

Second remark is about the relation between the class $\boldsymbol{G}$ and the class of symmetric Brownian resolvents in the sense of T. Shiga and T. Watanabe [**21**]. By a Brownian resolvent, we mean a resolvent kernel $\{G_\alpha(x, E),\ \alpha>0,\ x \in D,$ $E \subset D\}$ such that $G_\alpha f(x)=\int_D G_\alpha(x, dy)f(y)$ satisfies the equation

$$\Big(\alpha-\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}\Big)G_\alpha f(x)=f(x)\,, \qquad x \in D\,,$$

for any infinitely differentiable function f with compact support. A resolvent kernel $\{G_\alpha(x, E)\}$ is said symmetric if, for any non-negative measurable functions f and g, $\int_D G_\alpha f(x)g(x)dx=\int_D f(x)G_\alpha g(x)dx \leqq +\infty$. Any symmetric resolvent kernel defines a symmetric resolvent (operator) on $L^2(D)$ in the sense of section 2, so that we can associate with it a Dirichlet space relative to $L^2(D)$. It is obvious that each element of the class $\boldsymbol{G}$ is a density function of a conservative symmetric Brownian resolvent (kernel). Conversely, we can prove that any conservative symmetric Brownian resolvent is of the class $\boldsymbol{G}$, as is outlined in the following. It is implied in the remark preceding Proposition A. 6 of [**21**] that the decomposition theorem (Theorem 4.3) of the present paper is still valid for the Dirichlet space associated with any symmetric Brownian resolvent. Hence, starting with a conservative symmetric Brownian resolvent (without assuming the existence of a density function), we can go along the same line as in section 5 and we can reconstruct in section 7 the resolvent considered, by showing that it has a density function of the class $\boldsymbol{G}$.

I wish to express my hearty thanks to T. Shiga and T. Watanabe for their valuable advices. They have shown me the manuscript of [**21**] before publication. T. Watanabe admitted me to mention one of his unpublished results that the space $\mathcal{H}_\alpha$, in our context, is contained in the space of α-harmonic functions with finite Dirichlet integrals (Theorem 5.1). This made the arguments of section 5 simpler than those of the original version.

§ 2. Symmetric resolvents and Dirichlet spaces relative to L^2-spaces.

Let $(X, \mathcal{B}, m)$ be a measure space on a Hausdorff space X with the topological Borel field $\mathcal{B}$. We assume that m is finite: $m(X) < +\infty$. Denote by $L^2(X)$ the space of all real-valued square integrable functions on X with the inner product $(u, v)_X = \int_X u(x)v(x)m(dx)$.

DEFINITION 2.1. A *symmetric resolvent* on $L^2(X)$ is a family of symmetric linear operators $\{G_\alpha, \alpha > 0\}$ on $L^2(X)$ such that $G_\alpha u$ is non-negative for any non-negative $u \in L^2(X)$, $\alpha G_\alpha 1 \leqq 1$, $G_\alpha - G_\beta + (\alpha - \beta) G_\alpha G_\beta = 0$ and $G_\alpha u_n$ decreases to zero m-almost everywhere on X when $u_n \in L^2(X)$ decreases to zero.

DEFINITION 2.2. Let u and v be measurable functions on X. We call u a *normal contraction* of v if the following inequalities are valid on X;

$$|u(x)| \leqq |v(x)|, \qquad |u(x) - u(y)| \leqq |v(x) - v(y)|.$$

DEFINITION 2.3. A function space $(\mathcal{F}_X, \mathcal{E}_X(\,,\,))$ is called a *Dirichlet space relative to* $L^2(X)$, if the following three conditions are satisfied.

(2.1) $\mathcal{F}_X$ is a non-emply linear subset of $L^2(X)$ and $\mathcal{E}_X(\,,\,)$ is a non-negative symmetric bilinear form on $\mathcal{F}_X$.

(2.2) For some (or equivalently for every) $\alpha > 0$, $\mathcal{F}_X$ is a real Hilbert space with the inner product

$$\mathcal{E}_X^\alpha(u, v) = \mathcal{E}_X(u, v) + \alpha(u, v)_X\,,$$

two functions of $\mathcal{F}_X$ being identified if they coincide m-almost everywhere on X.

(2.3) Every normal contraction operates on $(\mathcal{F}_X, \mathcal{E}_X)$; if u is a normal contraction of $v \in \mathcal{F}_X$, then $u \in \mathcal{F}_X$ and $\mathcal{E}_X(u, u) \leqq \mathcal{E}_X(v, v)$.

Following Beurling and Deny [2] and Deny [4], let us state two theorems about a one-to-one correspondence between Dirichlet spaces and symmetric resolvents.

THEOREM 2.1. *Let $(\mathcal{F}_X, \mathcal{E}_X(\,,\,))$ be a Dirichlet space relative to $L^2(X)$.*

(i) *For each $\alpha > 0$ and $u \in L^2(X)$, there is a unique element $G_\alpha u$ of $\mathcal{F}_X$ such that*

$$\mathcal{E}_X^\alpha(G_\alpha u, v) = (u, v)_X \quad \textit{for any } v \in \mathcal{F}_X\,. \tag{2.4}$$

(ii) *The family of operators G_α, $\alpha > 0$, defined by* (2.4) *is a symmetric resolvent on $L^2(X)$.*

(iii) *For each $\alpha > 0$, $\{G_\alpha u\,;\, u \in L^2(X)\}$ is dense in $\mathcal{F}_X$ with respect to the norm $\mathcal{E}_X^\beta$ ($\beta > 0$ being arbitrary).*

We note that the non-negativity and the sub-Markov property of αG_α, where G_α is defined by the equation (2.4), follow from the condition (2.3) of the space $(\mathcal{F}_X, \mathcal{E}_X)$. Conversely, suppose that we are given a symmetric resolvent $\{G_\alpha, \alpha>0\}$ on $L^2(X)$. It is easy to see that G_α on $L^2(X)$ is a bounded operator with norm less than $1/\alpha$ and consequently $(G_\alpha u, u)_X$ is non-negative for any $u \in L^2(X)$[7]. Put for $\alpha \geqq 0$ and $u \in L^2(X)$,

$$\mathcal{E}^\alpha_{X,\beta}(u, u)=\beta(u-\beta G_{\beta+\alpha}u, u)_X \tag{2.5}$$

$$\mathcal{T}^\alpha_{X,\beta}(u, u)=(u-\beta G_{\beta+\alpha}u, u-\beta G_{\beta+\alpha}u)_X\,. \tag{2.6}$$

We then have,

$$\frac{\partial}{\partial\beta}\mathcal{E}^\alpha_{X,\beta}(u, u)=\mathcal{T}^\alpha_{X,\beta}(u, u) \quad \text{and} \quad \frac{\partial}{\partial\beta}\mathcal{T}^\alpha_{X,\beta}(u, u)\leqq 0, \quad \beta>0\,, \tag{2.7}$$

which leads us to the following theorem.

THEOREM 2.2. *Let $\{G_\alpha, \alpha>0\}$ be a symmetric resolvent on $L^2(X)$.*

(i) *$\mathcal{E}^\alpha_{X,\beta}(u, u)$ defined by* (2.5) *is non-negative and it is non-decreasing as β increases. If we set*

$$\mathcal{E}_X(u, u)=\lim_{\beta\to+\infty}\mathcal{E}^0_{X,\beta}(u, u)\,, \qquad u\in L^2(X)\,, \tag{2.8}$$

$$\mathcal{F}_X=\{u\,;\; u\in L^2(X)\,, \qquad \mathcal{E}_X(u, u)<+\infty\}\,, \tag{2.9}$$

then $(\mathcal{F}_X, \mathcal{E}_X(\,,\,))$ is a Dirichlet space relative to $L^2(X)$.

(ii) *For $u\in\mathcal{F}_X$ and $\alpha>0$,*

$$\mathcal{E}^\alpha_X(u, u)(=\mathcal{E}_X(u, u)+\alpha(u, u)_X)=\lim_{\beta\to+\infty}\mathcal{E}^\alpha_{X,\beta}(u, u)\,.$$

(iii) *G_α satisfies the equation* (2.4) *for the space $(\mathcal{F}_X, \mathcal{E}_X(\,,\,))$ defined by* (2.8) *and* (2.9).

Assertions (i) and (ii) of the theorem can be proved easily from (2.5) and (2.7). As for the statement (iii), note a consequence of (2.7): $\beta G_\beta v$ converges to v strongly in $L^2(X)$ if v is in $\mathcal{F}_X$. Hence we can conclude that the equation in statement (iii) is valid for every $v\in\mathcal{F}_X$.

The following lemma will be used in section 5.

LEMMA 2.1. *Suppose that $(\mathcal{F}_X, \mathcal{E}_X)$ is a Dirichlet space and $u\in\mathcal{F}_X$. Denote by u_n the truncation of u: $u_n(x)=u(x)$ for $|u(x)|<n$, $u_n(x)=n$ for $u(x)\geqq n$ and $u_n(x)=-n$ for $u(x)\leqq -n$. Then,*

(i) *$u_n\in\mathcal{F}_X$, and $\mathcal{E}_X(u_n, u_n)$ increases to $\mathcal{E}_X(u, u)$ as n tends to infinity.*

(ii) *$(u_n)^2\in\mathcal{F}_X$ and $\mathcal{E}_X((u_n)^2, (u_n)^2)\leqq 4n^2\mathcal{E}_X(u, u)$.*

PROOF. Since u_n is a normal contraction of u, u_n is an element of u. Obviously $\mathcal{E}_X(u_n, u_n)$ is increasing and its limit is no greater than $\mathcal{E}_X(u, u)$. Define G_β and $\mathcal{E}^0_{X,\beta}$ by (2.4) and (2.5) successively. Theorem 2.1 and 2.2 imply that, for any $v\in\mathcal{F}_X$, $\mathcal{E}^0_{X,\beta}(v, v)$ increases to $\mathcal{E}_X(v, v)$ as $\beta\to+\infty$. Hence, we

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7) By the resolvent equation, $\frac{d}{d\alpha}(G_\alpha u, u)_X=-(G_\alpha u, G_\alpha u)_X\leqq 0$.

have $\mathcal{E}^0_{X,\beta}(u_n, u_n) \leqq \lim_{n\to+\infty} \mathcal{E}_X(u_n, u_n)$. Letting n and β tend to infinity successively, we arrive at the statement (i). Assertion (ii) is an immediate consequence of the fact that $\left(\frac{1}{2n}u_n\right)^2$ is a normal contraction of $\frac{1}{2n}u_n$.

From now on, we treat only the cases that the underlying space X is an Euclidean domain or its Martin boundary.

Suppose that $G_\alpha(x, y)$, $\alpha > 0$, $x, y \in D$, $x \neq y$ is a symmetric resolvent density on a bounded Euclidean domain D. Then, by

$$G_\alpha u(x) = \int_D G_\alpha(x, y)u(y)dy, \qquad \alpha > 0, \qquad u \in L^2(D), \tag{2.10}$$

we have a symmetric resolvent $\{G_\alpha, \alpha > 0\}$ on $L^2(D)$.

DEFINITION 2.4. With the resolvent (2.10), we define a Dirichlet space $(\mathcal{F}_D, \mathcal{E})$ relative to $L^2(D)$ by formulae (2.8) and (2.9). We call $(\mathcal{F}_D, \mathcal{E})$ the *Dirichlet space associated with the resolvent density $G_\alpha(x, y)$ on D.*

Denote by $\boldsymbol{B}(D)(\boldsymbol{C}_0^\infty(D))$ the space of all bounded measurable functions on D (resp. all infinitely differentiable functions with compact supports). By Theorem 2.2 (iii), we have

LEMMA 2.2. *Let $G_\alpha(x, y)$ be a symmetric resolvent on D. Then, $\{G_\alpha u, u \in \boldsymbol{C}_0^\infty(D)\}$ and $\{G_\alpha u, u \in \boldsymbol{B}(D)\}$ are the dense subsets of the associated Dirichlet space $\mathcal{F}_D$ with metric $\mathcal{E}^\beta(\,,\,)(\beta > 0$ being arbitrary).*

§ 3. Space of BLD functions which are square integrable. Integrations by the Feller kernel.

Properties of BLD functions were profoundly investigated by Deny and Lions [5] and Doob [7]. In this section, we will study BLD functions in terms of the associated Dirichlet spaces and the Feller kernels defined on the Martin boundary. Theorem 3.1 will state that the space of BLD functions of potential type is identical with the Dirichlet space associated with the resolvent density of the absorbing barrier Brownian motion. We will give two applications of this theorem to exhibit the properties of the Feller kernel. Finally, we will present some results concerning boundary properties of α-harmonic functions with finite Dirichlet integrals, analogous to those by Doob [7]. Inequalities in Lemma 3.1 and equalities in the proof of the lemma will play basic roles in the following sections.

Throughout this section to section 8, we fix an arbitrary bounded domain D of R^N.

DEFINITION 3.1. Denote by $\widehat{\text{BLD}}$ the space of all BLD functions which are square integrable on D. Precisely, $u \in \widehat{\text{BLD}}$, if and only if $u \in L^2(D)$, every first partial derivatives of u (in the sense of Schwartz's distribution) are in

$L^2(D)$ and u is fine continuous quasi-everywhere on D[8].

For $u, v \in \widehat{\mathrm{BLD}}$, put

$$(u, v)_{D,1} = \frac{1}{2}\int_D (\mathrm{grad}\, u,\ \mathrm{grad}\, v)(x)dx\,.$$

The pair $(\widehat{\mathrm{BLD}}, (\ ,\ )_{D,1})$ is a Dirichlet space relative to $L^2(D)$ in the sense of Definition 2.3.

DEFINITION 3.2. Denote by BLD_0 the closure of $C_0^\infty(D)$ in the space $(\widehat{\mathrm{BLD}}, (\ ,\ )_{D,1})$.

Note that, for each $\alpha > 0$, $(u, u)_{D,1} + \alpha(u, u)_D$ gives a metric equivalent to $(u, u)_{D,1}$ for the space BLD_0([5]). In accordance with Doob [7], a function of BLD_0 will be called a BLD function of potential type.

Let $(\mathcal{F}_D^{(0)}, \mathcal{E}^{(0)})$ be the Dirichlet space associated with the resolvent density $G_\alpha^0(x, y)$ of the absorbing barrier Brownian motion on D (see Definition 2.4). We put

$$\mathcal{F}^{(0)} = \{u \in \mathcal{F}_D^{(0)},\ u \text{ is fine-continuous quasi-everywhere on } D\}\,. \tag{3.1}$$

We call $\mathcal{F}^{(0)}$ the refinement of the space $\mathcal{F}_D^{(0)}$.

THEOREM 3.1.

(i) *For each function u of $\mathcal{F}_D^{(0)}$, there exists a function of $\mathcal{F}^{(0)}$, which is equal to u almost everywhere.*

(ii) $\mathcal{F}^{(0)} = \mathrm{BLD}_0$ *and* $\mathcal{E}^{(0)}(u, u) = (u, u)_{D,1}$, $u \in \mathcal{F}^{(0)}$.

PROOF. On account of Lemma 2.2 and the remark in the preceding paragraph, it is sufficient to show that, for a fixed $\alpha > 0$,

(a) $\mathcal{R}^{(0)} = \{G_\alpha^0 u\,;\ u \in C_0^\infty(D)\}$ is contained in BLD_0 and, for $v \in \mathcal{R}^{(0)}$, $\mathcal{E}^{(0),\alpha}(v, v) = (v, v)_{D,1} + \alpha(v, v)_D$.

(b) $\mathcal{R}^{(0)}$ is dense in the space BLD_0 with respect to the norm $(\ ,\ )_{D,1} + \alpha(\ ,\ )_D$.

Consider a sequence of domains D_n which increases to D. Assume that boundaries ∂D_n are of class C^2. Approximate the function $v = G_\alpha^0 u$, $u \in C_0^\infty(D)$ by functions

$$v_n(x) = \begin{cases} G_\alpha^{(n)} u(x) & x \in D_n \\ 0 & x \in D - D_n, \qquad n = 1, 2, \cdots, \end{cases}$$

where $G_\alpha^{(n)} u$ is defined by (2.10) for the resolvent density of absorbing Brownian motion on D_n. We can see that $v_n \in \mathrm{BLD}_0$[9]. By the equality

$$\alpha v_n(x) = \frac{1}{2}\sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} v_n(x) + u(x), \qquad x \in D_n\,,$$

we have

$$(v_n, v_m)_{D,1} + \alpha(v_n, v_m)_D = (u, v_m)_D, \qquad n \geqq m\,. \tag{3.2}$$

8) By "quasi-everywhere" we means "except for a set of capacity zero".

9) $G_\alpha^{(n)} u$ is in BLD_0 for the domain D_n and hence, $v_n \in \mathrm{BLD}_0$ for D [5].

Since v_n converges to v uniformly on each compact set of D, the formula (3.2) implies that v_n is convergent in norm $\sqrt{(\ ,\ )_{D,1}+\alpha(\ ,\ )_D}$ and the limiting function in BLD_0 coincides with v almost everywhere. Hence, $v\in\mathrm{BLD}_0$ and $(v, v)_{D,1}+\alpha(v, v)_D=(u, v)_D=\mathcal{E}^{(0),\alpha}(v, v)$, completing the proof of assertion (a).

As for (b), assume that $w\in\mathrm{BLD}_0$ satisfies $(w, v)_{D,1}+\alpha(w, v)_D=0$ for all $v=G_\alpha^0 u\in\mathcal{R}^{(0)}$. Find $w_n\in C_0^\infty(D)$ which converges to w in BLD_0, then we see that the left-hand side of the above equation is equal to $\lim_{n\to+\infty}((w_n, v)_{D,1}+\alpha(w_n, v)_D)$ $=\lim_{n\to+\infty}(w_n, u)_D=(w, u)_D$. Thus, w must vanish. The proof of the theorem is complete.

Now we are in a position to introduce several notions related to the Martin boundary M of the domain D. Let $\mu(E)$ be the harmonic measure of the Borel set E of M relative to the fixed reference point $x_0\in D$.

DEFINITION 3.3. If a function u on D has a fine limit $\varphi(\xi)$ at μ-almost every $\xi\in M$, we denote φ by γu and call it a boundary function of u.

Doob [7] has proved that every BLD function has a boundary function in $L^2(M)$ and that u is an element of BLD_0 if and only if u is a BLD-function and $(\gamma u)(\xi)=0$ for almost all $\xi\in M$. Thus,

COROLLARY TO THEOREM 3.1. *u belongs to $\mathcal{F}^{(0)}$ if and only if u is a* BLD *function and u has a boundary function vanishing μ-almost everywhere on M.*

Let $K(x, \xi)=K^\xi(x)$, $x\in D$, be the Martin kernel associated with $\xi\in M$.

Define, for $\alpha>0$.

$$K_\alpha(x, \xi)=K_\alpha^\xi(x)=K^\xi(x)-\alpha\int_D G_\alpha^0(x, y)K^\xi(y)dy\,. \tag{3.3}$$

Put for $\xi,\ \eta\in M,\ \alpha>0$,

$$U_\alpha(\xi, \eta)=\alpha(K^\xi, K_\alpha^\eta)_D\leqq+\infty\,. \tag{3.4}$$

$U_\alpha(\xi, \eta)$ is non-decreasing in α and we put

$$U(\xi, \eta)=\lim_{\alpha\to+\infty}U_\alpha(\xi, \eta)\leqq+\infty\,. \tag{3.5}$$

We call U_α and U the Feller kernels[10]. For functions φ and ψ on M, we define

$$U_\alpha(\varphi, \psi)=\int_M\int_M U_\alpha(\xi, \eta)\varphi(\xi)\psi(\eta)\mu(d\xi)\mu(d\eta)\,, \tag{3.6}$$

$$U(\varphi, \psi)=\int_M\int_M U(\xi, \eta)\varphi(\xi)\psi(\eta)\mu(d\xi)\mu(d\eta)\,. \tag{3.7}$$

Finally, we set for $\varphi\in L^1(M)$,

$$H\varphi(x)=\int_M K(x, \xi)\varphi(\xi)\mu(d\xi)\,,\qquad x\in D\,, \tag{3.8}$$

10) These kernels are symmetric μ-almost everywhere (see [13] and footnote 15)).

$$H_\alpha\varphi(x)=\int_M K_\alpha(x,\xi)\varphi(\xi)\mu(d\xi)\,,\qquad x\in D\,. \tag{3.9}$$

If $\varphi\in L^1(M)$, then we have $\gamma(H\varphi)=\varphi$[11].

Here are two applications of Theorem 3.1.

THEOREM 3.2. *Let φ be a non-negative bounded measurable functions on M. Then, it holds that*

$$\boldsymbol{U}(\varphi,\varphi)=\mathcal{E}^{(0)}(H\varphi,H\varphi)\,. \tag{3.10}$$

Moreover, if $\boldsymbol{U}(\varphi,1)$ is finite, then φ must vanish almost everywhere on M.

PROOF. It is evident that $H\varphi\in L^2(D)$. Identity (3.10) follows from $\boldsymbol{U}_\alpha(\varphi,\varphi)=\alpha(H_\alpha\varphi,H\varphi)_D=\alpha(H\varphi-\alpha G^0_\alpha H\varphi,H\varphi)_D=\mathcal{E}^{(0),0}_\alpha(H\varphi,H\varphi)$. Assume that $\boldsymbol{U}(\varphi,1)$ is finite. Then $\boldsymbol{U}(\varphi,\varphi)$ is finite, and identity (3.10) implies that $H\varphi$ must be an element of $\mathcal{F}^{(0)}$. Corollary to Theorem 3.1 now implies that $\gamma(H\varphi)=\varphi=0$.

Theorem 3.2 will be used in the next section. In section 8, we will refer to the following theorem.

Let $\tilde{D}=D\cup\{\infty\}$ be the one point compactification of D. For a Borel subset A of the Martin boundary M, we set $\Pi^A_\beta(x)=H_\beta\chi_A(x)$, $\chi_A(\xi)$ being the indicator function of the set A. Define a probability measure V^A_β on $\tilde{D}$ by

$$\begin{cases} V^A_\beta(E)=\dfrac{\int_E \Pi^A_\beta(x)dx}{(\Pi^A_\beta,1)_D}\,, & \text{if } E \text{ is a Borel set of } D \\ V^A_\beta(\{\infty\})=0\,. \end{cases} \tag{3.11}$$

THEOREM 3.3. *Suppose that $\mu(A)>0$. As β tends to infinity, the sequence of measures $V^A_\beta(dx)$ on $\tilde{D}=D\cup\{\infty\}$ converges weakly to the δ-measure concentrated at $\{\infty\}$.*

PROOF. By virtue of Theorem 3.2, $\beta(\Pi^A_\beta,1)_D=\boldsymbol{U}_\beta(\chi_A,1)\to+\infty$ as β tends to infinity. Hence, it suffices to prove that, for each open set E the closure of which is compact in D, $\beta\int_E\Pi^A_\beta(x)dx$ is bounded in $\beta>0$. Choose a non-negative $u\in \boldsymbol{C}^\infty_0(D)$ with $u=1$ on the set E. Let v be an element of $\boldsymbol{C}^\infty_0(D)$ which is less than $H\chi_A$ everywhere on D and equal to $H\chi_A$ on the support of u. Then,

$$\begin{aligned}\beta\int_E\Pi^A_\beta(x)dx &\leqq \beta(\Pi^A_\beta,u)_D=\beta(H\chi_A,u)_D-\beta^2(G^0_\beta H\chi_A,u)_D\\ &\leqq \beta(v,u)_D-\beta^2(G^0_\beta v,u)_D\,.\end{aligned}$$

Owing to Theorem 3.1, the last term converges to $(v,u)_{D,1}$ as $\beta\to+\infty$. The proof of Theorem 3.3 is complete.

Turning to the study of boundary properties of α-harmonic functions, let

11) Cf. Doob [**6**].

us introduce new spaces of functions on M. For a function φ on M, we put $U_\alpha\varphi(\xi)=\int_M U_\alpha(\xi,\eta)\varphi(\eta)\mu(d\eta)$. Define a new measure μ' on M by

$$\mu'(A)=\int_A U_1 1(\xi)\mu(d\xi) \tag{3.12}$$

for the Borel set A of M. For functions φ and ψ on M, set

$$\begin{cases} (\varphi,\psi)_M=\int_M \varphi(\xi)\psi(\xi)\mu(d\xi) \\ (\varphi,\psi)'_M=\int_M \varphi(\xi)\psi(\xi)\mu'(d\xi) \end{cases} \tag{3.13}$$

$$\boldsymbol{D}(\varphi,\psi)=\frac{1}{2}\int_M\int_M(\varphi(\xi)-\varphi(\eta))(\psi(\xi)-\psi(\eta))U(\xi,\eta)\mu(d\xi)\mu(d\eta)\,. \tag{3.14}$$

Denote by $L^2(M)(L^2(M)')$ the space of measurable functions φ on M such as $(\varphi,\varphi)_M<+\infty$ (resp. $(\varphi,\varphi)'_M<+\infty$). We set

$$\boldsymbol{H}_M=\{\varphi\,;\,\varphi\in L^2(M)' \text{ and } \boldsymbol{D}(\varphi,\varphi)<+\infty\}\,. \tag{3.15}$$

$\boldsymbol{B}(D)$ ($\boldsymbol{B}(M)$) will stand for the space of all bounded measurable functions on D (resp. on M).

The next lemma collects the basic relations among these spaces and norms.

LEMMA 3.1.

(i) $\boldsymbol{B}(M)\subset L^2(M)'\subset L^2(M)$ *and there is a constant* $C>0$ *such that*

$$(\varphi,\varphi)_M\leqq C(\varphi,\varphi)'_M\,,\qquad \textit{for every } \varphi\in L^2(M)'\,. \tag{3.16}$$

(ii) *For* $\varphi\in L^2(M)'$ *and* $\alpha>0$,

$$0\leqq \boldsymbol{U}_\alpha(\varphi,\varphi)\leqq(\alpha\vee 1)(\varphi,\varphi)'_M\,. \tag{3.17}$$

(iii) *For* $\varphi\in\boldsymbol{H}_M$ *and* $\alpha>0$,

$$(\varphi,\varphi)'_M\leqq\left(1\vee\frac{1}{\alpha}\right)\{\boldsymbol{D}(\varphi,\varphi)+\boldsymbol{U}_\alpha(\varphi,\varphi)\}\,. \tag{3.18}$$

PROOF. The first inclusion in assertion (i) follows from $(1_M,1_M)'_M=(H1_M, H_1 1_M)_D\leqq$ the Lebesgue measure of D[12]. $U_1 1(\xi)$ is finite for μ-almost all $\xi\in M$. It is lower semi-continuous and strictly positive everywhere on M. Hence, it suffices to set $C=1/\inf_{\xi\in M}U_1 1(\xi)$ to obtain estimate (3.16). Next, observe that $U_\alpha 1(\xi)$ is increasing and $\frac{1}{\alpha}U_\alpha 1(\xi)$ is decreasing as α increases. The first and second inequalities in (3.17) and inequality (3.18) are the consequences of the following equalities (3.19), (3.20) and (3.21) respectively.

12) 1_M denotes the function which is identically one on M.

(3.19) $$U_\alpha(\varphi, \varphi)=\alpha(H_\alpha\varphi, H_\alpha\varphi)_D+\alpha^2(G^0_{0+}H_\alpha\varphi, H_\alpha\varphi)_D,$$

(3.20) $$U_\alpha(\varphi, \varphi)+\frac{1}{2}\int_M\int_M(\varphi(\xi)-\varphi(\eta))^2U_\alpha(\xi, \eta)\mu(d\xi)\mu(d\eta)$$
$$=\int_M\varphi(\xi)^2U_\alpha 1(\xi)\mu(d\xi), \qquad \varphi\in L^2(M)',$$

(3.21) $$D(\varphi, \varphi)+U_\alpha(\varphi, \varphi)=\frac{1}{2}\int_M\int_M(\varphi(\xi)-\varphi(\eta))^2$$
$$\{U(\xi, \eta)-U_\alpha(\xi, \eta)\}\mu(d\xi)\mu(d\eta)+\int_M\varphi(\xi)^2U_\alpha 1(\xi)\mu(d\xi), \qquad \varphi\in \boldsymbol{H}_M.$$

Now, denote by $\widehat{\mathrm{BLD}}_{\alpha,h}$ the space of all α-harmonic functions belonging to $\widehat{\mathrm{BLD}}$. It is easy to see that $\widehat{\mathrm{BLD}}_{\alpha,h}$ is the orthogonal complement of BLD_0 in the Hilbert space $(\widehat{\mathrm{BLD}}, (\ ,\ )_{D,1}+\alpha(\ ,\ )_D)$.

Our final assertions in this section are as follows.

THEOREM 3.4. *Fix an* $\alpha>0$.

(i) *Every bounded α-harmonic function u on D has its boundary function γu in $\boldsymbol{B}(M)$ and $u(x)=H_\alpha(\gamma u)(x)$, $x\in D$.*

(ii) *Every function u of $\widehat{\mathrm{BLD}}_{\alpha,h}$ has its boundary function γu in $\boldsymbol{H}_M$ and $u(x)=H_\alpha(\gamma u)(x)$, $x\in D$. Further, it holds that*

(3.22) $$(u, u)_{D,1}+\alpha(u, u)_D=D(\gamma u, \gamma u)+U_\alpha(\gamma u, \gamma u).$$

(iii) *For $\varphi\in L^2(M)'$, $H_\alpha\varphi$ has φ as its boundary function. In particular, if $\varphi\in\boldsymbol{H}_M$, then $H_\alpha\varphi\in\widehat{\mathrm{BLD}}_{\alpha,h}$ and equality* (3.22) *holds for $u=H_\alpha\varphi$ and $\gamma u=\varphi$.*

(iv) *For $u\in\widehat{\mathrm{BLD}}_{\alpha,h}$, the following inequality holds.*

(3.23) $$(\gamma u, \gamma u)'_M\leqq\left(1\vee\frac{1}{\alpha}\right)\{(u, u)_{D,1}+\alpha(u, u)_D\},$$

γu being the boundary function of u.

PROOF. (i) set

(3.24) $$u_1=u+\alpha G^0_{0+}u.$$

u_1 is a bounded harmonic function. Hence, u_1 has its boundary function, say φ, in $\boldsymbol{B}(M)$ and $u_1(x)=H\varphi(x)$, $x\in D$[13]. Since $\gamma(G^0_{0+}u)=0$, u has φ as its boundary function. By virtue of the equality $u=u_1-\alpha G^0_\alpha u_1=H\varphi-\alpha G^0_\alpha H\varphi$ and identity (3.3), we obtain $u=H_\alpha\varphi$.

(ii) For $u\in\widehat{\mathrm{BLD}}_{\alpha,h}$, define u_1 by (3.24). Note that G^0_{0+} is a bounded operator on $L^2(D)$, so that $u_1\in L^2(D)$. Therefore $G^0_{0+}u=G^0_\alpha u_1\in\mathrm{BLD}_0$ (Theorem 3.1) and $u_1\in\widehat{\mathrm{BLD}}$. Hence we have

13) Cf. Doob [6].

$$(u_1, u_1)_{D,1}+\alpha(u_1, u_1)_D=(u, u)_{D,1}+\alpha(u, u)_D+(\alpha G^0_{0+}u, \alpha G^0_{0+}u)_{D,1}$$
$$+\alpha(\alpha G^0_{0+}u, \alpha G^0_{0+}u)_D=(u, u)_{D,1}+\alpha(u, u)_D+(\alpha G^0_\alpha u_1, \alpha G^0_{0+}u)_{D,1}$$
$$+\alpha(\alpha G^0_\alpha u_1, \alpha G^0_{0+}u)_D=(u, u)_{D,1}+\alpha(u, u)_D+(\alpha u_1, \alpha G^0_{0+}u)_D\,,$$

so that

$$(u, u)_{D,1}+\alpha(u, u)_D=(u_1, u_1)_{D,1}+\alpha(u_1, u)_D\,. \tag{3.25}$$

Owing to Doob [7] and Fukushima [13], u_1 has the boundary function (say φ) in $L^2(M)$ with $(u_1, u_1)_{D,1}=\boldsymbol{D}(\varphi, \varphi)$. Corollary Theorem 3.1 implies that $\gamma(G^0_{0+}u)=0$. Thus, in the same way as in the proof of statement (i), we have $\gamma u=\varphi$ and $u=H_\alpha\varphi$. Identity (3.25) now implies (3.22). Further, in view of equality (3.22) and the preceding lemma, $\varphi(=\gamma u)$ must be an element of $\boldsymbol{H}_M$.

(iii) By virtue of formulae (3.17) and (3.19), we see that $H_\alpha\varphi\in L^2(D)$ for $\varphi\in L^2(M)'$. Hence, $G^0_{0+}(H_\alpha\varphi)\in \mathrm{BLD}_0$ and $\gamma(H_\alpha\varphi)=\gamma(H\varphi)=\varphi$. If, in addition, $\boldsymbol{D}(\varphi, \varphi)$ is finite, then $u_1=H\varphi$ is BLD harmonic with $(u_1, u_1)_{D,1}=\boldsymbol{D}(\varphi, \varphi)$ ([7]) and identity (3.25) is valid for $u=H_\alpha\varphi$.

(iv) is only the restatement of Lemma 3.1 (iii).

§ 4. An expression of the symmetric resolvent density $G_\alpha(x, y)$ and a decomposition of the Dirichlet space associated with $G_\alpha(x, y)$.

Throughout § 4, 5 and 6, we assume that we are given a resolvent $G_\alpha(x, y)$ of $\boldsymbol{G}$: $G_\alpha(x, y)=G_\alpha(x, y)+R_\alpha(x, y)$ is a conservative symmetric resolvent density and $R_\alpha(x, y)$ satisfies the conditions (G, a) and (G, b) stated in the beginning of section 1.

Our first task in this section is to give an expression of $R_\alpha f$, $f\in\boldsymbol{B}(D)$, which is analogous to that of Feller [11].

For a function φ on M, $H_\alpha\varphi$ defined by (3.9) can be rewritten in terms of the measure μ' (see (3.12)) as

$$H_\alpha\varphi(x)=H^x_\alpha\varphi=\int_M K'_\alpha(x, \xi)\varphi(\xi)\mu'(d\xi)\,, \qquad x\in D\,, \tag{4.1}$$

with the function $K'_\alpha(x, \xi)$, $\alpha>0$, $x\in D$, $\xi\in M$, defined by

$$K'_\alpha(x, \xi)=\begin{cases} K_\alpha(x, \xi)/U_1 1(\xi) & \text{if } U_1 1(\xi)<+\infty\,, \\ 0 & \text{if } U_1 1(\xi)=+\infty\,. \end{cases} \tag{4.2}$$

14) Theorem 1 of [13] states that $U(\xi, \eta)=\frac{\mathrm{q}}{2}\theta(\xi, \eta)$ when ξ and η are exit boundary points. Here, θ is Naim's kernel [18] and q denotes either 2π (if $N=2$) or $(N-2)\times$\{area of the unit sphere\}. Since D is bounded, μ-almost all points of M are exit (see footnote 15)) and Theorem 9.2 of [7] leads to this expression of the Dirichlet integral of the harmonic function. For one dimensional case, this expression is trivially true (see footnote 33)).

Indeed, $U_1 1(\xi)$ is strictly positive everywhere and finite almost everywhere on M. For a signed measure $\nu(dy)$ on D, let us put

$$\hat{H}_\alpha \nu(\xi) = \int_D K'_\alpha(x, \xi)\nu(dy), \qquad \xi \in M. \tag{4.3}$$

$\hat{H}_\alpha$ brings signed measure on D into functions on M. For $x \in D$, $\hat{H}_\alpha^x$ will stand for $K'_\alpha(x, \xi)$. When ν has a density function $f \in \boldsymbol{B}(D)$, $\hat{H}_\alpha \nu$ will be denoted by $\hat{H}_\alpha f$. Obviously, $\hat{H}_\alpha f(\xi) = \int_D \hat{H}_\alpha^x(\xi) f(x) dx$.

LEMMA 4.1. (i) *$\hat{H}_\alpha$ is a bounded linear operator from $\boldsymbol{B}(D)$ into $\boldsymbol{B}(M)$, and*

$$|\hat{H}_\alpha f(\xi)| \leqq \left(1 \vee \frac{1}{\alpha}\right) \sup_{x \in D} |f(x)| \tag{4.4}$$

for $f \in \boldsymbol{B}(D)$, $\xi \in M$.

(ii) *The equation*

$$\hat{H}_\alpha f - \hat{H}_\beta f + (\alpha - \beta)\hat{H}_\alpha G_\beta^0 f = 0 \tag{4.5}$$

holds for every $f \in \boldsymbol{B}(D)$, $\alpha, \beta > 0$.

(iii) *The identity*

$$(\alpha - \beta)(\varphi, \hat{H}_\alpha(H_\beta \psi))'_M = \boldsymbol{U}_\alpha(\varphi, \psi) - \boldsymbol{U}_\beta(\varphi, \psi) \tag{4.6}$$

holds for every $\varphi, \psi \in \boldsymbol{B}(M)$, $\alpha, \beta > 0$.

PROOF. Note that μ-almost all points $\xi \in M$ are exit in the sense that $K_\alpha(x, \xi) > 0$ for some $\alpha > 0$ and some $x \in D$[15]. $U_\alpha(\xi, \eta)$ is symmetric if ξ and η are exit ([**13**; Lemma 2]). Therefore, $\int_D K'_\alpha(y, \xi) dy = \hat{H}_\alpha(H1)(\xi)$ is either

$$\frac{1}{\alpha} \frac{U_\alpha 1(\xi)}{U_1 1(\xi)} \qquad \text{or zero}.$$

Inequality (4.4) follows from this. The definition (3.3) of $K_\alpha(x, \xi)$ and the resolvent equation for G_α^0 lead to equation (4.5) for bounded f with compact support. Identity (4.5) is valid for every $f \in \boldsymbol{B}(D)$ by means of the bounded convergence theorem. Identity (4.6) follows from (3.3) and (4.5).

Now, let us state a representation theorem for

$$R_\alpha f(x) = \int_D R_\alpha(x, y) f(y) dy, \qquad f \in \boldsymbol{B}(D).$$

Set $\boldsymbol{A}(M) = \hat{H}_\alpha(\boldsymbol{B}(D))$ for an $\alpha > 0$. In view of the preceding lemma, $\boldsymbol{A}(M)$ is independent of $\alpha > 0$ and it is a linear subset of $\boldsymbol{B}(M)$.

To avoid confusion, we denote by 1_D (resp. 1_M) the function on D (resp. M) which is identically unity there. Note that 1_M is, up to a set of μ-measure

15) Let E be the set of all non-exit boundary points and put $u = H\chi_E$. Then $\alpha G_\alpha^0 u = u$ for every $\alpha > 0$. Letting α tend to zero, we obtain $u = 0$ and $\mu(E) = 0$.

zero, an element of $A(M)$ because of $1_M = \hat{H}_1 1_D$ μ-almost everywhere.

THEOREM 4.1. *For each* $\alpha > 0$, *the function* $R_\alpha f$, $f \in \boldsymbol{B}(D)$, *is expressed as*

$$R_\alpha f(x) = H_\alpha^x \tilde{R}^\alpha(\hat{H}_\alpha f), \qquad x \in D, \tag{4.7}$$

with a non-negative linear operator $\tilde{R}^\alpha$ *from* $A(M)$ *into* $L^\infty(M)$ *satisfying the following conditions:*

$$\mu\text{-ess}\sup_{\xi \in M} |\tilde{R}^\alpha \varphi(\xi)| \leq \left(1 \vee \frac{1}{\alpha}\right) \sup_{\xi \in M} |\varphi(\xi)|, \qquad \varphi \in A(M), \tag{4.8}$$

$$(\varphi, \tilde{R}^\alpha \psi)'_M = (\tilde{R}^\alpha \varphi, \psi)'_M, \qquad \varphi, \psi \in A(M) \tag{4.9}$$

$$\lim_{\alpha \to +\infty} (1_M, \tilde{R}^\alpha 1_M)'_M = 0. \tag{4.10}$$

PROOF. On account of the conditions (G, a) and (G, b), for each $x \in D$, $R_\alpha(x, y)$ is bounded and α-harmonic in $y \in D$. By Theorem 3.4 (i), there exists a boundary value

$$R_\alpha(x, \xi) = \mathit{fine}\text{-}\lim_{y \to \xi} R_\alpha(x, y) \tag{4.11}$$

for μ-almost all $\xi \in M$ and

$$R_\alpha(x, y) = \int_M R_\alpha(x, \xi) \hat{H}_\alpha^y(\xi) \mu'(d\xi) \qquad \text{for every } y \in D. \tag{4.12}$$

We set for $\varphi \in A(M)$

$$R^\alpha \varphi(x) = \int_M R_\alpha(x, \xi) \varphi(\xi) \mu'(d\xi). \tag{4.13}$$

For any $\varphi \in A(M)$, $R^\alpha \varphi(x)$ is bounded and α-harmonic on D. Indeed, φ can be written as $\hat{H}_\alpha f$, $f \in \boldsymbol{B}(D)$, and, in view of identity (4.12), $R^\alpha \varphi$ is expressed with this f as

$$R^\alpha \varphi(x) = R_\alpha f(x), \qquad x \in D. \tag{4.14}$$

Thus, owing to Theorem 3.4 (i), there is a well defined function

$$\tilde{R}^\alpha \varphi = \gamma(R^\alpha \varphi), \qquad \varphi \in A(M) \tag{4.15}$$

and this $\tilde{R}^\alpha$ is a non-negative linear operator from $A(M)$ into $L^\infty(M)$. Since $1_M = \hat{H}_1 1_D \leq (\alpha \vee 1) \hat{H}_\alpha 1_D$ μ-almost everywhere, we see by (4.14) that $R^\alpha 1_M(x) \leq (\alpha \vee 1) R_\alpha 1_D(x) \leq 1 \vee \frac{1}{\alpha}$, $x \in D$, and that $\tilde{R}^\alpha 1_M(\xi) \leq 1 \vee \frac{1}{\alpha}$ μ-almost everywhere, which implies the estimate (4.8). Equality (4.9) is an immediate consequence of the expression (4.7) and the symmetry of $R_\alpha(x, y)$. Let us prove (4.10). We have for all $\alpha > 0$

$$U_\alpha(\tilde{R}^\alpha 1_M, 1_M) = (1_M, 1_M)'_M < +\infty, \tag{4.16}$$

because the left-hand side of (4.16) is equal to

$$\alpha(\tilde{R}^{\alpha}1_M,\ \hat{H}_\alpha 1_D)'_M=(1_M,\ \alpha\tilde{R}^{\alpha}(\hat{H}_\alpha 1_D))'_M=(1_M,\ \gamma(\alpha R_\alpha 1_D))'_M\,.$$

Let $\{\alpha_n,\ n=1, 2, \cdots\}$ be an arbitrary sequence of real numbers increasing to infinity. Then $\tilde{R}^{\alpha_n}1_M(\xi)$ decreases to a non-negative function $\varphi(\xi)$ for every $\xi\in M$ except on a set of μ-measure zero. We set $\varphi(\xi)=0$ on the exceptional set. From (4.16), we have $U_\alpha(\varphi,1_M)\leqq(1_M,1_M)'_M$ for all $\alpha>0$. Letting α tend to infinity, we obtain $U(\varphi, 1_M)<+\infty$. Theorem 3.2 now implies that φ vanishes almost everywhere. Therefore, $\lim_{n\to+\infty}(1_M,\ \tilde{R}^{\alpha_n}1_M)'_M=(1_M,\ \varphi)'_M=0$, completing the proof of (4.10).

Next, let $(\mathcal{F}_D, \mathcal{E})$ be the Dirichlet space associated with our resolvent density $G_\alpha(x, y)=G_\alpha^0(x, y)+R_\alpha(x, y)$. Let us represent $(\mathcal{F}_D, \mathcal{E})$ as a direct sum of a potential part and an α-harmonic part. Our procedure is based on Theorem 4.1 and we will never use any classical tool such as Green's formula.

Put for $\alpha>0$.

$$\mathcal{F}^*=\{G_\alpha^0 f,\ f\in \boldsymbol{B}(D)\} \tag{4.17}$$

$$\mathcal{H}_\alpha^*=\{R_\alpha f,\ f\in \boldsymbol{B}(D)\}\,.$$

Note that $\mathcal{F}^*$ is independent of $\alpha>0$. Let us show the following basic lemma.

LEMMA 4.2. (i) *$\mathcal{F}^*\subset\mathcal{F}_D$ and*

$$\mathcal{E}(u, u)=\mathcal{E}^{(0)}(u, u) \qquad \textit{for } u\in\mathcal{F}^*\,.$$

Here, $\mathcal{E}^{(0)}$ is the norm for the Dirichlet space $\mathcal{F}_D^{(0)}$ associated with the resolvent density $G_\alpha^0(x, y)$.

(ii) *For each $\alpha>0$, $\mathcal{F}^*$ and $\mathcal{H}_\alpha^*$ are orthogonal with respect to the inner product*

$$\mathcal{E}^{\alpha}(u, v)=\mathcal{E}(u, v)+\alpha(u, v)_D\,.$$

PROOF. (i) Set $u=G_1^0 f,\ f\in\boldsymbol{B}(D)$. Then $u\in L^2(D)$ and

$$\begin{aligned}\mathcal{E}_\beta^0(u, u)&=\beta(u-\beta G_\beta^0 u,\ u)_D-\beta^2(R_\beta u,\ u)_D\\&=\mathcal{E}_\beta^{(0),0}(u, u)-\beta^2(R_\beta u,\ u)_D\,.\end{aligned}$$

On the other hand, by virtue of Theorem 4.1 and Lemma 4.1,

$$\begin{aligned}\beta^2(R_\beta u,\ u)_D&=\beta^2(H_\beta\tilde{R}^{\beta}(\hat{H}_\beta G_1^0 f),\ G_1^0 f)_D=\beta^2(\tilde{R}^{\beta}(\hat{H}_\beta G_1^0 f),\ \hat{H}_\beta G_1^0 f)'_M\\&=\frac{\beta^2}{(\beta-1)^2}(\tilde{R}^{\beta}\varphi_\beta,\ \varphi_\beta)'_M, \qquad \text{with } \varphi_\beta=(\hat{H}_1-\hat{H}_\beta)f\,.\end{aligned}$$

By Theorem 4.1, we have

$$|(\tilde{R}^{\beta}\varphi_\beta,\ \varphi_\beta)'_M|\leqq(\tilde{R}^{\beta}1_M,\ 1_M)'_M(\sup_{x\in D}|f(x)|)^2\xrightarrow[\beta\to+\infty]{}0\,.$$

Thus, $u\in\mathcal{F}_D$ and $\mathcal{E}(u, u)=\mathcal{E}^{(0)}(u, u)$.

(ii) Owing to the preceding assertion (i), we have for $f, g \in \boldsymbol{B}(D)$ that

$$\begin{aligned}\mathcal{E}^{\alpha}(G_{\alpha}^{0}f, R_{\alpha}g) &= \mathcal{E}^{\alpha}(G_{\alpha}^{0}f, G_{\alpha}g) - \mathcal{E}^{\alpha}(G_{\alpha}^{0}f, G_{\alpha}^{0}g) \\ &= (G_{\alpha}^{0}f, g)_{D} - \mathcal{E}^{(0),\alpha}(G_{\alpha}^{0}f, G_{\alpha}^{0}g) \\ &= (G_{\alpha}^{0}f, g)_{D} - (G_{\alpha}^{0}f, g)_{D} = 0\,.\end{aligned}$$

The proof of Lemma 4.2 is complete.

According to Lemma 2.2, functions

$$\{G_{\alpha}f = G_{\alpha}^{0}f + R_{\alpha}f,\ f \in \boldsymbol{B}(D)\} \quad (\text{resp. } \{G_{\alpha}^{0}f,\ f \in \boldsymbol{B}(D)\})$$

are dense in the Hilbert space $\{\mathcal{F}_{D}, \mathcal{E}^{\alpha}\}$ (resp. $\{\mathcal{F}_{D}^{(0)}, \mathcal{E}^{(0),\alpha}\}$). Hence, we immediately obtain the next theorem from Lemma 4.2.

THEOREM 4.2. (i) *$\mathcal{F}_{D}^{(0)} \subset \mathcal{F}_{D}$ and*

$$\mathcal{E}(u, u) = \mathcal{E}^{(0)}(u, u) \qquad \text{for } u \in \mathcal{F}_{D}^{(0)}\,.$$

(ii) *For each $\alpha > 0$, the Hilbert space $(\mathcal{F}_{D}, \mathcal{E}^{\alpha})$ can be decomposed as a direct sum:*

$$\mathcal{F}_{D} = \mathcal{F}_{D}^{(0)} \oplus \mathcal{H}_{\alpha}\,,$$

$\mathcal{H}_{\alpha}$ being the closure of $\{R_{\alpha}f,\ f \in \boldsymbol{B}(D)\}$ in this space.

In order to refine Theorem 4.2, let us introduce the space

(4.18) $\mathcal{F} = \{u \in \mathcal{F}_{D}\,;\ u$ is fine continuous quasi-everywhere on $D\}$.

We will call this the refinement of the Dirichlet space $\mathcal{F}_{D}$. We have then

THEOREM 4.3. (i) *For each function u of $\mathcal{F}_{D}$, there exists a function of $\mathcal{F}$ which coincides with u almost everywhere on D.*

(ii) *$\mathcal{F}^{(0)} \subset \mathcal{F}$ and $\mathcal{E}(u, u) = \mathcal{E}^{(0)}(u, u)$, $u \in \mathcal{F}^{(0)}$, $\mathcal{F}^{(0)}$ being the refinement of the space $\mathcal{F}_{D}^{(0)}$ ((3.1)).*

(iii) *For each $\alpha > 0$, the space $(\mathcal{F}, \mathcal{E}^{\alpha})$ is represented as*

$$\mathcal{F} = \mathcal{F}^{(0)} \oplus \mathcal{H}_{\alpha}\,,$$

with $\mathcal{H}_{\alpha} = \{u \in \mathcal{F}\,;\ u$ is α-harmonic on $D\}$. $R_{\alpha}(\boldsymbol{B}(D))$ is dense in $(\mathcal{H}_{\alpha}, \mathcal{E}^{\alpha})$.

PROOF. Let $\mathcal{H}_{\alpha}$ be the space of Theorem 4.2 (ii). Any function in $\mathcal{H}_{\alpha}$ is α-harmonic[16], and so, continuous on D. Hence, in view of Theorem 3.1 (i) and Theorem 4.2, we can see that statements (i) and (ii) of the present theorem hold and that $\mathcal{F} = \mathcal{F}^{(0)} \oplus \mathcal{H}_{\alpha}$. Take any α-harmonic function u of $\mathcal{F}$ and decompose u as $u = u^{(1)} + u^{(2)}$, $u^{(1)} \in \mathcal{F}^{(0)}$, $u^{(2)} \in \mathcal{H}_{\alpha}$. Then, $u^{(1)}$ is α-harmonic and belongs to the space BLD_0 (Theorem 3.1 (ii)). Hence, $u^{(1)} \in \mathrm{BLD}_0 \cap \widehat{\mathrm{BLD}}_{\alpha,h}$ and $u^{(1)} = 0$. This proves the last assertion of Theorem 4.3.

16) Any $u \in \mathcal{H}_{\alpha}$ is a limit of α-harmonic functions $R_{\alpha}f_{n}$, $f_{n} \in \boldsymbol{B}(D)$, in $L^{2}(D)$. Hence, $R_{\alpha}f_{n}(x)$ converges to $u(x)$ uniformly on each compact subset of D and u is α-harmonic (see [**15**; Lemma 2.2]).

§ 5. The Dirichlet space $(\mathcal{F}_M, \mathcal{E}_M)$ induced by $(\mathcal{H}_\alpha, \mathcal{E}^\alpha)$.

In this section, the Hilbert space $(\mathcal{H}_\alpha, \mathcal{E}^\alpha)$ appeared in Theorem 4.3 will be identified with a Dirichlet space formed by functions on the Martin boundary M.

For this purpose, we will employ the next theorem due to T. Watanabe, which permits us to conclude that each function u of $\mathcal{H}_\alpha$ has its boundary function γu in $L^2(M)'$ and that $u=H_\alpha(\gamma u)$.

Take any symmetric Brownian resolvent $\{\hat{G}_\alpha, \alpha>0\}$ (see the final part of section 1 for the definition) and consider its associated Dirichlet space $(\hat{\mathcal{F}}, \hat{\mathcal{E}})$ relative to $L^2(D)$ in the sence of section 2. Set $\hat{\mathcal{H}}_\alpha=\{u\in\hat{\mathcal{F}}\,;\ u \text{ is } \alpha\text{-harmonic}\}$. Then,

THEOREM 5.1 (T. Watanabe).

$$\hat{\mathcal{H}}_\alpha\subset \widehat{\mathrm{BLD}}_{\alpha,h} \quad \textit{and} \quad \hat{\mathcal{E}}(u, u)\geqq(u, u)_{D,1} \qquad \textit{for } u\in\hat{\mathcal{H}}_\alpha .$$

Combining this with Theorem 3.4 (ii), we are led to

COROLLARY. *Every function u of $\hat{\mathcal{H}}_\alpha$ has its boundary function γu in $\boldsymbol{H}_M$ (consequently in $L^2(M)'$) and $u(x)=H_\alpha(\gamma u)(x)$, $x\in D$. Further we have, for $u\in\hat{\mathcal{H}}_\alpha$, $\hat{\mathcal{E}}^\alpha(u, u)\geqq \boldsymbol{D}(\gamma u, \gamma u)+\boldsymbol{U}_\alpha(\gamma u, \gamma u)$.*

Let us sketch the proof of Theorem 5.1. Take a function $u\in\hat{\mathcal{H}}_\alpha$ and set $\hat{\mathcal{E}}^0_\beta(u, u)=\beta(u-\beta\hat{G}_\beta u, u)_D$. Then, by definition, $\hat{\mathcal{E}}(u, u)=\lim_{\beta\to+\infty}\hat{\mathcal{E}}^0_\beta(u, u)<+\infty$. It suffices for us to derive the inequality $\lim_{\beta\to+\infty}\hat{\mathcal{E}}^0_\beta(u, u)\geqq\left(\frac{1}{4}\Delta(u^2), 1\right)_D-\alpha(u, u)_D$, since the right-hand side is nothing but $(u, u)_{D,1}$. $\hat{\mathcal{E}}^0_\beta(u, u)$ can be expressed as $\hat{\mathcal{E}}^0_\beta(u, u)=\frac{1}{2}(f_\beta, 1)_D$, with $f_\beta=2\beta u(u-\beta\hat{G}_\beta u)-\beta(u^2-\beta\hat{G}_\beta u^2)+\beta u^2(1-\beta\hat{G}_\beta 1)_D$. It is easy to see that f_β is a non-negative function on D for each $\beta>0$. On the other hand, $\{\hat{G}_\beta, \beta>0\}$, being a Brownian resolvent, has the following property. If both $|g|$ and $\hat{G}_\beta g$ are locally integrable, then $\beta(g-\beta\hat{G}_\beta g)\underset{\beta\to+\infty}{\longrightarrow}-\frac{1}{2}\Delta g$ in the sence of Schwartz's distribution. Therefore, f_β converges (as distribution) to $2u\left(-\frac{1}{2}\Delta u\right)+\frac{1}{2}\Delta(u^2)$ and we have, for any $h\in \boldsymbol{C}_0^\infty(D)$ such as $0\leqq h\leqq 1$, $\lim_{\beta\to+\infty}\hat{\mathcal{E}}^0_\beta(u, u)\geqq\frac{1}{2}\left(\frac{1}{2}\Delta(u^2)-u\cdot\Delta u, h\right)_D$. The desired inequality follows from this.

Now, let us consider the space $(\mathcal{H}_\alpha, \mathcal{E}^\alpha)$, $\alpha>0$, of Theorem 4.3. Our main assertions are as follows.

THEOREM 5.2. (i) *For each $\alpha>0$, any function u of $\mathcal{H}_\alpha$ has its boundary function γu in $L^2(M)'$ and $u=H_\alpha(\gamma u)$.*

(ii) *The function space*

(5.1) $$\mathcal{F}_M = \gamma\mathcal{H}_\alpha = \{\varphi\,;\ \varphi = \gamma u,\ u \in \mathcal{H}_\alpha\}$$

is independent of $\alpha > 0$. *Set for* $\varphi, \psi \in \mathcal{F}_M$,

(5.2) $$\mathcal{E}_M^{[\alpha]}(\varphi, \psi) = \mathcal{E}^\alpha(H_\alpha\varphi, H_\alpha\psi)$$

(5.3) $$\mathcal{E}_M(\varphi, \psi) = \mathcal{E}_M^{[\alpha]}(\varphi, \psi) - \boldsymbol{U}_\alpha(\varphi, \psi)$$

then $\mathcal{E}_M(\varphi, \psi)$ *is independent of* $\alpha > 0$,

(iii) *For each* $\alpha > 0$, *the space* $(\mathcal{F}_M, \mathcal{E}_M^{[\alpha]})$ *is a Dirichlet space relative to* $L^2(M)'$.

(iv) *The space* $(\mathcal{F}_M, \mathcal{E}_M)$ *is a Dirichlet space relative to* $L^2(M)'$ *and it satisfies the conditions* (B. 1), (B. 2) *and* (B. 3) *stated in section* 1. *Moreover, the bilinear form* $N(\ ,\ )$ *in* (B. 2) *and* (B. 3) *is given by the following formula.*

(5.4) $$N(\varphi, \varphi) = \lim_{n\to+\infty}\ \lim_{\alpha\to+\infty}\ \lim_{\mu\to+\infty} \frac{1}{2}\mu^2 \int_M\int_M \tilde{R}_\mu^\alpha(d\xi, d\eta)(\varphi_n(\xi) - \varphi_n(\eta))^2 .$$

Here, $\tilde{R}^\alpha(d\xi, d\eta)$ *is a Radon measure on* $M \times M$ *satisfying*

(5.5) $$\int_M\int_M \tilde{R}_\mu^\alpha(d\xi, d\eta)\varphi(\xi)\cdot\psi(\eta) = (\tilde{R}_\mu^\alpha\varphi, \psi)'_M ,$$

$\varphi, \psi \in L^2(M)'$, *for the symmetric resolvent* $\{\tilde{R}_\mu^\alpha, \mu > 0\}$ *on* $L^2(M)'$ *associated with the Dirichlet space* $(\mathcal{F}_M, \mathcal{E}_M^{[\alpha]})$. φ_n *is a truncation of* $\varphi \in F_M$: $\varphi = (\varphi\wedge n)\vee(-n)$.

PROOF OF THEOREM 5.2. The first assertion is involved in Corollary to Theorem 5.1, since $\mathcal{H}_\alpha$ of Theorem 4.3 consists of all α-harmonic functions in $(\mathcal{F}_D, \mathcal{E})$, which is a Dirichlet space associated with a symmetric Brownian resolvent having a density function in $\boldsymbol{G}$.

The first part of (ii) is a consequence of Theorem 4.3 and Corollary to Theorem 3.1. Indeed, we have $\mathcal{F}_M = \gamma(\mathcal{F}^{(0)} \oplus \mathcal{H}_\alpha) = \gamma\mathcal{F}$, which is independent of $\alpha > 0$.

Now, let us prove the remaining assertions of Theorem 5.2 by a series of Lemmas.

The second part of statement (ii) is contained in Lemma 5.1. The third assertion will be proved in Lemma 5.3 by making use of Lemma 5.2. The last assertion is just Lemma 5.4.

LEMMA 5.1. (i) *If* $\varphi_n \in \mathcal{F}_M$ *converges to* $\varphi \in \mathcal{F}_M$ *in norm* $\mathcal{E}_M^{[\alpha]}$, *then* $\boldsymbol{U}_\alpha(\varphi_n, \varphi_n)$ *converges to* $\boldsymbol{U}_\alpha(\varphi, \varphi)$.

(ii) $\mathcal{E}_M(\varphi, \psi)$ *defined by* (5.3) *for* $\varphi, \psi \in \mathcal{F}_M$ *is independent of* $\alpha > 0$.

(iii) $1_M \in \mathcal{F}_M$ *and* $\mathcal{E}_M(1_M, \varphi) = 0$ *for any* $\varphi \in \mathcal{F}_M$.

PROOF. (i) From the definition of $\mathcal{E}_M^{[\alpha]}(\ ,\ )$, it follows that $H_\alpha\varphi_n$ converges to $H_\alpha\varphi$ in $L^2(D)$. Then, on account of identity (3.19) and the estimate $(u, G_{0+}^0 u)_D \leq \sup_{x\in D} G_{0+}^0 1(x)\cdot(u, u)_D$ for $u \in L^2(D)$, we can see that statement (i) is valid.

17) Cf. footnote 16).

(ii) The desired identity is

$$\mathcal{E}_M^{[\alpha]}(\varphi, \varphi)-\boldsymbol{U}_\alpha(\varphi, \varphi)=\mathcal{E}_M^{[\beta]}(\varphi, \varphi)-\boldsymbol{U}_\beta(\varphi, \varphi)$$

for α, $\beta>0$ and $\varphi\in\mathcal{F}_M$. Set $\tilde{\mathfrak{R}}=\gamma(R_\alpha(\boldsymbol{B}(D)))$, then $\tilde{\mathfrak{R}}=\gamma(G_\alpha(\boldsymbol{B}(D)))$ and this is independent of $\alpha>0$. Further, $\tilde{\mathfrak{R}}$ is dense in the space $(\mathcal{F}_M, \mathcal{E}_M^{[\alpha]})$ for an arbitrary $\alpha>0$, since $R_\alpha(\boldsymbol{B}(D))$ is dense in $(\mathcal{H}_\alpha, \mathcal{E}^\alpha(\ ,\ ))$. Therefore, taking into account of the first assertion of this lemma, it suffices for us to show the above identity for $\varphi\in\tilde{\mathfrak{R}}$. Let φ be $\gamma(R_\alpha f)$ with an $\alpha>0$ and an $f\in\boldsymbol{B}(D)$. Then, it holds that

$$\text{(5.6)}\qquad \begin{aligned}\mathcal{E}_M^{[\alpha]}(\varphi, \psi)&=\mathcal{E}^\alpha(R_\alpha f, H_\alpha\psi)=\mathcal{E}^\alpha(G_\alpha f, H_\alpha\psi)\\ &=(f, H_\alpha\psi)_D=(\hat{H}_\alpha f, \psi)'_M \quad \text{for any } \psi\in\mathcal{F}_M.\end{aligned}$$

On the other hand, the resolvent equation for G_α implies that φ can be expressed as $\gamma(R_\beta g)$ with $\beta>0$ and

$$\text{(5.7)}\qquad g=f+(\beta-\alpha)G_\alpha^0 f+(\beta-\alpha)H_\alpha\varphi.$$

Hence, equations (4.5), (4.6) and (5.6) lead us to

$$\begin{aligned}\mathcal{E}_M^{[\beta]}(\varphi, \varphi)-\boldsymbol{U}_\beta(\varphi, \varphi)&=(\hat{H}_\beta g, \varphi)'_M-\boldsymbol{U}_\beta(\varphi, \varphi)\\ &=(\hat{H}_\alpha f, \varphi)'_M-\boldsymbol{U}_\alpha(\varphi, \varphi)=\mathcal{E}_M^{[\alpha]}(\varphi, \varphi)-\boldsymbol{U}_\alpha(\varphi, \varphi).\end{aligned}$$

(iii) From equation (5.6) and the identity $\gamma(\alpha R_\alpha 1_D)=\gamma(\alpha G_\alpha 1_D)=1_M$, we have $\mathcal{E}_M^{[\alpha]}(1_M, \psi)=\alpha(\hat{H}_\alpha 1_D, \psi)'_M=\alpha(H1_M, H_\alpha\psi)_D=\boldsymbol{U}_\alpha(1_M, \psi)$, $\psi\in\mathcal{F}_M$. The proof of Lemma 5.1 is complete.

Next, set for α, $\lambda>0$ and u, $v\in\mathcal{F}$,

$$\text{(5.8)}\qquad \mathcal{E}^{\alpha,\lambda}(u, v)=\mathcal{E}^\alpha(u, v)+\lambda(\gamma u, \gamma v)'_M.$$

LEMMA 5.2. *Let us consider the space* $(\mathcal{F}, \mathcal{E}^{\alpha,\lambda}(\ ,\ ))$ *with* $\alpha>0$ *and* $\lambda>0$ *fixed.*

(i) *It is a real Hillert space.*

(ii) *It is decomposed as a direct sum:*

$$\mathcal{F}=\mathcal{F}^{(0)}\oplus\mathcal{H}_\alpha.$$

Especially $\mathcal{H}_\alpha$ *is a closed subspace.*

(iii) *If a function* v *on* D *is a normal contraction of a function* $u\in\mathcal{F}$, *then* $v\in\mathcal{F}$ *and* $\mathcal{E}^{\alpha,\lambda}(v, v)\leq\mathcal{E}^{\alpha,\lambda}(u, u)$.

PROOF. Theorem 4.3 and Corollary to Theorem 3.1 imply that each element u of $\mathcal{F}$ is a sum of functions $u^{(0)}\in\mathcal{F}^{(0)}$ and $u^{(1)}\in\mathcal{H}_\alpha$ and that $\mathcal{E}^{\alpha,\lambda}(u^{(0)}, u^{(1)})=\mathcal{E}^\alpha(u^{(0)}, u^{(1)})+\lambda(\gamma u^{(0)}, \gamma u^{(1)})'_M=0$.

Since $\mathcal{E}^{\alpha,\lambda}(u, u)=\mathcal{E}^\alpha(u, u)$ for $u\in\mathcal{F}^{(0)}$, the space $\mathcal{F}^{(0)}$ is closed in norm $\mathcal{E}^{\alpha,\lambda}$. Therefore, for the proof of assertions (i) and (ii), it suffices to show that $\mathcal{H}_\alpha$

is complete with metric $\mathcal{E}^{\alpha,\lambda}$. Suppose that $\{u_n\}$ forms a Cauchy sequence in $\{\mathcal{H}_\alpha, \mathcal{E}^{\alpha,\lambda}\}$. Then, u_n converges to a function $u \in \mathcal{H}_\alpha$ with metric $\mathcal{E}^\alpha$ and γu_n converges in $L^2(M)'$-sence to a function φ. Since u_n converges in $L^2(D)$-sence, the convergence is the pointwise sence[18]. On the other hand, $u_n(x) = H_\alpha(\gamma u_n)(x) \underset{n\to+\infty}{\to} H_\alpha\varphi(x)$ for each $x \in D$. Hence $u = H_\alpha\varphi$ and $\gamma u = \varphi$. The last statement of Lemma 5.2 follows from the facts that $(\mathcal{F}, \mathcal{E})$ is a Dirichlet space and that $|\gamma v(\xi)| \leqq |\gamma u(\xi)|$ for μ-almost all $\xi \in M$.

We will mention here the consequences of Lemma 5.2. Let φ be in $L^2(M)'$. Owing to Lemma 5.2 (i), there exists a unique element $u_\varphi^{\alpha,\lambda}$ of $\mathcal{F}$ such that the equation

$$\mathcal{E}^{\alpha,\lambda}(u_\varphi^{\alpha,\lambda}, v) = (\varphi, \gamma v)'_M \qquad \text{holds for all } v \in \mathcal{F}\,. \tag{5.9}$$

By virtue of Lemma 5.2 (ii), we can conclude that

$$u_\varphi^{\alpha,\lambda} \in \mathcal{H}_\alpha\,, \tag{5.10}$$

since (5.9) implies $\mathcal{E}^{\alpha,\lambda}(u_\varphi^{\alpha,\lambda}, v) = 0$ for all $v \in \mathcal{F}^{(0)}$. Furthermore, $u_\varphi^{\alpha,\lambda}$ enjoys the property:

$$0 \leqq u_\varphi^{\alpha,\lambda} \leqq 1 \qquad \text{if } 0 \leqq \varphi \leqq 1\,. \tag{5.11}$$

We can see this from the final statement of Lemma 5.2 and the fact that $u_\varphi^{\alpha,\lambda}$ is the unique element of $\mathcal{F}$ minimizing the functional $\Phi(v) = \mathcal{E}^\alpha(v, v) + \lambda\left(\gamma v - \frac{1}{\lambda}\varphi, \gamma v - \frac{1}{\lambda}\varphi\right)'_M$.

Set, for $\varphi \in L^2(M)'$,

$$\tilde{R}_\lambda^\alpha \varphi = \gamma u_\varphi^{\alpha,\lambda} (\in \mathcal{F}_M)\,, \tag{5.12}$$

then we have

LEMMA 5.3. *Fix an* $\alpha > 0$.

(i) *For each* $\lambda > 0$ *and* $\varphi \in L^2(M)'$, $\tilde{R}_\lambda^\alpha \varphi$ *defined by* (5.12) *is the unique element of* $\mathcal{F}_M$ *for which the equation*

$$\mathcal{E}_M^{[\alpha]}(\tilde{R}_\lambda^\alpha \varphi, \psi) + \lambda(\tilde{R}_\lambda^\alpha \varphi, \psi)'_M = (\varphi, \psi)'_M \tag{5.13}$$

holds for every $\psi \in \mathcal{F}_M$.

(ii) $\{\tilde{R}_\lambda^\alpha, \lambda > 0\}$ *is a symmetric resolvent on* $L^2(M)'$ *(see Definition* 2.1.).

(iii) $(\mathcal{F}_M, \mathcal{E}_M^{[\alpha]})$ *is just the Dirichlet space relative to* $L^2(M)'$ *associated with the above resolvent. In other words,* $\varphi \in L^2(M)'$ *is an element of* $\mathcal{F}_M$ *if and only if* $\lim_{\mu\to+\infty} \mathcal{E}_{M,\mu}^{[\alpha]}(\varphi, \varphi)$ *is finite, and in this case the limit necessarily coincides with* $\mathcal{E}_M^{[\alpha]}(\varphi, \varphi)$. *Here,*

$$\mathcal{E}_{M,\mu}^{[\alpha]}(\varphi, \psi) = \mu(\varphi - \mu\tilde{R}_\mu^\alpha\varphi, \psi)'_M\,. \tag{5.14}$$

18) Cf. footnote 16).

PROOF. (i) By (5.9), (5.10) and (5.12), the equation $\mathcal{E}^{\alpha,\lambda}(H_\alpha \tilde{R}_\lambda^\alpha \varphi, H_\alpha \psi) = (\varphi, \psi)'_M$ holds for every $\psi \in \mathcal{F}_M$. Rewrite the left-hand side to obtain (5.13), which obviously characterize $\tilde{R}_\lambda^\alpha \varphi$ in $\mathcal{F}_M$.

(ii) In view of (5.13), $\tilde{R}_\lambda^\alpha$ is a bounded linear operator on $L^2(M)'$. Further, by (5.11), we have $\lambda \tilde{R}_\lambda^\alpha 1 \leqq 1$ and $\tilde{R}_\lambda^\alpha \varphi \geqq 0$ for $\varphi \geqq 0$. Symmetry and the resolvent equation for $\{\tilde{R}_\lambda^\alpha, \lambda > 0\}$ follow from assertion (i).

(iii) Note that, for each $\lambda > 0$, the space $(\mathcal{F}_M, \mathcal{E}_M^{[\alpha]}(\ ,\ ) + \lambda(\ ,\ )'_M)$ is a real Hilbert space, since the space $(\mathcal{H}_\alpha, \mathcal{E}^{\alpha,\lambda})$ is. Identity (5.13) for the resolvent $\{\tilde{R}_\lambda^\alpha, \lambda > 0\}$ now implies assertion (iii).

LEMMA 5.4. (i) *For* $\varphi \in \mathcal{F}_M$, $\mathcal{E}_M(\varphi, \varphi)$ *is expressed as* $\mathcal{E}_M(\varphi, \varphi) = \boldsymbol{D}(\varphi, \varphi) + \boldsymbol{N}(\varphi, \varphi)$ *with* $\boldsymbol{D}(\varphi, \varphi)$ *and* $\boldsymbol{N}(\varphi, \varphi)$ *defined by* (3.14) *and* (5.4) *respectively. In particular,* $\mathcal{F}_M$ *is a linear subspace of* $\boldsymbol{H}_M$.

(ii) $\mathcal{F}_M$ *contains constant functions and* $\boldsymbol{N}(1, 1) = 0$.

(iii) *For each* $\lambda > 0$, $(\mathcal{F}_M, \mathcal{E}_M(\ ,\ ) + \lambda(\ ,\ )'_M)$ *is a real Hilbert space.*

(iv) *If* φ *is a normal contraction of* $\psi \in \mathcal{F}_M$, *then* $\varphi \in \mathcal{F}_M$ *and* $\boldsymbol{N}(\varphi, \varphi) \leqq \boldsymbol{N}(\psi, \psi)$.

PROOF. (i) Take φ in $\mathcal{F}_M$ and define φ_n by $\varphi_n = (\varphi \wedge n) \vee (-n)$, $n = 1, 2, \cdots$. Then, $\varphi_n \in \mathcal{F}_M$ and

$$\lim_{n \to +\infty} \mathcal{E}_M(\varphi_n, \varphi_n) = \mathcal{E}_M(\varphi, \varphi). \tag{5.15}$$

Indeed, for any $\alpha > 0$, $(\mathcal{F}_M, \mathcal{E}_M^{[\alpha]})$ is a Dirichlet space (Lemma 5.3 (iii)) and therefore Lemma 2.1 implies that $\varphi_n \in \mathcal{F}_M$ and $\lim_{n \to +\infty} (\mathcal{E}_M(\varphi_n, \varphi_n) + \boldsymbol{U}_\alpha(\varphi_n, \varphi_n)) = \mathcal{E}_M(\varphi, \varphi) + \boldsymbol{U}_\alpha(\varphi, \varphi)$. On the other hand, $\boldsymbol{U}_\alpha(\varphi_n, \varphi_n)$ converges to $\boldsymbol{U}_\alpha(\varphi, \varphi)$ because of identity (3.20).

We will compute $\mathcal{E}_M(\varphi_n, \varphi_n)$. Owing to Lemma 5.3 (iii), it holds that

$$\mathcal{E}_M(\varphi_n, \varphi_n) = \lim_{\mu \to +\infty} \mathcal{E}_{M,\mu}^{[\alpha]}(\varphi_n, \varphi_n) - \boldsymbol{U}_\alpha(\varphi_n, \varphi_n) \tag{5.16}$$

with $\mathcal{E}_{M,\mu}^{[\alpha]}$ defined by (5.14). The right-hand side of (5.16) is independent of $\alpha > 0$ (Lemma 5.1 (ii)). Rewrite $\mathcal{E}_{M,\mu}^{[\alpha]}(\varphi_n, \varphi_n)$ as[19)]

$$\mathcal{E}_{M,\mu}^{[\alpha]}(\varphi_n, \varphi_n) = \frac{1}{2}\mu^2 \int_M \int_M \tilde{R}_\mu^\alpha(d\xi, d\eta)(\varphi_n(\xi) - \varphi_n(\eta))^2 + \mu(1 - \mu \tilde{R}_\mu^\alpha 1, \varphi_n^2)'_M. \tag{5.17}$$

On account of Lemma 2.1 and Lemma 5.1 (iii), we see that $\varphi_n^2 \in \mathcal{F}_M$ and

$$\mathcal{E}_{M,\mu}^{[\alpha]}(1, \varphi_n^2) = (1 - \mu \tilde{R}_\mu^\alpha 1, \varphi_n^2)'_M \xrightarrow[\mu \to +\infty]{} \mathcal{E}_M(1, \varphi_n^2) + \boldsymbol{U}_\alpha(1, \varphi_n^2) = \boldsymbol{U}_\alpha(1, \varphi_n^2). \tag{5.18}$$

Combining (5.16) with (5.17) and (5.18) and employing equality (3.26), we arrive at

19) Cf. [2].

$$(5.19)\qquad \mathcal{E}_M(\varphi_n, \varphi_n) = \frac{1}{2}\int_M\int_M (\varphi_n(\xi)-\varphi_n(\eta))^2 U_\alpha(\xi, \eta)\mu(d\xi)\mu(d\eta) + \lim_{\mu\to+\infty}\mu^2\int_M\int_M \tilde{R}^\alpha_\mu(d\xi, d\eta)(\varphi_n(\xi)-\varphi_n(\eta))^2$$

for each n and $\alpha>0$.

Statement (i) of our lemma can be derived from (5.19) by letting α and then n tend to infinity.

(ii) This assertion is immediate from Lemma 5.1 (iii) and formula (5.4).

(iii) By Lemma 5.3 (iii), the space $\mathcal{F}_M$ is complete with metric $\mathcal{E}_M(\ ,\ ) + U_\alpha(\varphi, \varphi) + \lambda(\ ,\ )'_M$ for $\alpha>0$. In view of inequality (3.17), we arrive at conclusion (iii).

(iv) This is a consequence of Lemma 5.3 (iii) and formula (5.4).

The proof of Theorem 5.2 is now complete. We should point out here that the space $(\mathcal{F}_M, \mathcal{E}_M)$ characterizes our resolvent density. Precisely,

THEOREM 5.3. *Consider two elements $G^{(i)}_\alpha(x, y)$ of the class $\boldsymbol{G}$, $i=1, 2$. We associate the space $(\mathcal{F}^{(i)}_M, \mathcal{E}^{(i)}_M)$ with $G^{(i)}_\alpha(x, y)$ by means of Theorem 5.2, $i=1, 2$. Assume that $(\mathcal{F}^{(1)}_M, \mathcal{E}^{(1)}_M)=(\mathcal{F}^{(2)}_M, \mathcal{E}^{(2)}_M)$, then $G^{(1)}_\alpha(x, y)=G^{(2)}_\alpha(x, y)$, $\alpha>0$, $x, y\in D$.*

PROOF. Let $(\mathcal{F}^{(i)}, \mathcal{E}^{(i),\alpha})$ and $(\mathcal{H}^{(i)}_\alpha, \mathcal{E}^{(i),\alpha})$ be the spaces of Theorem 4.3 associated with $G^{(i)}_\alpha(x, y)$, $i=1, 2$. We have by assumption $\mathcal{H}^{(1)}_\alpha = H_\alpha(\mathcal{F}^{(1)}_M) = H_\alpha(\mathcal{F}^{(2)}_M) = \mathcal{H}^{(2)}_\alpha$ and $\mathcal{E}^{(1),\alpha}(u, u)=\mathcal{E}^{(1)}_M(\gamma u, \gamma u)+U_\alpha(\gamma u, \gamma u)=\mathcal{E}^{(2)}_M(\gamma u, \gamma u)+U_\alpha(\gamma u, \gamma u)=\mathcal{E}^{(2),\alpha}(u, u)$ for $u\in\mathcal{H}^{(1)}_\alpha$. By Theorem 4.3, we see that $(\mathcal{F}^{(1)}, \mathcal{E}^{(1),\alpha})=(\mathcal{F}^{(2)}, \mathcal{E}^{(2),\alpha})$ and that, for every $u, v\in L^2(D)$, $(G^{(1)}_\alpha u, v)_D = \mathcal{E}^{(2),\alpha}(G^{(1)}_\alpha u, G^{(2)}_\alpha v) = \mathcal{E}^{(1),\alpha}(G^{(1)}_\alpha u, G^{(2)}_\alpha v) = (u, G^{(2)}_\alpha v)_D$, from which the conclusion of Theorem 5.3 follows.

§ 6. Boundary condition.

In the preceding two sections, we have investigated the structure of the Dirichlet space $(\mathcal{F}_D, \mathcal{E})$ associated with a given element $G_\alpha(x, y)$ in $\boldsymbol{G}$. Consider the space $(\widehat{\mathrm{BLD}}, (\ ,\ )_{D,1})$ in section 3. On the ground of Theorem 3.1, 3.4, 4.3, 5.1 and 5.2, we can state the relation of the refinement $\mathcal{F}$[20] of $\mathcal{F}_D$ to the space $\widehat{\mathrm{BLD}}$ as follows.

THEOREM 6.1. (i) $\mathrm{BLD}_0\subset\mathcal{F}\subset\widehat{\mathrm{BLD}}$,

(ii) *Each function u of $\mathcal{F}$ has its boundary function γu in $\boldsymbol{H}_M$ and it holds that*

$$(6.1)\qquad \mathcal{E}(u, u) = (u, u)_{D,1} + N(\gamma u, \gamma u),$$

with a bilinear non-negative form N on $\gamma\mathcal{F}$. Moreover, if v is a normal contraction of $u\in\mathcal{F}$, then $v\in\mathcal{F}$ and $N(\gamma v, \gamma v)\leqq N(\gamma u, \gamma u)$.

PROOF. The first assertion is immediate from Theorem 3.1, 4.3 and 5.1.

20) See (4.18).

Indeed, for a fixed $\alpha>0$, the Hilbert space $(\mathcal{F}, \mathcal{E}^\alpha)$ is a direct sum of the space $\mathcal{F}^{(0)}=\mathrm{BLD}_0$ and the space $\mathcal{H}_\alpha$, the latter being a subspace of $\widehat{\mathrm{BLD}}_{\alpha,h}$. In order to prove equality (6.1), decompose $u \in \mathcal{F}$ as $u=u_1+u_2$, $u_1 \in \mathcal{F}^{(0)}$, $u_2 \in \mathcal{H}_\alpha$. Then,

$$(u_1, u_2)_{D,1}+\alpha(u_1, u_2)_D=0\,. \tag{6.2}$$

By Theorem 3.1 and 4.3 (ii),

$$\mathcal{E}^\alpha(u_1, u_1)=(u_1, u_1)_{D,1}+\alpha(u_1, u_1)_D\,. \tag{6.3}$$

Combining Theorem 5.2 with Theorem 3.4 (iii), we see that u_2 has its boundary function $\gamma u_2=\gamma u$ in $\boldsymbol{H}_M$ and

$$\mathcal{E}^\alpha(u_2, u_2)=(u_2, u_2)_{D,1}+\alpha(u_2, u_2)_D+\boldsymbol{N}(\gamma u, \gamma u)\,. \tag{6.4}$$

Formula (6.2), (6.3) and (6.4) lead us to equality (6.1). The properties of $\boldsymbol{N}$ stated in this theorem are implied in Theorem 5.2.

Our next task is concerned with an expression of the boundary condition for the class $\boldsymbol{G}$.

DEFINITION 6.1. If, for a function $u \in \widehat{\mathrm{BLD}}$, there exists an $f \in L^2(D)$ such that the equation

$$(u, v)_{D,1}=(f, v)_D \tag{6.5}$$

holds for every $v \in \mathrm{BLD}_0$, then we will write

$$\frac{1}{2}\Delta u=-f\,. \tag{6.6}$$

The set of functions u satisfying the above property will be denoted by $\mathcal{D}(\Delta)$. We call such Δ the *generalized Laplacian* with domain $\mathcal{D}(\Delta)$.

We notice that the equation (6.5) holds for all $v \in \mathrm{BLD}_0$ if and only if it does for all $v \in \boldsymbol{C}_0^\infty(D)$ (see the paragraph following Definition 3.2). Thus, u is an element of $\mathcal{D}(\Delta)$ if and only if $u \in \widehat{\mathrm{BLD}}$ and $\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} u$ in the sense of Schwartz's distribution is a function of $L^2(D)$. The notion Δ in (6.6) is nothing but the Laplacian in the distribution sense.

For a given element $G_\alpha(x, y)$ of $\boldsymbol{G}$, let us put

$$\mathcal{D}=G_\alpha(L^2(D))=\{u\,;\, u=G_\alpha f=\int_D G_\alpha(\cdot, y)f(y)dy\,, \qquad f \in L^2(D)\}\,. \tag{6.7}$$

The space $\mathcal{D}$ does not depend on $\alpha>0$. Let $\mathcal{F}_M$ be the space of Theorem 5.2 and $\boldsymbol{N}$, the form of (5.4). The next theorem will characterize the space $\mathcal{D}$ (and consequently, the element of $\boldsymbol{G}$).

THEOREM 6.2. *A function u belongs to $\mathcal{D}$ if and only if*

(1) *$u \in \mathcal{D}(\Delta)$, and*

(2) *u has its boundary function γu in $\mathcal{F}_M$ and it satisfies, for every $\phi \in \mathcal{F}_M$,*

(6.8) $$\boldsymbol{D}(\gamma u, \phi)+\boldsymbol{N}(\gamma u, \phi)+\left(\frac{1}{2}\Delta u,\ H\phi\right)_D=0\,.$$

PROOF. Take a function u of $\mathcal{D}$. For $\alpha>0$, u is equal to $G_\alpha f=G_\alpha^0 f+R_\alpha f$ with an $f\in L^2(D)$. The function u is an element of $\mathcal{F}$ and we have $\mathcal{E}^\alpha(u, v)=\mathcal{E}(u, v)+\alpha(u, v)_D=\mathcal{E}^{(0),\alpha}(G_\alpha^0 f, v)=(f, v)_D$ for every $v\in \mathrm{BLD}_0$. On the other hand, according to the preceding theorem, u belongs to $\widehat{\mathrm{BLD}}$ and $\mathcal{E}(u, v)=(u, v)_{D,1}$ for every $v\in\mathrm{BLD}_0$. Therefore, $u\in\mathcal{D}(\Delta)$ and

(6.9) $$\frac{1}{2}\Delta u=\alpha u-f\,.$$

Next, by making use of the identity

(6.10) $$\mathcal{E}^\alpha(u, H_\alpha\phi)=\mathcal{E}^\alpha(R_\alpha f, H_\alpha\phi)\,,\qquad \phi\in\mathcal{F}_M\,,$$

we will derive formula (6.8). The left-hand side of (6.10) is equal to $(f, H_\alpha\phi)_D$ and the right-hand side can be expressed in terms of $\gamma u=\gamma(R_\alpha f)$ as $\boldsymbol{D}(\gamma u, \phi)+\boldsymbol{U}_\alpha(\gamma u, \phi)+\boldsymbol{N}(\gamma u, \phi)$ (Theorem 5.2). Hence is suffices to show

(6.11) $$(f, H_\alpha\phi)_D-\boldsymbol{U}_\alpha(\gamma u, \phi)=-\left(\frac{1}{2}\Delta u,\ H\phi\right)_D.$$

Note that $(|g|, |H\phi|)_D$ is finite for $g\in L^2(D)$ and $\phi\in L^2(M)'$. In fact, it is no greater than $(|g|, |H_\alpha\phi|)_D+\alpha(|g|, G_{0+}^0|H_\alpha\phi|)_D$, which is finite because $H_\alpha\phi\in L^2(D)$ (see (3.19)) and $(|g|, G_{0+}^0|H_\alpha\phi|)_D^2\leq(\sup_{x\in D} G_{0+}^0 1(x))^2(g, g)_D(H_\alpha\phi, H_\alpha\phi)_D$. Equation (6.11) now follows from (6.9) and a formal computation as follows:

$$(f, H_\alpha\phi)_D-\boldsymbol{U}_\alpha(\gamma u, \phi)=(f, H\phi)_D-\alpha(G_\alpha^0 f, H\phi)_D-\alpha(R_\alpha f, H\phi)_D\,.$$

Conversely, suppose that a function u satisfies conditions (1) and (2) of our theorem. Set $f=\alpha u-\frac{1}{2}\Delta u$, $v=G_\alpha f$ and $w=u-v$. Then, we have $\frac{1}{2}\Delta w=\alpha w$ or equivalently,

(6.12) $$(w, v')_{D,1}+\alpha(w, v')_D=0\qquad\text{for every } v'\in\mathrm{BLD}_0\,.$$

Hence, $w\in\widehat{\mathrm{BLD}}_{\alpha,h}$ and $w=H_\alpha(\gamma w)$ (Theorem 3.4). However, w satisfies the condition (6.8) for all $\phi\in\mathcal{F}_M$. Set $\phi=\gamma w$. Then we have $\boldsymbol{U}_\alpha(\gamma w, \gamma w)=0$ which implies that $w=H_\alpha(\gamma w)=0$ in view of identity (3.19). Thus, u must be an element of $\mathcal{D}$.

§ 7. Construction of the symmetric resolvent density.

In the present section we are concerned with the converse problem to that of sections 4 and 5. For a given space $(\mathcal{F}_M, \mathcal{E}_M)$ described just below, does there its associated resolvent density $G_\alpha(x, y)$ of $\boldsymbol{G}$ exist? The answer is affirmative.

Define the bilinear form $\boldsymbol{D}$ and the function space $\boldsymbol{H}_M$ by (3.14) and (3.15) respectively. We start with a function space $\mathcal{F}_M$ and a non-negative symmetric bilinear form $\boldsymbol{N}$ on $\mathcal{F}_M$ satisfying the following conditions: (B. 1) $\mathcal{F}_M$ is a linear subspace of $\boldsymbol{H}_M$ and it contains constant functions, (B. 2) $\{\mathcal{F}_M, \boldsymbol{D}(\ ,\ )+\boldsymbol{N}(\ ,\ )\}$ is a Dirichlet space relative to $L^2(M)'$ and $\boldsymbol{N}(1, 1)=0$ and (B. 3) if ψ is a normal contraction of $\varphi\in\mathcal{F}_M$, then $\psi\in\mathcal{F}_M$ and $\boldsymbol{N}(\psi, \psi)\leqq\boldsymbol{N}(\varphi, \varphi)$.

For φ and $\psi\in\mathcal{F}_M$, set

$$\mathcal{E}_M(\varphi, \psi)=\boldsymbol{D}(\varphi, \psi)+\boldsymbol{N}(\varphi, \psi)\,, \tag{7.1}$$

$$\mathcal{E}_M^{[\alpha]}(\varphi, \psi)=\mathcal{E}_M(\varphi, \psi)+\boldsymbol{U}_\alpha(\varphi, \psi)\,, \qquad \alpha>0\,. \tag{7.2}$$

By the assumption, the space $\mathcal{F}_M$ is complete with the metric $\mathcal{E}_M(\ ,\ )+\lambda(\ ,\ )'_M$ for each $\lambda>0$. On the other hand, inequality (3.18) leads us to

$$(\varphi, \varphi)'_M\leqq\Big(1\vee\frac{1}{\alpha}\Big)\mathcal{E}_M^{[\alpha]}(\varphi, \varphi)\,, \qquad \varphi\in\mathcal{F}_M,\ \alpha>0\,. \tag{7.3}$$

By virtue of inequality (7.3) and Lemma 3.1 (ii), $\mathcal{E}_M^{[\alpha]}$ defines a metric on $\mathcal{F}_M$ equivalent to $\mathcal{E}_M(\ ,\ )+\lambda(\ ,\ )'_D$, $\lambda>0$. Hence, the space $(\mathcal{F}_M, \mathcal{E}_M^{[\alpha]})$ is a real Hilbert space for each $\alpha>0$. Further, (7.3) implies

LEMMA 7.1. *Fix an $\alpha>0$. For each $\varphi\in L^2(M)'$, there is a unique element $\tilde{R}^\alpha\varphi\in\mathcal{F}_M$ such that*

$$\mathcal{E}_M^{[\alpha]}(\tilde{R}^\alpha\varphi, \psi)=(\varphi, \psi)'_M \tag{7.4}$$

for every $\psi\in\mathcal{F}_M$.

For $\alpha>0$ and $y\in D$, the function $K'_\alpha(y, \xi)$ defined by (4.2) is in $\boldsymbol{B}(M)$ and so in $L^2(M)'$ as a function of $\xi\in M$ (Lemma 3.1 (i)).

DEFINITION 7.1. For $x, y\in D$ and $\alpha>0$, set

$$R_\alpha(x, y)=H_\alpha^x\tilde{R}^\alpha\hat{H}_\alpha^y \tag{7.5}$$

with $\tilde{R}^\alpha$ of the preceding lemma (see section 4 for notations H_α^x and $\hat{H}_\alpha^y$). Further, we set

$$G_\alpha(x, y)=G_\alpha^0(x, y)+R_\alpha(x, y) \tag{7.6}$$

with above R_α and the resolvent density G_α^0 of the absorbing barrier Brownian motion on D.

We will show the following theorem.

THEOREM 7.1. *Suppose that a function space $\mathcal{F}_M$ and a non-negative definite symmetric bilinear form $\boldsymbol{N}$ on $\mathcal{F}_M$ satisfying conditions* (B. 1), (B. 2) *and* (B. 3) *are given. Then, the following statements hold.*

(i) *$G_\alpha(x, y)$ defined by Definition 7.1 is an element of $\boldsymbol{G}$; it is a conservative, symmetric resolvent density satisfying conditions* (G. a) *and* (G. b).

(ii) *Let* $(\mathcal{F}_D, \mathcal{E})$ *be the Dirichlet space relative to* $L^2(D)$ *associated with this* $G_\alpha(x, y)$. *For each* $\alpha > 0$, *decompose* $\{\mathcal{F}_D, \mathcal{E}^\alpha(\ ,\ ) = \mathcal{E}(\ ,\ ) + \alpha(\ ,\ )_D\}$ *as* $\mathcal{F}_D = \mathcal{F}_D^{(0)} + \mathcal{H}_\alpha$ *by means of Theorem* 4.2.

Then, we have

$$\gamma\mathcal{H}_\alpha = \mathcal{F}_M \quad and \quad \mathcal{E}^\alpha(u, v) = \mathcal{E}_M^{[\alpha]}(\gamma u, \gamma v) \qquad for\ u, v \in \mathcal{H}_\alpha .$$

Owing to Theorem 6.2, we obtain

COROLLARY TO THEOREM 7.1. *Under the assumption of Theorem* 7.1, *there exists a unique element* $G_\alpha(x, y)$ *of* $\boldsymbol{G}$ *such that every function of* $G_\alpha(L^2(D))$ *satisfies the boundary condition* (6.8).

Before proceeding to the proof of Theorem 7.1, we prepare two lemmas. For $\varphi \in \boldsymbol{B}(M)$ and $\alpha > 0$, we set

$$U'_\alpha\varphi(\xi) = \begin{cases} U_\alpha\varphi(\xi)/U_1 1(\xi) & \text{if } U_1 1(\xi) < +\infty , \\ 0 & \text{if } U_1 1(\xi) = +\infty . \end{cases} \tag{7.7}$$

Following the argument in the proof of Lemma 4.1,

$$U'_\alpha\varphi(\xi) = \alpha\hat{H}_\alpha(H\varphi)(\xi) \qquad \text{for } \mu\text{-almost all } \xi \in M. \tag{7.8}$$

Further we have easily

$$1_M \leqq \left(1 \vee \frac{1}{\alpha}\right) U'_\alpha 1_M \qquad \mu\text{-almost everywhere.} \tag{7.9}$$

LEMMA 7.2. *Consider the operator* $\tilde{R}^\alpha$ *of Lemma* 7.1.

(i) *For each* $\alpha > 0$, $\tilde{R}^\alpha$ *is a positive linear operator.*

(ii) $\tilde{R}^\alpha U'_\alpha 1_M = 1_M$, $\alpha > 0$.

(iii) $\tilde{R}^\alpha$ *is a bounded operator on* $\boldsymbol{B}(M)$ *with norm less than* $1 \vee \frac{1}{\alpha}$.

(iv) $\tilde{R}^\alpha\varphi - \tilde{R}^\beta\varphi + \tilde{R}^\alpha(U'_\alpha - U'_\beta)\tilde{R}^\beta\varphi = 0, \quad \alpha, \beta > 0,$
$\varphi \in \boldsymbol{B}(M)$[21].

PROOF. (i) We can see from condition (B. 3) and identity (3. 21) that every normal contraction operates on $(\mathcal{F}_M, \mathcal{E}_M^{[\alpha]})$; if φ is a normal contraction of $\psi \in \mathcal{F}_M$, then $\varphi \in \mathcal{F}_M$ and $\mathcal{E}_M^{[\alpha]}(\varphi, \varphi) \leqq \mathcal{E}_M^{[\alpha]}(\psi, \psi)$. Thus, $\tilde{R}^\alpha$ must be positive. (ii) and (iv). For $\varphi, \psi \in \mathcal{F}_M$, since $\mathcal{E}_M(\varphi, 1_M) = 0$ and $\mathcal{E}_M^{[\alpha]}(\varphi, \psi) = \mathcal{E}_M(\varphi, \psi) + (\varphi, U'_\alpha\psi)'_M$, equalities of (ii) and (iv) follow from equation (7.4) through simple computations. Assertion (iii) is a consequence of (i), (ii) and inequality (7.9).

LEMMA 7.3. *Suppose that a function* $\varphi^x(\xi)$, $x \in D$, $\xi \in M$, *is jointly measurable in* (x, ξ) *and bounded in* ξ *for each* $x \in D$. *Let* ν *be a signed measure on* D *such that* $\varphi^\nu(\xi) = \int_D \varphi^x(\xi)\nu(dx)$ *is bounded in* $\xi \in M$. *Then, it holds that*

21) See Neveu [20] for an analogous formula.

$$\int_D (\psi, \tilde{R}^{\alpha}\varphi^x)'_M \nu(dx) = (\psi, \tilde{R}^{\alpha}\varphi^{\nu})'_M$$

for every $\psi \in L^2(M)'$.

PROOF. By equation (7.4), $\tilde{R}^{\alpha}$ is symmetric on $L^2(M)'$. Integrating the identity

$$(\psi, \tilde{R}^{\alpha}\varphi^x)'_M = (\tilde{R}^{\alpha}\psi, \varphi^x)'_M$$

by ν, we have

$$\int_D (\psi, \tilde{R}^{\alpha}\varphi^x)'_M \nu(dx) = (\tilde{R}^{\alpha}\psi, \varphi^{\nu})'_M = (\psi, \tilde{R}^{\alpha}\varphi^{\nu})'_M .$$

PROOF OF THEOREM 7.1 (i).

Condition (G. a). $R_\alpha(x, y) = H^x_\alpha \tilde{R}^{\alpha} \hat{H}^y_\alpha$ is α-harmonic in $x \in D$ for each $y \in D$. Its non-negativity is due to Lemma 7.2 (i).

Condition (G. b). Take a compact set K of D. In view of Lemma 7.2 (iii),

$$\sup_{x \in D, y \in K} R_\alpha(x, y) \leqq \sup_{\xi \in M, y \in K} \tilde{R}^{\alpha}\hat{H}^y_\alpha(\xi) \leqq \left(1 \vee \frac{1}{\alpha}\right) \sup_{\xi \in M, y \in K} \hat{H}^y_\alpha(\xi) < +\infty .$$

Symmetry. $R_\alpha(x, y) = (\hat{H}^x_\alpha, \tilde{R}^{\alpha}\hat{H}^y_\alpha)'_M$ is symmetric in $x, y \in D$, since $\tilde{R}^{\alpha}$ is symmetric on $L^2(M)'$.

Conservativity. By Lemma 7.3, identity (7.8) and Lemma 7.2 (ii),

$$\begin{aligned} \alpha R_\alpha 1_D(x) &= \alpha \int_D (\hat{H}^x_\alpha, \tilde{R}^{\alpha}\hat{H}^y_\alpha)'_M dy \\ &= \alpha(\hat{H}^x_\alpha, \tilde{R}^{\alpha}(\hat{H}_\alpha 1_D))'_M = (\hat{H}^x_\alpha, \tilde{R}^{\alpha} U'_\alpha 1_M)'_M \\ &= H^x_\alpha 1_M(x) = 1 - \alpha G^0_\alpha 1_D(x) \end{aligned}$$

and, therefore, $\alpha G_\alpha 1_D(x) = 1$, $x \in D$.

Resolvent equation. By making use of the resolvent equation for G^0_α and equation (4.5), we can see that

$$G_\alpha(x, y) - G_\beta(x, y) + (\alpha - \beta)\int_D G_\alpha(x, z) G_\beta(z, y) dz$$

is equal to

$$\begin{aligned} (7.10) \qquad & (\hat{H}^x_\alpha, \tilde{R}^{\alpha}\hat{H}^y_\alpha)'_M - (\hat{H}^x_\alpha, \tilde{R}^{\beta}\hat{H}^y_\beta)'_M \\ & + (\alpha - \beta)\int_D (\hat{H}^x_\alpha, \tilde{R}^{\alpha}\hat{H}^z_\alpha)'_M (\hat{H}^z_\beta, \tilde{R}^{\beta}\hat{H}^y_\beta)'_M dz \\ & + (\alpha - \beta)\int_D (\hat{H}^x_\alpha, \tilde{R}^{\alpha}\hat{H}^z_\alpha)'_M G^0_\beta(z, y) dz . \end{aligned}$$

By virtue of Lemma 7.3 and equations (4.5) and (4.6), (7.10) is seen to be identical with

$$(7.11) \qquad (\hat{H}^x_\alpha, \tilde{R}^{\alpha}\hat{H}^y_\beta)'_M - (\hat{H}^x_\alpha, \tilde{R}^{\beta}\hat{H}^y_\beta)'_M + (\hat{H}^x_\alpha, \tilde{R}^{\alpha}(U'_\alpha - U'_\beta)\tilde{R}^{\beta}\hat{H}^y_\beta)'_M$$

which vanishes according to Lemma 7.2 (iv).

PROOF OF THEOREM 7.1 (ii). Set $\mathcal{H}'_\alpha = H_\alpha(\mathcal{F}_M) = \{u\,;\, u = H_\alpha\varphi,\ \varphi \in \mathcal{F}_M\}$ and $\mathcal{E}'^{,\alpha}(u, v) = \mathcal{E}_M^{[\alpha]}(\gamma u, \gamma v)$ for $u, v \in \mathcal{H}'_\alpha$. It suffices to prove that $(\mathcal{H}'_\alpha, \mathcal{E}'^{,\alpha})$ coincides with the space $(\mathcal{H}_\alpha, \mathcal{E}^\alpha)$. Space $(\mathcal{H}'_\alpha, \mathcal{E}'^{,\alpha})$ is a real Hilbert space since $(\mathcal{F}_M, \mathcal{E}_M^{[\alpha]})$ is. We can see that $R_\alpha f$, $f \in \boldsymbol{B}(D)$, belongs to $\mathcal{H}'_\alpha$ and satisfies

$$\mathcal{E}'^{,\alpha}(R_\alpha f, v) = (f, v)_D \qquad \text{for every } v \in \mathcal{H}'_\alpha\,. \tag{7.12}$$

Indeed, according to Lemma 7.3, $R_\alpha f(x) = H_\alpha^*(\tilde{R}^\alpha \hat{H}_\alpha f)$. Hence $R_\alpha f \in \mathcal{H}'_\alpha$ and

$$\begin{aligned}\mathcal{E}'^{,\alpha}(R_\alpha f, H_\alpha\psi) &= \mathcal{E}_M^{[\alpha]}(\tilde{R}^\alpha \hat{H}_\alpha f, \psi)\\ &= (\hat{H}_\alpha f, \psi)'_M = (f, H_\alpha\psi)_D \qquad \text{for } \psi \in \mathcal{F}_M\,.\end{aligned}$$

Evidently, $R_\alpha f$, $f \in \boldsymbol{B}(D)$, is an element of $\mathcal{H}_\alpha$ and equation (7.12) is still valid if $\mathcal{E}'^{,\alpha}$ is replaced by $\mathcal{E}^\alpha$ and $\mathcal{H}'_\alpha$, by $\mathcal{H}_\alpha$. Thus, $R_\alpha(\boldsymbol{B}(D))$ being dense in both spaces $\mathcal{H}'_\alpha$ and $\mathcal{H}_\alpha$, $(\mathcal{H}'_\alpha, \mathcal{E}'^{,\alpha})$ must be identical with $(\mathcal{H}_\alpha, \mathcal{E}^\alpha)$.

§8. A class of diffusions including the reflecting Brownian motion.

In the preceding sections we have established a one-to-one correspondence between the class $\boldsymbol{G}$ of symmetric resolvent densities and the class of pairs $(\mathcal{F}_M, \boldsymbol{N})$ satisfying conditions (B. 1), (B. 2) and (B. 3).

Denote by $\boldsymbol{G}_1$ the totality of $G_\alpha(x, y)$ in $\boldsymbol{G}$ such that the corresponding form $\boldsymbol{N}(\,,\,)$ vanishes identically on the corresponding space $\mathcal{F}_M$. According to those arguments in the preceding two sections, we can assert as follows.

THEOREM 8.1.

(i) *There is a one-to-one correspondence between the class* $\boldsymbol{G}_1$ *and the class of function spaces* $\mathcal{F}_M$ *satisfying*

(B_1. 1) $\mathcal{F}_M$ *contains every constant function on* M *and* $\mathcal{F}_M$ *is a linear subspace of* $\boldsymbol{H}_M$.

(B_1. 2) $\mathcal{F}_M$ *is closed with the norm* $\boldsymbol{D}(\,,\,) + \lambda(\,,\,)'_M$ *for a* $\lambda > 0$.

(B_1. 3) *Every normal contraction of an element of* $\mathcal{F}_M$ *is also an element of* $\mathcal{F}_M$.

(ii) *A linear space* $\mathcal{F}$ *of functions on* D *with a bilinear form* $\mathcal{E}(\,,\,)$ *is the refinement*[22] *of a Dirichlet space associated with an element of* $\boldsymbol{G}_1$ *if and only if* $\mathcal{F}$ *contains every constant function on* D, $\mathrm{BLD}_0 \subset \mathcal{F} \subset \widehat{\mathrm{BLD}}$, $\mathcal{E}(u, u) = (u, u)_{D,1}$ *for every* $u \in \mathcal{F}$, $\mathcal{F}$ *is closed with norm* $\mathcal{E}^\alpha(u, u) = (u, u)_{D,1} + \alpha(u, u)_D$ *for an* $\alpha > 0$, *and finally, every normal contraction of an element of* $\mathcal{F}$ *is also an element of* $\mathcal{F}$.

(iii) *For any element* $G_\alpha(x, y)$ *of* $\boldsymbol{G}_1$, *the function* $u = G_\alpha f$ $(f \in \boldsymbol{B}(D),\ \alpha > 0)$ *belongs to the space* $\widehat{\mathrm{BLD}}$ *and*

22) See (4.18).

(8.1) $$(u, u)_{D,1}+\alpha(u, u)_D=(u, f)_D\,.$$

PROOF. Conditions (B_1. 1), (B_1. 2) and (B_1. 3) of the first assertion are nothing but conditions (B. 1), (B. 2) and (B. 3) with $N(\,,\,)=0$. Suppose that a space $(\mathcal{F}, \mathcal{E})$ satisfies all conditions of statement (ii). Set $\mathcal{F}_M=\gamma\mathcal{F}$. Owing to Theorem 3.4, $H_\alpha(\mathcal{F}_M)$ is the projection of $(\mathcal{F}, \mathcal{E}^\alpha)$ to the space $\widehat{\mathrm{BLD}}_{\alpha,h}$ and $\mathcal{F}_M$ is closed with norm $\boldsymbol{D}(\,,\,)+\boldsymbol{U}_\alpha(\,,\,)$. Hence, $\mathcal{F}_M$ satisfies conditions (B_1. 1) and (B_1. 2) (see the argument preceding Lemma 7.1). The same procedure as in Lemmas 5.2 and 5.3 can be applied to obtain the property (B_1. 3) for $\mathcal{F}_M$[23]. Let $G_\alpha(x, y)$ be the element of $\boldsymbol{G}_1$ which corresponds to this $\mathcal{F}_M$ by means of assertion (i). Then, by virtue of Theorem 4.3 and 7.1, we can see that $(\mathcal{F}, (\,,\,)_{D,1})$ is the refinement of the Dirichlet space associated with this $G_\alpha(x, y)$. Property (iii) follows from statement (ii).

Our main interest of this section lies on those Markov processes associated with elements of $\boldsymbol{G}_1$. As was seen in the final argument of section 1, all the results of section 3 in the article [**15**] are valid for every resolvent of the class $\boldsymbol{G}$.

Further, as far as the elements of the class $\boldsymbol{G}_1$ are concerned, all the statements of [**15**; Section 4] are valid, since we never used in [**15**] any special property of the resolvent density of the reflecting barrier Brownian motion expect the above idensity (8.1) (see [**15**; (4.11)]). Thus, we have the following generalization of [**15**; Theorem 2].

THEOREM 8.2. *For each element $G_\alpha(x, y)$ of $\boldsymbol{G}_1$, there exists a diffusion process (a strong Markov process with continuous paths) $\boldsymbol{X}=(X_t, P_x, x\in D^*)$ on an extended state space D^* and $\boldsymbol{X}$ has properties ($\boldsymbol{X}$.1), ($\boldsymbol{X}$.2) and ($\boldsymbol{X}$.3) mentioned in the final part of section* 1.

Here are two extreme cases of elements in $\boldsymbol{G}_1$.

(I). *Resolvent density of the reflecting Brownian motion.* This resolvent $G_\alpha(x, y)=G_\alpha^0(x, y)+R_\alpha(x, y)$ was defined in [**15**] by means of the equation

(8.2) $$(R_\alpha(x, \cdot), v)_{D,1}+\alpha(R_\alpha(x, \cdot), v)_D=v(x)$$

for every $v\in\widehat{\mathrm{BLD}}_{\alpha,h}$. This is fitted for the case that $\mathcal{F}_M=\boldsymbol{H}_M$ and $\mathcal{F}=\widehat{\mathrm{BLD}}$. Indeed, the same procedure as in the proof of [**15**; Lemma 2.10] is applicable to get from (8.2) the following equation for $R_\alpha f$, $f\in\boldsymbol{B}(D)$,

(8.3) $$(R_\alpha f, v)_{D,1}+\alpha(R_\alpha f, v)_D=(f, v)_D$$

for all $v\in\widehat{\mathrm{BLD}}_{\alpha,h}$. Equation (8.3) implies that the Dirichlet space $\mathcal{F}$ associated with the resolvent satisfying (8.2) is just the space $\widehat{\mathrm{BLD}}$. Note that the result

23) All assertions of Lemma 5.2 and 5.3 are still valid when we replace $\mathcal{F}$ by the space in the latter statement of Theorem 8.1 (ii) and $\mathcal{E}^\alpha(\,,\,)$, by $(\,,\,)_{D,1}+\alpha(\,,\,)_D$. Thus, $\mathcal{F}_M$ $(=\gamma\mathcal{F})$ is a Dirichlet space associated with a resolvent on $L^2(M)'$.

in the preceding section gives another method to construct the resolvent density satisfying (8.2)[24].

The boundary condition (6.8) for $u \in G_\alpha(L^2(D))$ is now

$$\mathbf{D}(\gamma u, \psi)+\left(\frac{1}{2}\Delta u, H\psi\right)_D=0 \qquad \text{for every } \psi \in \mathbf{H}_M. \tag{8.4}$$

Formula (8.4) means that $u \in G_\alpha(L^2(D))$ has, as its generalized normal derivative of Doob [7] (in a slightly modified sense), a function identically vanishing on the boundary M.

(II). *The case when $\mathcal{F}_M$ is trivial.* Let $\mathcal{F}_M$ be the set of all constant functions on M. $\mathcal{F}_M$ satisfies conditions (B_1. 1), (B_1. 2) and (B_1. 3) of Theorem 8.1 trivially. The corresponding resolvent in $\mathbf{G}_1$ is

$$G_\alpha(x, y)=G_\alpha^0(x, y)+\frac{\Pi_\alpha(x)\Pi_\alpha(y)}{\alpha(\Pi_\alpha, 1_D)_D}, \tag{8.5}$$

with $\Pi_\alpha(x)=H_\alpha 1_M(x)$[25]. In fact, by Definition 7.1, $R_\alpha(x, y)$ is equal to $H_\alpha^x \tilde{R}^\alpha \hat{H}_\alpha^y$ with $\tilde{R}^\alpha \hat{H}_\alpha^y$ in $\mathcal{F}_M$ satisfying equation (7.4) for $\varphi=\hat{H}_\alpha^y$. Hence $\tilde{R}^\alpha \hat{H}_\alpha^y$ is a constant and

$$\tilde{R}^\alpha \hat{H}_\alpha^y=\frac{(\hat{H}_\alpha^y, 1_M)'_M}{U_\alpha(1_M, 1_M)}=\frac{\Pi_\alpha(y)}{\alpha(\Pi_\alpha, 1)_D}.$$

By virtue of Theorem 8.2, the corresponding process $\mathbf{X}$ to (8.5) is a diffusion. However, it may generally include branching points on D^*-D in Ray's sense[26]. Suppose that, the relative boundary ∂D of D is so smooth that $G_\alpha^0(x, y)\to 0$ and $\Pi_\alpha(x)\to 1$ $(\alpha>0)$ as x goes out of any compact subset of D. Then, the Martin-Kuramochi type completion D^* of D with respect to $\{G_1(x, y)\}$ of (8.5) is just the one point compactification $D\cup\{\infty\}$ of D and the extended resolvent density is given by

$$G_\alpha(\{\infty\}, y)=\lim_{x\to\{\infty\}} G_\alpha(x, y)=\frac{\Pi_\alpha(y)}{\alpha(\Pi_\alpha, 1_D)_D}, \qquad \alpha>0. \tag{8.6}$$

Hence, owing to Theorem 3.3, the measure $\alpha G_\alpha(\{\infty\}, y)dy$ converges on $D\cup\{\infty\}$ to the δ-measure concentrated at $\{\infty\}$ as α tends to infinity. Thus, we can conclude, under the assumption on the smoothness of D, that to the resolvent (8.5) corresponds a continuous Hunt process on $D\cup\{\infty\}$ (including no branching point).

Finally, we note that, besides above extreme cases (I) and (II), there may

24) Cf. [14].

25) The space $\mathcal{D}=G_\alpha(L^2(D))$ for this resolvent is characterized as follows. $u\in\mathcal{D}$ if and only if $u\in\mathcal{D}(\Delta)$, u has a constant boundary function and $\int_D \Delta u(x)dx=0$.

26) Cf. [15].

be many elements of G_1[27]. For instance,

(III) *Brownian motion on a torus.* Consider an open square $D=\{(x_1, x_2);\ 0<x_i<1,\ i=1, 2\} \subset R^2$. The Martin boundary M of D consists of all its sides. Put

$$\mathcal{F}_M^T=\{\varphi \in \boldsymbol{H}_M\,;\ \varphi((x_1, 0))=\varphi((x_1, 1)) \text{ and }$$
$$\varphi((0, x_2))=\varphi((1, x_2)) \quad \mu\text{-almost everywhere}\}.$$

$\mathcal{F}_M^T$ satisfies conditions $(B_1. 1)$, $(B_1. 2)$, $(B_1. 3)$ of Theorem 8.1. Therefore, we can associate an element, say $G_\alpha^*(x, y)$, of the class $\boldsymbol{G}_1$ with the space $\mathcal{F}_M^T$. Let us show that the corresponding diffusion in Theorem 8.2 is the Brownian motion on the torus $K=[0, 1)\times[0, 1)$. Denote the resolvent density of the latter by $G_\alpha^T(x, y)$[28]. Since the Martin-Kuramochi type completion of the domain D with respect to functions $\{G_1^T(\cdot, y),\ y \in D\}$ is just the torus K, it suffices for us to show that $G_\alpha^*(x, y)=G_\alpha^T(x, y)$, $x, y \in D$. $u(x)=\int_D G_\alpha^T(x, y)f(y)dy$, with $f((x_1, x_2)) =f_1(x_1)\cdot f_2(x_2)$, $f_i \in \boldsymbol{C}_0(0, 1)$, $i=1, 2$, has the following properties.

(T.1) u and its first derivatives can be continuously extended to $[0, 1]\times[0, 1]$, periodically such as $u((x_1, 0))=u((x_1, 1))$, $u((0, x_2))=u((1, x_2))$, $u_{x_2}((x_1, 0))= u_{x_2}((x_1, 1))$, $u_{x_1}((0, x_2))=u_{x_1}((1, x_2))$, for every $x_1, x_2 \in [0, 1]$[29].

(T.2) $$\frac{1}{2}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)u(x)=\alpha u(x)-f(x), \qquad x \in D.$$

Hence, $u \in \mathfrak{D}(\Delta)$ and $\gamma u \in \mathcal{F}_M^T$. Further,

$$\boldsymbol{D}(\gamma u, \psi)+\left(\frac{1}{2}\Delta u, H\psi\right)_D=(H(\gamma u), H\psi)_{D,1}+\frac{1}{2}(\Delta u, H\psi)_D$$
$$=\int_M \frac{\partial u}{\partial \boldsymbol{n}}(\xi)\psi(\xi)\sigma(d\xi)=0 \qquad \text{for any } \psi \in \mathcal{F}_M^T \text{ [30]}.$$

Thus, by Theorem 6.2, we have $u=G_\alpha^* f$ and consequently, $G_\alpha^*(x, y)=G_\alpha^T(x, y)$, $x, y \in D$.

27) It is plausible that the class $\boldsymbol{G}_1$ is characterized by a family of partitions of the boundary M.

28) $G_\alpha^T(x, y)$ is the Laplace transform of the transition density $p(t, x, y)= \sum_{m,n=-\infty}^{\infty} g(t, x, (y_1+m, y_2+n))$. Here, $g(t, x, y)$ is the two dimensional Gauss kernel.

29) u_{x_i} denotes the derivative of u with respect to the variable x_i $(i=1, 2)$.

30) The second equality for $\psi \in \boldsymbol{H}_M$ is obtained in the similar manner as [**15**; footnote 5]). $\boldsymbol{n}$ denotes the normal and σ denotes the linear Lebesgue measure on M. σ is absolutely continuous with respect to the measure μ.

§ 9. Cases of a circular disk and an interval.

(I) *The case of a circular disk.*

Let us examine the case when D is an open disk of radius 1. The Martin boundary M of D is, in this case, identified with its circle whose points can be characterized by the parameter θ; $0 \leqq \theta < 2\pi$. The Feller kernel $U(\ ,\ )$ is a constant multiple of $\frac{1}{1-\cos(\theta-\theta')}$ and the space $\boldsymbol{H}_M$ is given by

$$(9.1) \qquad \boldsymbol{H}_M = \{\varphi \in L^2(d\theta)\,;\ \boldsymbol{D}(\varphi, \varphi) = C\int_0^{2\pi}\int_0^{2\pi}(\varphi(\theta)-\varphi(\theta'))^2 \frac{d\theta d\theta'}{1-\cos(\theta-\theta')} < +\infty\},$$

C being a positive constant.

Suppose that the functions $\sin\theta$ and $\cos\theta$ belong to the space $\mathfrak{F}_M$ of Theorem 5.2. Then, by making use of formula (5.4), the bilinear form $N(\ ,\ )$ of this theorem can be expressed explicitly as follows. For any continuously differentiable function $\varphi \in \mathfrak{F}_M$,

$$(9.2) \qquad N(\varphi, \varphi) = \int_0^{2\pi}\varphi'(\theta)^2\nu(d\theta) + \int_0^{2\pi}\int_0^{2\pi}(\varphi(\theta)-\varphi(\theta'))^2\Phi(d\theta, d\theta'),$$

where, ν is a finite measure on M and Φ is a symmetric Radon measure on $M\times M$ off the diagonal such that, for any $\delta > 0$,

$$(9.3) \qquad \iint_{|e^{i\theta}-e^{i\theta'}|>\delta}\Phi(d\theta, d\theta') < +\infty,$$

$$(9.4) \qquad \iint_{|e^{i\theta}-e^{i\theta'}|\leqq\delta}(1-\cos(\theta-\theta'))\Phi(d\theta, d\theta') < +\infty.$$

We note that the convergence condition (9.4) for the Levy measure Φ may not be satisfied in general[31]. For instance, choose a measurable function $a(\theta)$ bounded below and above by strictly positive constants and set

$$\Phi^{\#}(\theta, \theta') = \frac{1}{(1-\cos(\theta-\theta'))\left(1-\cos\left(\frac{\theta+\theta'}{2}\right)\right)},$$

$$N^{\#}(\varphi, \varphi) = \int_0^{2\pi}\varphi'(\theta)^2 a(\theta)d\theta + \int_0^{2\pi}\int_0^{2\pi}(\varphi(\theta)-\varphi(\theta'))^2\Phi^{\#}(\theta, \theta')d\theta d\theta',$$

$$\mathfrak{F}_M^{\#} = \{\varphi \in \boldsymbol{H}_M\,;\ \varphi \text{ is absolutely continuous and } N^{\#}(\varphi, \varphi) \text{ is finite}\}.$$

The space $\mathfrak{F}_M^{\#}$ is non-trivial, since it contains the function $\sin^2\theta$. The measure

31) In this sense, our boundary condition (6.8) for the disk is never included by the Wentzell boundary condition [**23**].

$\Phi^{\#}(\theta, \theta')d\theta d\theta'$ satisfies condition (9.3), but does not satisfy (9.4). However, the pair $(\mathfrak{F}_M^{\#}, N^{\#})$ clearly satisfies conditions (B. 1), (B. 2) and (B. 3), and hence, on account of Theorem 7.1, we can construct a resolvent $G_\alpha(x, y)$ of the class $\boldsymbol{G}$ which corresponds to this pair (in the manner of Theorem 5.2). I don't know whether the closed disk $\bar{D}$ is identified with the state space D^* (the Martin-Kuramochi type completion of D with respect to $G_1(x, y)$) on which the associated strong Markov process moves.

(II) *One-dimensional case.*

In this case, D is a finite open interval (a, b) and the Martin boundary consists of two points a and b. We can express explicitly all the resolvents in the class $\boldsymbol{G}$.

(II$_1$) *The case when $\mathfrak{F}_M$ is trivial; circular Brownian motion.*

This is one-dimensional case of section 8 (II). The corresponding resolvent is expressed as (8.5). The boundary condition is $u(a)=u(b)$ and $u'(a)=u'(b)$[32]. The corresponding process is a conservative diffusion on the one-point compactification of (a, b) and, as one easily sees, it is nothing but the Brownian motion on a circle.

(II$_2$) *The case when $\mathfrak{F}_M$ is non-trivial.*

The space $\mathfrak{F}_M$ satisfying (B. 1) and (B. 2) necessarily consists of all functions on $\{a, b\}$. $N(\varphi, \varphi)$ satisfying (B. 2) and (B. 3) is written as

$$N(\varphi, \varphi)=\kappa(\varphi(a)-\varphi(b))^2 \tag{9.5}$$

with a non-negative constant κ. Thus, this case is completely determined by each $\kappa \geqq 0$. Take a $\kappa \geqq 0$. By means of one-dimensional Brownian measure and Brownian hitting time to a and b, we set $H_\alpha^x(a)=E_x(e^{-\alpha\sigma_a}; \sigma_a<\sigma_b)$ and $H_\alpha^x(b)=E_x(e^{-\alpha\sigma_b}; \sigma_b<\sigma_a)$, $a<x<b$. Rewriting formulae (7.4) and (7.5), we can derive the following expression of the corresponding resolvent density.

$$G_\alpha(x, y)=G_\alpha^0(x, y)+(H_\alpha^x(a), H_\alpha^x(b))\Lambda^\alpha\begin{pmatrix} H_\alpha^y(a) \\ H_\alpha^y(b) \end{pmatrix}, \tag{9.6}$$

where Λ^α is the inverse of the regular matrix

$$\mathcal{M}^\alpha=\begin{pmatrix} U^{ab}+\kappa+U_\alpha^{aa}, & -U^{ab}-\kappa+U_\alpha^{ab} \\ -U^{ab}-\kappa+U_\alpha^{ab}, & U^{ab}+\kappa+U_\alpha^{bb} \end{pmatrix}. \tag{9.7}$$

Here, $U^{ab}=U(a, b)\mu(\{a\})\mu(\{b\})$, $U_\alpha^{aa}=U_\alpha(a, a)\mu(\{a\})\mu(\{a\})$ and so on[33]. Theorem

32) See footnote 25).

33) $U_\alpha^{ab}=U_\alpha^{ba}=\frac{b-a}{2}(1-\sqrt{2\alpha}\ \mathrm{cosech}\ \sqrt{2\alpha}(b-a))$,

$U_\alpha^{aa}=U_\alpha^{bb}=\frac{b-a}{2}(\sqrt{2\alpha}\ \coth\ \sqrt{2\alpha}(b-a)-1)$ and

$U^{ab}=U^{ba}=\frac{b-a}{2}$.

6.2 states that $u \in G_\alpha(L^2(a, b))$ if and only if (1) $u \in \mathcal{D}(\Delta)$, (2) $u(x)$ has limits at a and b and

$$2(u(a)-u(b))(U^{ab}+\kappa)+\int_a^b \Delta u(x)H_0^{\pi}(a)dx=0$$

$$-2(u(a)-u(b))(U^{ab}+\kappa)+\int_a^b \Delta u(x)H_0^{\pi}(b)dx=0.$$

It is easy to see that these conditions (1) and (2) are equivalent to the following simple conditions: (1)' u, u' and u'' are square integrable. Here u' and u'' are the Radon-Nikodym derivatives. (2)' u and u' have limits at a and b and

$$u'(a)+(u(b)-u(a))\kappa=0$$

$$u'(b)+(u(a)-u(b))\kappa=0.$$

Thus, as for the one-dimensional case, the boundary condition (6.8) is reduced to Feller's one [12] applied to the class G.

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REGULAR REPRESENTATIONS OF DIRICHLET SPACES

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Abstract. We construct a regular and a strongly regular Dirichlet space which are equivalent to a given Dirichlet space in the sense that their associated function algebras are isomorphic and isometric. There is an appropriate strong Markov process called a Ray process on the underlying space of each strongly regular Dirichlet space.

1. **Introduction.** A. Beurling and J. Deny [1] introduced the notion of Dirichlet spaces and developed the general theory of kernel-free potentials. Recently the author [6] adopted Dirichlet spaces relative to L^2-spaces (we will call them L^2-Dirichlet spaces or D-spaces as an abbreviation) to describe boundary conditions for multidimensional Brownian motions.

A D-space is a certain space of functions that are defined on an underlying measure space (X, m). When (X, m) is fixed, there is a one-to-one correspondence between the set of all symmetric sub-Markov resolvent operators on $L^2(X; m)$ and the set of all D-spaces. In particular, any sub-Markov resolvent kernel on X which is symmetric with respect to m generates a D-space. The present paper and the subsequent one [9] concern the problem of whether conversely any D-space guarantees the existence of a suitable strong Markov process or not.

The present paper aims at constructing a regular and a strongly regular D-space which are equivalent to a given D-space. A D-space is called regular if it densely contains sufficiently many continuous functions vanishing at infinity on its underlying space. There corresponds a potential theory of a type of Beurling-Deny to each regular D-space. A strongly regular D-space is a regular one which is generated by a Ray resolvent kernel. According to D. Ray [15], there is a right continuous strong Markov process on the underlying space of each strongly regular D-space.

Suppose that we are given a D-space with underlying space (X, m). Theorem 2 in §5 states that there exists then a regular D-space with some modified underlying space (X', m') in such a way that these two D-spaces are equivalent to each other as function spaces. The latter D-space will be called a *regular representation* of the given one. The regular representation will be carried out depending on a subalgebra L of $L^\infty(X; m)$ satisfying a certain condition denoted by (C). Actually we will take as X' the space of all regular maximal ideals of L.

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There are generally many possibilities to find L satisfying (C). In §6, a special L possessing an additional property denoted by (R) will be constructed by making use of the method of F. Knight [11] and H. Kunita and T. Watanabe [12]. We can regard the condition (R) as a generalization of Ray's hypothesis for a sub-Markov resolvent [15]. Theorem 3 in §6 asserts that the regular representation with respect to such an L turns out to be a strongly regular D-space.

§3 consists of typical examples of D-spaces related to the multidimensional Brownian motion. Those D-spaces except for the last example took the fundamental roles in the investigations of boundary problems by J. L. Doob [4] and by the author [5], [6]. The last example is a rather sophisticated one of regular D-spaces(1). Much stress on the roles of regular ones will be laid in [9].

The appendix is referred to only in §3.

2. **Basic properties of D-spaces.**

DEFINITION 2.1. We call $(X, m, \mathscr{F}, \mathscr{E})$ an L^2-*Dirichlet space* (or a *D-space*, for short) if the following conditions are satisfied.

(D.1) X is a locally compact, Hausdorff, and separable space. m is a Radon measure on X.

(D.2) $\mathscr{F}$ is a linear subspace of the real $L^2(X)=L^2(X; m)$, two functions of $\mathscr{F}$ being identified if they coincide m-a.e. on X. $\mathscr{E}$ is a symmetric nonnegative definite bilinear form on $\mathscr{F}$ and, for each $\alpha>0$, $\mathscr{F}$ is a real Hilbert space with respect to the inner product

$$\mathscr{E}^{\alpha}(u, v) = \mathscr{E}(u, v)+\alpha(u, v)_X, \qquad u, v \in \mathscr{F}, \tag{2.1}$$

where $(u, v)_X$ denotes the inner product of $L^2(X)$.

(D.3) Every normal contraction operates on $(\mathscr{F}, \mathscr{E})$: if $u \in \mathscr{F}$ and a m-measurable function v satisfies inequalities

$$|v(x)| \leqq |u(x)|, \qquad |v(x)-v(y)| \leqq |u(x)-u(y)|$$

m-a.e. on X, then $v \in \mathscr{F}$ and $\mathscr{E}(v, v) \leqq \mathscr{E}(u, u)$.

The present definition of D-space was given in [6]. (X, m) is called the *underlying space* of the D-space. According to §2 of [6], let us state a theorem about a one-to-one correspondence between D-spaces and L^2-resolvents.

DEFINITION 2.2. Let (X, m) satisfy condition (D.1). A system $\{G_\alpha, \alpha>0\}$ of linear, bounded and symmetric operators on $L^2(X)$ is called an L^2-*resolvent* if it has the following properties.

(G.1) Sub-Markov property: if $u \in L^2(X)$ and $0 \leqq u \leqq 1$ m-a.e. then $0 \leqq \alpha G_\alpha u \leqq 1$ m-a.e., for any $\alpha>0$.

(G.2) Resolvent equation: $G_\alpha - G_\beta + (\alpha-\beta) G_\alpha G_\beta = 0$, $\alpha, \beta > 0$.

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(1) N. Ikeda suggested to the author the last example of §3 and theorem of the appendix.

THEOREM 1. *Let us fix* (X, m) *satisfying condition* (D.1). *For a given D-space* $(\mathcal{F}, \mathcal{E})$ *with underlying space* (X, m), *there exists a unique* L^2*-resolvent* $\{G_\alpha, \alpha>0\}$ *on* $L^2(X)$ *satisfying the equation*

$$\mathcal{E}^\alpha(G_\alpha u, v) = (u, v)_X \tag{2.2}$$

for any $v \in \mathcal{F}$, *where* $\alpha>0$ *and* $u \in L^2(X)$ *are arbitrarily fixed. Conversely, for a given* L^2*-resolvent* $\{G_\alpha, \alpha>0\}$ *on* $L^2(X)$, *a D-space is defined by*

$$\mathcal{F} = \left\{u \in L^2(X);\ \lim_{\beta\to+\infty} \beta(u-\beta G_\beta u, u)_X < +\infty\right\}, \tag{2.3}$$

$$\mathcal{E}(u, v) = \lim_{\beta\to+\infty} \beta(u-\beta G_\beta u, v)_X, \qquad u, v \in \mathcal{F}. \tag{2.4}$$

The correspondence defined by (2.2) *and that defined by* (2.3) *and* (2.4) *are reciprocal to each other.*

REMARK 2.1. (i) The proof of Theorem 1 was sketched in §2 of [6]. The essential ideas for the proof can be found in Beurling-Deny [1] and Deny [2]. So far as this theorem and the next lemma are concerned, condition (D.1) for (X, m) can be much weakened. These have been proved in [7] without the separability assumption for X (see also [8]). T. Shiga and T. Watanabe [16] gave a detailed proof of Theorem 1 under the assumption that, instead of (D.1), the underlying space (X, m) is merely a σ-finite measure space.

(ii) Condition (D.3) in the definition of D-space can be replaced with the following apparently weaker but equivalent condition (D.3)′ [16].

(D.3)′ Every unit contraction operates on $(\mathcal{F}, \mathcal{E})$: if $u \in \mathcal{F}$ then $v=(0 \vee u) \wedge 1$ is also in $\mathcal{F}$ and $\mathcal{E}(v, v) \leqq \mathcal{E}(u, u)$. Here, the lattice operations $\vee$ and $\wedge$ for functions on X are defined by $(u_1 \vee u_2)(x) = \max(u_1(x), u_2(x))$ and $u_1 \wedge u_2 = -((-u_1) \vee (-u_2))$.

The next lemma states the basic properties of D-spaces which we need in the later discussions. Notice that, for a D-space, $\mathcal{E}^\alpha$ and $\mathcal{E}^\beta$ define equivalent metrics on $\mathcal{F}$ for any $\alpha, \beta>0$.

LEMMA 2.1. *Let* $(X, m, \mathcal{F}, \mathcal{E})$ *be a D-space and* $\{G_\alpha, \alpha>0\}$ *be its associated* L^2*-resolvent. Fix an* $\alpha_0>0$.

(i) *If S is a dense subset of* $L^2(X)$, *then, for any* $\alpha>0$, $G_\alpha(S)$ *is dense in* $\mathcal{F}$ *with respect to metric* $\mathcal{E}^{\alpha_0}$.

(ii) *For* $u, v \in \mathcal{F}$,

$$\mathcal{E}^\alpha(u, v) = \lim_{\beta\to+\infty} \beta(u-\beta G_{\beta+\alpha} u, v)_X. \tag{2.5}$$

(iii) *For any* $u \in \mathcal{F}$, $\lim_{\beta\to+\infty} \beta G_\beta u = u$ *strongly in norm* $\mathcal{E}^{\alpha_0}$ *and hence strongly in* $L^2(X)$ *sense.*

(iv) $\mathcal{F}$ *is a function lattice: if* $u, v \in \mathcal{F}$, *then* $u \vee v$, $u \wedge v \in \mathcal{F}$. *Further* $u \wedge 1 \in \mathcal{F}$ *for* $u \in \mathcal{F}$.

(v) *If u and v are both in $\mathscr{F}$ and m-essentially bounded, then the product $u \cdot v$ is also in $\mathscr{F}$.*

(vi) *For $u \in \mathscr{F}$, put $u_n = ((-n) \vee u) \wedge n$.*
Then $\lim_{n \to +\infty} u_n = u$ strongly in norm $\mathscr{E}^{\alpha_0}$.

Proof. (i) is a consequence of the equation (2.2).

(ii) is a consequence of Lemma 1 of [8].

(iii) For $\beta > \alpha_0$,

$$\begin{aligned} \mathscr{E}^{\alpha_0}(\beta G_\beta u - u, \beta G_\beta u - u) &\leqq \mathscr{E}^{\beta}(\beta G_\beta u - u, \beta G_\beta u - u) \\ &= \beta^2(\beta G_\beta u, u)_X - 2\beta(u, u)_X + \mathscr{E}^\beta(u, u) \\ &= -\beta(u - \beta G_\beta u, u)_X + \mathscr{E}(u, u) \to 0, \qquad \beta \to +\infty. \end{aligned}$$

(iv) Since $|u|$ and $u \wedge 1$ are normal contractions of u, they are in $\mathscr{F}$ if u is. Note that

$$u \vee v = \tfrac{1}{2}((u+v) + |u-v|), \qquad u \wedge v = \tfrac{1}{2}((u+v) - |u-v|).$$

(v) If $u \in \mathscr{F}$ and $|u| \leqq M$ m-a.e. for some constant M, then u^2 is a normal contraction of $2Mu$ and hence $u^2 \in \mathscr{F}$. Note that $u \cdot v = \frac{1}{4}((u+v)^2 - (u-v)^2)$.

(vi) By Lemma 2.1 of [6], $\mathscr{E}^{\alpha_0}(u_n, u_n)$ increases to $\mathscr{E}^{\alpha_0}(u, u)$ as n tends to infinity. On the other hand,

$$\mathscr{E}^{\alpha_0}(u_n, G_{\alpha_0} w) = (u_n, w)_X \xrightarrow[n \to +\infty]{} (u, w)_X = \mathscr{E}^{\alpha_0}(u, G_{\alpha_0} w)$$

for any $w \in L^2(X)$. These facts combined with the first statement of this lemma imply that u_n converges to u weakly and after all strongly with respect to the inner product $\mathscr{E}^{\alpha_0}$.

We will now give definitions and remarks concerning regularity of D-spaces. For a locally compact space X, denote by $C(X)$ (resp. $C_0(X)$) the space of all continuous functions vanishing at infinity (resp. with compact supports). $C^+(X)$ (resp. $C_0^+(X)$) will denote the set of all nonnegative elements of $C(X)$ (resp. $C_0(X)$). We say a measure m on X to be *everywhere dense* if $m(E)$ is not zero for any non-empty open set $E \subset X$.

DEFINITION 2.3. A D-space $(X, m, \mathscr{F}, \mathscr{E})$ is called *regular* if m is everywhere dense and $\mathscr{F} \cap C(X)$ is dense both in $\mathscr{F}$ with norm $\mathscr{E}^{\alpha_0}$ and in $C(X)$ with uniform norm. Here, $\alpha_0 > 0$ is arbitrarily fixed.

Next, consider (X, m) satisfying condition (D.1). For a sub-Markov resolvent kernel([2]) $\{G_\alpha(x, E), \alpha > 0\}$ on X, we set

$$G_\alpha u(x) = \int_X G_\alpha(x, dy) u(y), \qquad u \in C(X). \tag{2.6}$$

([2]) $G_\alpha(x, E)$ is called a kernel on X if, for a fixed $x \in E$, $G_\alpha(x, \cdot)$ is a Borel measure on X and, for a fixed Borel set $E \subset X$, $G_\alpha(\cdot, E)$ is a measurable function on X.

DEFINITION 2.4. (i) A sub-Markov resolvent kernel $\{G_\alpha(x, E), \alpha>0\}$ on X is called *m-symmetric* if

$$\int_X G_\alpha u(x)\cdot v(x)m(dx) = \int_X u(x)\cdot G_\alpha v(x)m(dx) \leqq +\infty$$

for any $u, v \in C^+(X)$. (ii) A sub-Markov resolvent kernel $\{G_\alpha(x, E), \alpha>0\}$ on X is called a *Ray resolvent* if it satisfies the following conditions.

(R.a) $G_\alpha(C(X)) \subset C(X)$ for any $\alpha>0$.

(R.b) There exists a countable subcollection C_1 of $C^+(X)$ such that (α) C_1 separates points of X, and, for any $x \in X$, there exists a $u \in C_1$ whose value at x is not zero, (β) for some $\alpha_0>0$, every function $u \in C_1$ satisfies the inequality $\beta G_{\alpha_0+\beta} u \leqq u$, $\beta>0$.

Consider any m-symmetric sub-Markov resolvent kernel $\{G_\alpha(x, E), \alpha>0\}$ on X. It satisfies the inequality $(\alpha G_\alpha u, \alpha G_\alpha u)_X \leqq (u, u)_X$ for all $u \in L^2(X; m) \cap C(X)$ [16]. Therefore it determines a unique L^2-resolvent. The Dirichlet space associated with this L^2-resolvent will be said *to be generated by the resolvent kernel* $\{G_\alpha(x, E), \alpha>0\}$.

We will say the set C_1 appearing in the definition of Ray resolvent *to be attached to the given Ray resolvent.*

DEFINITION 2.5. A D-space $(X, m, \mathscr{F}, \mathscr{E})$ is called *strongly regular* if m is everywhere dense on X, $(\mathscr{F}, \mathscr{E})$ is generated by an m-symmetric Ray resolvent on X and $\mathscr{F} \cap C(X)$ contains the set C_1 attached to this Ray resolvent.

REMARK 2.2. (i) A strongly regular D-space is regular. To see this, let $(X, m, \mathscr{F}, \mathscr{E})$ be a strongly regular D-space and $\{G_\alpha(x, E), \alpha>0\}$ be its associated Ray resolvent. $\mathscr{F} \cap C(X)$ contains $G_\alpha(L^2(X) \cap C(X))$, which is dense in $(\mathscr{F}, \mathscr{E}^{\alpha_0})$ by virtue of Lemma 2.1(i). Owing to the fifth statement of the lemma, $\mathscr{F} \cap C(X)$ is a function algebra. Since it contains the set C_1 attached to $\{G_\alpha, \alpha>0\}$, it is dense in $C(X)$ by Stone-Weierstrass theorem.

(ii) Consider a Ray resolvent $\{G_\alpha(x, E), \alpha>0\}$ on a locally compact Hausdorff separable space X. Let $\bar{X}=X \cup \{\infty\}$ be the one point compactification of X if X is not compact. If X is compact, let $\{\infty\}$ be an isolated point. Define a new kernel $\{\bar{G}_\alpha(x, E), \alpha>0\}$ on $\bar{X}$ by $\bar{G}_\alpha(x, E)=G_\alpha(x, E \cap X)+((1-\alpha G_\alpha(x, X))/\alpha)\delta_{\{\infty\}}(E)$, $x \in X$, $\bar{G}_\alpha(\{\infty\}, E)=(1/\alpha)\delta_{\{\infty\}}(E)$. Then $\{\bar{G}_\alpha, \alpha>0\}$ is a conservative Ray resolvent on the compactum $\bar{X}$. By Ray's theory [15], [12], this defines on $\bar{X}$ a right continuous conservative strong Markov process for which the point $\{\infty\}$ is a trap. Thus, we obtain a right continuous strong Markov process $(X_t, \zeta, P_x, x \in X)$ on X such that

$$G_\alpha(x, E) = E_x\left(\int_0^\zeta e^{-\alpha t}\chi_E(X_t)\, dt\right), \tag{2.7}$$

χ_E being the indicator function of the Borel set E. We will call the process on X so obtained the *Ray process* associated with the Ray resolvent $\{G_\alpha, \alpha>0\}$ on X.

There is a Ray process on the underlying space of any strongly regular D-space.

3. **Examples.** Denote by D a domain of Euclidean N-space R^N ($N \geqq 1$).

EXAMPLE 1. Let us put

$$\mathscr{E}^1_{L^2}(D) = \{u; u \in L^2(D), \partial u/\partial x_i \in L^2(D), i = 1, 2, \ldots, N\},$$

$$(u, v)_{D,1} = \frac{1}{2} \int_D' \sum_{i=1}^{N} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx.$$

Here, derivatives are taken in Schwartz distribution sense and dx denotes the Lebesgue measure on R^N.

$(D, dx, \mathscr{E}^1_{L^2}(D), (\ ,\ )_{D,1})$ is a D-space in our sense. Condition (D.3)′ for this space can be verified easily (see Proposition A.1 of [16] or Théorème 3.1 of [3]). This space is not regular except when it coincides with $\mathscr{D}^1_{L^2}(D)$ of the next example. Denote by $\bar{D}$ the closure of D in R^N. Let $C^\infty(\bar{D})$ be the space of restrictions to $\bar{D}$ of functions which are infinitely differentiable on R^N. If $\partial D = \bar{D} - D$ is a closed hypersurface of class C^1, then $\mathscr{E}^1_{L^2}(D) \cap C^\infty(\bar{D})$ is dense in $\mathscr{E}^1_{L^2}(D)$ [14]. Therefore, in this case, $(\bar{D}, dx, \mathscr{E}^1_{L^2}(D), (\ ,\ )_{D,1})$(3) is a regular D-space. When D is bounded, the space $(\mathscr{E}^1_{L^2}(D), (\ ,\ )_{D,1})$ is generated by the continuous resolvent density constructed in [5] and in §8(I) of [6].

EXAMPLE 2. Denote by $C_0^\infty(D)$ the space of infinitely differentiable functions on D with compact supports. Let $\mathscr{D}^1_{L^2}(D)$ be the closure of $C_0^\infty(D)$ in

$$(\mathscr{E}^1_{L^2}(D), (\ ,\ )_{D,1} + (\ ,\ )_D).$$

$(D, dx, \mathscr{D}^1_{L^2}(D), (\ ,\ )_{D,1})$ is a regular D-space. Since $\mathscr{D}^1_{L^2}(D)$ coincides with the completion of $\mathscr{E}^1_{L^2}(D) \cap C_0(D)$ with respect to metric $(\ ,\ )_{D,1} + (\ ,\ )_D$, we can apply Corollary 3 of Appendix to show that it is a regular D-space. It is generated by a continuous resolvent density of the absorbing barrier Brownian motion on D [6]. It is strongly regular when each point of the boundary ∂D is regular with respect to the Dirichlet problem for D.

EXAMPLE 3. Let M be the Martin boundary of the domain D and μ be the harmonic measure on M with respect to a reference point x_0 of D. J. L. Doob [4] introduced the space $\boldsymbol{H}_h'$ of measurable functions φ on M for which the integral

$$\boldsymbol{D}_M(\varphi, \varphi) = \frac{q}{4} \int_M \int_M (\varphi(\xi) - \varphi(\eta))^2 \theta(\xi, \eta) \mu(d\xi) \mu(d\eta)$$

is finite. Here, $\theta(\xi, \eta)$ is Naim's kernel on M and q is 2π if $N=2$ or the product of $N-2$ and the unit ball boundary area if $N>2$. It was proved in [4] that $\boldsymbol{H}_h' \subset L^2(M; \mu)$. We can easily see that $(M, \mu, \boldsymbol{H}_h', \boldsymbol{D}_M(\ ,\ ))$ is a D-space. This is regular when D is a disk (§18 of [4]). Let $\boldsymbol{H}_h$ be the space of all harmonic functions on D with finite integrals $(u, u)_{D,1}$. Then, $(\boldsymbol{H}_h', \boldsymbol{D}_M(\ ,\ ))$ is the trace on M of the space $(\boldsymbol{H}_h, (\ ,\ )_{D,1})$ in the following sense: each function u of $\boldsymbol{H}_h$ has a fine boundary

(3) We regard here $\mathscr{E}^1_{L^2}(D)$ as a subspace of $L^2(\bar{D})$ $(=L^2(D))$. See Remark 5.2.

limit function γu in H'_h and γ define a unitary map from $(H_h, (\ ,\ )_{D,1})$ onto (H'_h, D_M). This is the reason why functions of H'_h were called in [4] BLD boundary functions.

A modification of the space (H'_h, D_M) was introduced in [6] in order to describe the space of all α-harmonic functions of $\mathscr{E}^{1,2}_L(D)$. Suppose that D is bounded. Let $U_\alpha(\xi, \eta)$ and $U(\xi, \eta)$ be Feller kernels on M. $U(\xi, \eta)$ is equal to $(q/2)\cdot\theta(\xi, \eta)$ μ-a.e. Denote by μ' the measure $U_1 1\cdot\mu$ on M and put $H_M = H'_h \cap L^2(M; \mu')$. Then, $(M\ \mu'\ H_M, D_M)$ is a D-space. By virtue of Lemma 3.1 and equality (3.21) of [6], it is clear that $(M, \mu', H_M, D^{[\alpha]}_M)$ is also a D-space for each $\alpha > 0$, where

$$D^{[\alpha]}_M(\varphi, \psi) = D_M(\varphi, \psi) + \int_M \int_M \varphi(\xi) U_\alpha(\xi, \eta)\psi(\eta)\mu(d\xi)\mu(d\eta).$$

Let $\mathscr{H}_\alpha$ be the orthogonal complement of $\mathscr{D}^{1,2}_L(D)$ in the Hilbert space

$$(\mathscr{E}^{1,2}_L(D), (\ ,\ )_{D,1} + \alpha(\ ,\ )_D).$$

The space $(H_M, D^{[\alpha]}_M)$ is nothing but the trace on M of the space

$$(\mathscr{H}_\alpha, (\ ,\ )_{D,1} + \alpha(\ ,\ )_D)\ (^4).$$

EXAMPLE 4(5). Assume that D is bounded. Let $\Delta = \bigcup_{p \in P} E_p$ be a measurable partition of the Martin boundary M. Then, Δ defines a Dirichlet subspace $(\mathscr{F}^\Delta_M, D_M)$ of (H_M, D_M) by $\mathscr{F}^\Delta_M = \{\varphi \in H_M$; there exists a set E_φ such that $\mu'(E_\varphi) = 0$ and φ is a constant on $E_p - E_\varphi$ for each $p \in P\}$. Even when D is a unit disk, $(M, \mu', \mathscr{F}^\Delta_M, D_M)$ is no longer regular except for a trivial case that $\mathscr{F}^\Delta_M$ is equal to H_M.

EXAMPLE 5. Consider the whole plane R^2 and put

$$\mathscr{E}(u, v) = \frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)dx\,dy + \frac{1}{2}\int_{-\infty}^{\infty}\frac{\partial u(x, 0)}{\partial x}\frac{\partial v(x, 0)}{\partial x}dx,$$

$$\mathscr{A} = \{u \in C_0(R^2);\ u(x, y) \text{ is absolutely continuous in each variable } x \text{ and } y \text{ and } \mathscr{E}(u, u) < +\infty\}.$$

For this space $(\mathscr{A}, \mathscr{E})$, let us check the conditions of Theorem of Appendix. ($\mathscr{A}$.1) and ($\mathscr{A}$.2) are evident. To see ($\mathscr{A}$.3), assume that a sequence $u_n \in \mathscr{A}$ satisfies $(u_n, u_n)_{R^2} \to 0$ and $\mathscr{E}(u_n - u_m, u_n - u_m) \to 0$. We have to prove $\mathscr{E}(u_n, u_n) \to 0$. Since u_n converges to zero in $\mathscr{D}^{1,2}_L(R^2)$ with metric $(\ ,\ )_{R^2,1} + (\ ,\ )_{R^2}$, we can select a subsequence u_{n_k} such that $u_{n_k}(x, y)$ converges to zero for every (x, y) except on a 2-dimensional Brownian polar set(6) [3]. Especially, $u_{n_k}(x, 0)$ converges to zero for every x except on a set of linear Lebesgue measure zero.

Now it is easy to see that $\int_{-\infty}^{\infty} (\partial u_n(x, 0)/\partial x)^2\, dx \to 0, n \to \infty$. Hence $\mathscr{E}(u_n, u_n) \to 0$ as was to be proved.

(4) Theorem 3.4 of [6].

(5) See footnote 27 of [6].

(6) The subsequent paper [9] will provide general discussions of this point.

By means of Theorem of Appendix we get a D-space $(\mathcal{F}, \mathcal{E})$ on R^2, $(\mathcal{F}, \mathcal{E}^\alpha)$ being the completion of $(\mathcal{A}, \mathcal{E}^\alpha)$ for each $\alpha>0$. This D-space is regular because $C_0^\infty(R^2)\subset\mathcal{A}$ (Corollary 1 of Appendix).

4. **Equivalence of *D*-spaces.** Consider a D-space $(X, m, \mathcal{F}, \mathcal{E})$. For $u\in L^\infty(X)$ $(=L^\infty(X; m))$, put $\|u\|_\infty = m\text{-ess sup}_{x\in X}|u(x)|$. Let L be a closed subalgebra of $(L^\infty(X), \|\ \|_\infty)$. It is well known that L is then a function lattice and that $u\in L$ implies $u\wedge 1\in L$. Therefore, by making use of Lemma 2.1(iv) and (v), we get the next lemma.

LEMMA 4.1. *$\mathcal{F}\cap L$ is a function algebra and a function lattice. Further, $u\in\mathcal{F}\cap L$ implies $u\wedge 1\in\mathcal{F}\cap L$.*

Now we are in a position to define an equivalence relation in the set of all D-spaces.

DEFINITION 4.1. Two D-spaces $(X, m, \mathcal{F}, \mathcal{E})$ and $(X', m', \mathcal{F}', \mathcal{E}')$ are called *equivalent* if there is an algebraic isomorphism Φ from $\mathcal{F}\cap L^\infty(X)$ onto $\mathcal{F}'\cap L^\infty(X')$ and Φ preserves three kinds of metrics: $\|u\|_\infty=\|\Phi u\|'_\infty$, $\mathcal{E}(u, u)=\mathcal{E}'(\Phi u, \Phi u)$ and $(u, u)_X=(\Phi u, \Phi u)_{X'}$ for $u\in\mathcal{F}\cap L^\infty(X)$.

This definition of equivalence is the same as that of [8] where the definition is given in terms of the associated D-rings.

It is not difficult to see that the mapping Φ of Definition 4.1 turns out to be a lattice isomorphism and further Φ can be extended to a unitary map Φ_1 from $(\mathcal{F}, \mathcal{E})$ onto $(\mathcal{F}', \mathcal{E}')$ and a unitary map Φ_2 from $L_0^2(X)$ onto $L_0^2(X')$. Here, $L_0^2(X)$ (resp. $L_0^2(X')$) is the closure of $\mathcal{F}$ (resp. $\mathcal{F}'$) in the metric space $L^2(X)$ (resp. $L^2(X')$). We can use Lemma 2.1(vi) to define the extension Φ_1. The L^2-resolvents $\{G_\alpha, \alpha>0\}$ associated with equivalent D-spaces are mutually related by $G'_\alpha u'=\Phi_2 G_\alpha \Phi_2^{-1}u'$, $u'\in L_0^2(X')$, $\alpha>0$. This relation is proved in [8].

Before proceeding to the next sections, we will summarize here some facts related to Gelfand representations of subalgebras of L^∞. Let (X, m) be as above and L be a closed subalgebra of the real Banach algebra $(L^\infty(X; m), \|\ \|_\infty)$. A nonzero algebraic homomorphism χ from L into real numbers is called a (real) *character* on L. Denote by $\mathcal{M}$ the set of all characters on L. An algebraic homomorph Φ from L into real functions on $\mathcal{M}$ can be defined by

$$(4.1)\qquad \Phi u(\chi) = \chi(u), \qquad u\in L, \quad \chi\in\mathcal{M}.$$

We define a neighborhood of $\chi\in\mathcal{M}$ by

$$(4.2)\quad N(\chi; u_1, u_2, \dots, u_n; \varepsilon) = \{\chi'\in\mathcal{M}; |\Phi u_k(\chi')-\Phi u_k(\chi)| < \varepsilon, k = 1, 2, \dots, n\}$$

with any $\varepsilon>0$ and $u_1, u_2, \dots, u_n\in L$. The set $\mathcal{M}$ endowed with topology (4.2) will be called the *character space* of L.

LEMMA 4.2. (i) *The character space $\mathcal{M}$ of L is a locally compact Hausdorff space. If the algebra L is countably generated, then $\mathcal{M}$ is separable. $\mathcal{M}$ is compact if and only if $1\in L$.*

(ii) *The map* Φ *of* (4.1) *is an algebraic isomorphism and isometry from* $(L, \| \ \|_\infty)$ *onto* $C(\mathcal{M})$, $C(\mathcal{M})$ *being associated with the uniform norm.*

(iii) *Suppose that* m *is everywhere dense* $L \subset C_b(X)$ *(the space of continuous bounded funtions on* X*) and, for any* $x \in X$, *there is a* $u \in L$ *with* $u(x) \neq 0$. *There exists then a continuous mapping* q *from* X *onto a dense subset of* $\mathcal{M}$ *characterized by*

$$\Phi u(qx) = u(x), \qquad x \in X, \quad u \in L. \tag{4.3}$$

Proof. Consider the space $A = L + (-1)^{1/2}L$ with uniform norm $\| \ \|_\infty$. This is a complex Banach algebra closed under the operation of taking complex conjugate function. If $u \in A$, then

$$\frac{|u|^2}{1+|u|^2} = \frac{1}{1+a^2}\sum_{k=0}^{\infty} |u|^2\left(\frac{a^2-|u|^2}{1+a^2}\right)^k \in L,$$

where $a = \|u\|_\infty$. Therefore, A is a symmetric algebra and the character space $\mathcal{M}$ of L can be identified with the space of regular maximal ideals of A (Loomis [13, subsections 23A and 26C]). Now statements (i) and (ii) of our lemma are known facts. The statement (iii) is evident but we give its proof here for later conveniences. Fix an $x \in X$. A map $u \to u(x)$ is clearly a character on L which we denote by qx. q is continuous at $x \in X$ because any neighborhood $N(\chi; u_1, u_2, \ldots, u_n; \varepsilon)$ of $\chi = qx$ includes the set $q(U(x))$, where $U(x)$ is an open neighborhood of x defined by $U(x) = \{x' \in X; |u_k(x') - u_k(x)| < \varepsilon, k = 1, 2, \ldots, n\}$. Suppose that $q(X)$ is not dense in $\mathcal{M}$. There is then a nonvanishing $v \in C(\mathcal{M})$ such that $v = 0$ on $q(X)$. By (ii) and (4.3), we have $\|v\|_\infty = \|\Phi^{-1}v\|_\infty = \sup_{x\in X} |\Phi^{-1}v(x)| = \sup_{x \in X} |v(qx)| = 0$, which is a contradiction.

Finally we will state the following lemma according to 26J of [13].

LEMMA 4.3. *Suppose that* $\tilde{L}$ *is a dense ideal of* L *and every function in* $\tilde{L}$ *can be expressed as a difference of nonnegative functions in* $\tilde{L}$. *Then, for any positive linear functional* l *on* $\tilde{L}$, *there exists a unique Radon measure* μ *on* $\mathcal{M}$ *such that*

$$\begin{aligned} &\Phi(\tilde{L}) \subset L^1(\mathcal{M}; \mu), \\ &l(u \cdot v) = \int_{\mathcal{M}} \Phi u(\chi)\Phi v(\chi)\mu(d\chi), \qquad u \in \tilde{L}, \quad v \in L. \end{aligned} \tag{4.4}$$

5. **Regular representations.** Suppose that we are given a D-space $(X, m, \mathcal{F}, \mathcal{E})$. A closed subalgebra L of $L^\infty(X; m)$ will be said *to satisfy condition* (C) if it enjoys the following three properties.

(C.1) L is a countably generated closed subalgebra of $L^\infty(X; m)$.

(C.2) $\mathcal{F} \cap L$ is dense both in $(\mathcal{F}, \mathcal{E}^{\alpha_0})$ and in $(L, \| \ \|_\infty)$, α_0 being a fixed positive number.

(C.3) $L^1(X; m) \cap L$ is dense in $(L, \| \ \|_\infty)$.

THEOREM 2. (i) *There exists at least one* L *satisfying the condition* (C). (ii) *Let an* L *satisfying condition* (C) *be fixed and* X' *be its character space.* X' *is compact if* *and only if* $1 \in L$. *There exists a regular D-space whose underlying space is* X' *and which is equivalent to the given D-space.*

The regular D-space of Theorem 2(ii) will be called a *regular representation of the given D-space with respect to the algebra L.*

Proof of Theorem 2(i). We can find a countable subset D_0 of $C_0(X)$ such that each function in $C_0(X)$ can be uniformly approximated by a sequence of functions in D_0 whose supports are included in a suitable common compactum. D_0 is dense in $L^2(X; m)$. Let $\{G_\alpha, \alpha>0\}$ be the L^2-resolvent associated with the given $(\mathscr{F}, \mathscr{E})$. Then, $G_{\alpha_0}(D_0) \subset \mathscr{F} \cap L^\infty(X; m)$ and $G_{\alpha_0}(D_0)$ is dense in $(\mathscr{F}, \mathscr{E}^{\alpha_0})$ by Lemma 2.1(i). We define L as the closed subalgebra of $L^\infty(X; m)$ generated by $G_{\alpha_0}(D_0)$. It is clear that this L satisfies conditions (C.1) and (C.2). As for (C.3), observe that

$$G_{\alpha_0}(D_0) \subset L^1(X; m) \cap L$$

since

$$\begin{aligned}\int_X |G_{\alpha_0}u|\, dm &\leqq \int_X G_{\alpha_0}|u|\, dm = \sup_{0\leqq v\leqq 1,\, v\in C_0(X)} (v, G_{\alpha_0}|u|)_X \\ &\leqq \frac{1}{\alpha_0}\int_X |u|\, dm < +\infty, \qquad u \in D_0.\end{aligned}$$

Proof of Theorem 2(ii). Let L be a space satisfying condition (C) and X' be its character space. By (C.1) and Lemma 4.2(i), X' is a locally compact Hausdorff and separable space. X' is compact if and only if $1 \in L$. The map Φ of (4.1) is giving an algebraic isomorphism and isometry from L onto $C(X')$. Φ is consequently a lattice isomorph and it holds that $\Phi(u\wedge 1) = (\Phi u)\wedge 1$ for $u \in L$. Let us put

$$\mathscr{R} = \mathscr{F} \cap L, \qquad \mathscr{R}' = \Phi(\mathscr{R}). \tag{5.1}$$

Since $\mathscr{R}$ is dense in L by (C.2), $\mathscr{R}'$ is dense in $C(X')$. Further, by Lemma 4.1, $\mathscr{R}'$ is a lattice and $u'\wedge 1 \in \mathscr{R}'$ whenever $u' \in \mathscr{R}'$.

Keeping these in mind, we are now to construct, step by step, a regular representation $(X', m', \mathscr{F}', \mathscr{E}')$ by making use of the map Φ of (4.1).

(I) *A measure* m' *on* X'. There exists a unique Radon measure m' on X' which satisfies

$$\Phi(L^1(X; m) \cap L) \subset L^1(X'; m'),$$

(5.2)
$$\int_X u(x)v(x)m(dx) = \int_{X'} \Phi u(x')\Phi v(x')m'(dx'), \qquad u \in L^1(X; m) \cap L, \quad v \in L.$$

In fact, by virtue of (C.3), we can apply Lemma 4.3 to a dense ideal $\tilde{L} = L^1(X; m) \cap L$ and a positive linear functional

$$l(u) = \int_X u(x)m(dx), \qquad u \in \tilde{L}.$$

Consider the spaces $\mathscr{R}$ and $\mathscr{R}'$ of (5.1). Since condition (C.2) implies that $\mathscr{R}$ is dense in $\mathscr{F}$ in L^2-sense, we have

$$\mathscr{R} \subset L^2(X; m), \qquad \bar{\mathscr{R}} = L_0^2(X; m), \tag{5.3}$$

where the closure is taken in L^2-sense and $L_0^2(X; m)$ denotes $\bar{\mathscr{F}}$. Next we will prove

$$\mathscr{R}' \subset L^2(X'; m'), \qquad \bar{\mathscr{R}}' = L^2(X'; m'). \tag{5.4}$$

For any $u \in \mathscr{R}$, $(\Phi u)^2 = \Phi(u^2) \in \Phi(L \cap L^1(X; m))$ and hence $\Phi u \in L^2(X'; m')$ according to (5.2). In order to show that $\mathscr{R}'$ is dense in $L^2(X'; m')$, take a function u in $C_0^+(X')$. Since $\mathscr{R}'$ is uniformly dense in $C(X')$ and is a lattice, we can find a $v \in \mathscr{R}'$ and $u_n \in \mathscr{R}'$ such that $0 \leqq u_n \leqq v$ and u_n converges to u uniformly on X'. Hence, u_n converges to u in $L^2(X'; m')$.

Finally let us show

$$\int_X u(x)v(x)m(dx) = \int_X u'(x')v'(x')m'(dx'), \qquad u, v \in \mathscr{R}, \tag{5.5}$$

where $u' = \Phi u$ and $v' = \Phi v$. Take a nonnegative $v \in \mathscr{R}$. By condition (C.3) and the obvious fact that $L^1(X; m) \cap L$ is a lattice, we can select $v_n \in L^1(X; m) \cap L$ such as $0 \leqq v_n \leqq v$ m-a.e. and $\|v_n - v\|_\infty \to 0$. Since Φ is a lattice isomorph and preserves the uniform norm, the same relations hold for v_n' and v'. Now (5.2) for $u = v = v_n$ leads us to

$$\int_X v(x)^2 m(dx) = \int_{X'} v'(x')^2 m'(dx')$$

which implies (5.5) because each element of $\mathscr{R}$ is expressed as a difference of non-negative elements of $\mathscr{R}$ and Φ is an algebraic isomorphism.

(II) *Extended map Φ on $L_0^2(X; m)$.* In view of (5.3), (5.4), and (5.5) of the preceding paragraph, the algebraic and lattice isomorphism Φ from $\mathscr{R}$ to $\mathscr{R}'$ can be uniquely extended to

(Φ.1) A unitary map Φ from $L_0^2(X; m)$ onto $L^2(X'; m')$.

Let us study the features of this extended map Φ. It has the following properties.

(Φ.2) $L_0^2(X; m)$ is a lattice and Φ is a lattice isomorphism. $\Phi(u \wedge 1) = (\Phi u) \wedge 1$ whenever $u \in L_0^2(X; m)$.

(Φ.3) Φ is an algebraic isomorphism from $L_0^2(X; m) \cap L^\infty(X; m)$ onto

$$L^2(X'; m') \cap L^\infty(X'; m').$$

Further it holds that

$$\|u\|_\infty = \|\Phi u\|'_\infty, \qquad u \in L_0^2(X; m) \cap L^\infty(X; m). \tag{5.6}$$

To prove (Φ.2), take a $u \in L_0^2(X; m)$ and find a sequence $u_n \in \mathscr{R}$ which converges to u in L^2-sense. Since $|u_n| \in \mathscr{R}$ converges to $|u|$ in L^2-sense, $|u| \in L_0^2(X; m)$. Since Φ is a lattice isomorph on $\mathscr{R}$ and preserves L^2-norm, we have $\Phi|u| = \text{l.i.m. } \Phi|u_n| = \text{l.i.m. } |\Phi u_n| = |\Phi u|$. Thus we have proved the first half of (Φ.2). The latter half is similarly proved.

The property (Φ.3) follows from (Φ.2). In fact, for $u \in L_0^2(X; m)$ with $\|u\|_\infty = a < +\infty$, we have $|\Phi u| = \Phi(|u|) = \Phi(|u| \wedge a) = |\Phi u| \wedge a$ which means $\|\Phi u\|'_\infty \leqq \|u\|_\infty$. In the same way, we have $\|u'\|_\infty \geqq \|\Phi^{-1}u'\|_\infty$ for $u' \in L^2(X'; m') \cap L^\infty(X'; m')$. To see that Φ is an algebraic isomorphism, take a $u \in L_0^2(X; m) \cap L^\infty(X; m)$ and a sequence $u_n \in \mathscr{R}$ which converges to u in L^2-sense. We may assume that $|u_n| \leqq \|u\|_\infty$. Then u_n^2 (resp. $(\Phi u_n)^2$) converges to u^2 (resp. $(\Phi u)^2$) in L^2-sense. Since Φ is an algebraic isomorph on $\mathscr{R}$, $\Phi(u^2) = \text{l.i.m.}\ \Phi(u_n^2) = (\Phi u)^2$.

(III) *Induced D-space* $(X', m', \mathscr{F}', \mathscr{E}')$. By means of the preceding map Φ on $L_0^2(X; m) \supset \mathscr{F}$, we define

$$\mathscr{F}' = \Phi(\mathscr{F}),$$
$$\mathscr{E}'(u', v') = \mathscr{E}(\Phi^{-1}u', \Phi^{-1}v'), \qquad u', v' \in \mathscr{F}'. \tag{5.7}$$

Then, $(X', m', \mathscr{F}', \mathscr{E}')$ is a D-space.

Condition (D.1) for (X', m') has already been proved and (D.2) for $(\mathscr{F}', \mathscr{E}')$ is obvious by the property (Φ.1) of Φ. Instead of proving (D.3), let us check an equivalent condition (D.3)$'$ in Remark 2.1. Take $u' \in \mathscr{F}'$ and put $v' = (0 \vee u') \wedge 1$, $u = \Phi^{-1}u'$. Then we have $v' = 0 \vee \Phi u \wedge 1 = \Phi(0 \vee u \wedge 1)$ by (Φ.2). Since $v = 0 \vee u \wedge 1 \in \mathscr{F}$ and $\mathscr{E}(v, v) \leqq \mathscr{E}(u, u)$, $v' \in \mathscr{F}'$ and $\mathscr{E}'(v', v') \leqq \mathscr{E}'(u', u')$ proving (D.3)$'$.

(IV) $(X', m', \mathscr{F}', \mathscr{E}')$ *is equivalent to* $(X, m, \mathscr{F}, \mathscr{E})$. This is evident from ($\Phi$.1), ($\Phi$.3) and (5.7).

(V) $(X', m', \mathscr{F}', \mathscr{E}')$ *is regular.* Φ preserves $\mathscr{E}^{\alpha_0}$-norm and the uniform norm on $\mathscr{R} = \mathscr{F} \cap L$. Hence by virtue of condition (C.2), $\mathscr{R}' = \Phi(\mathscr{R})$ is dense both in $\mathscr{F}'$ and in $C(X')$. Since $\mathscr{R}$ is the intersection of $\mathscr{F}$ and the uniform closure of $\mathscr{R}$, the same relation holds for $\mathscr{R}'$ and $\mathscr{F}'$. Therefore

$$\mathscr{R}' = \mathscr{F}' \cap C(X'). \tag{5.8}$$

On the other hand we have by (5.6),

$$\sup_{x' \in X'} |u'(x')| = m'\text{-ess}\sup_{x' \in X'} |u'(x')|, \qquad u' \in \mathscr{F}' \cap C(X'). \tag{5.9}$$

Since $\mathscr{F}' \cap C(X')$ is dense in $C(X')$, (5.9) means that m' is everywhere dense on X'. The proof of (V) is complete.

The proof of Theorem 2 has ended.

The next remarks and lemma will state the meaning of Theorem 2 for special cases.

REMARK 5.1. Suppose that the given D-space $(X, m, \mathscr{F}, \mathscr{E})$ is regular. Since m is everywhere dense, $C(X)$ may be considered as a closed subalgebra of $L^\infty(X; m)$. Obviously $C(X)$ satisfies conditions (C.1) and (C.2). It also satisfies (C.3) because of $L^1(X; m) \cap C(X) \supset C_0(X)$. Therefore, we may consider the regular representation with respect to $C(X)$. However, as is well known, the character space of $C(X)$ coincides with X itself, and after all the regular representation goes back to the given regular D-space without any change.

LEMMA 5.1. *Suppose that m is everywhere dense. Suppose further that an algebra L satisfies not only conditions* (C.1), (C.2) *and* (C.3) *but also the following.*

(C.4) *$L \subset C_b(X)$, L separates points of X and, at any $x \in X$, there is a $u \in L$ such that $u(x) \neq 0$.*

Let $(X', m', \mathscr{F}', \mathscr{E}')$ be the regular representation with respect to this L. Then,

(i) *X is continuously embedded onto a dense subset of X'. By this embedding, any Borel set of X goes to a Borel set of X' and the restriction to X of any Borel set of X' is a Borel set of X (with respect to the original topology).*

(ii) *For any Borel subset A of X', $m'(A) = m(A \cap X)$. Therefore, the space $(L^2(X'; m'), (\ ,\ )_{X'})$ is identified with the space $(L^2(X; m), (\ ,\ )_X)$.*

(iii) *By the above identification, $(\mathscr{F}', \mathscr{E}')$ is equal to $(\mathscr{F}, \mathscr{E})$.*

Proof. By virtue of (C.4), the map q of (4.3) from X onto a dense subset of X' is not only continuous but also one-to-one. The rest of the lemma is obvious.

REMARK 5.2. Consider the situation of Example 1 of §3. If ∂D is of class C^1, then the space $L = \{u \in C_b(D);\ u$ is continuously extendable to $\bar{D}\}$ satisfies conditions (C.1)~(C.4). $\{\bar{D}, dx, \mathscr{E}^{1}_{L^2}, (\ ,\ )_{D,1}\}$ is just the regular representation of $\{D, dx, \mathscr{E}^{1}_{L^2}, (\ ,\ )_{D,1}\}$ with respect to this L. In this case, D is homeomorphically embedded into $\bar{D}$. Coming back to the general case of Lemma 5.1, X is homeomorphically embedded onto a dense subset of X' if and only if for any $x_0 \in X$ and $Y \subset X$ such as $x_0 \notin \bar{Y}$, there exists $u \in L$ such that $u(x_0) = 1$ and $u(x) = 0$ on Y.

6. **Strongly regular representations.** Suppose that we are given a D-space $(X, m, \mathscr{F}, \mathscr{E})$. Denote by $\{G_\alpha, \alpha > 0\}$ its associated L^2-resolvent.

LEMMA 6.1. (i) *G_α makes the space $L^2(X; m) \cap L^\infty(X; m)$ invariant and*

$$\|G_\alpha u\|_\infty \leqq \frac{1}{\alpha}\|u\|_\infty, \qquad u \in L^2 \cap L^\infty. \tag{6.1}$$

(ii) *G_α makes the space $L^\infty(X; m) \cap L^1(X; m)$ $(\subset L^2(X; m))$ invariant and*

$$\int_X |G_\alpha u(x)| m(dx) \leqq \frac{1}{\alpha}\int_X |u(x)| m(dx), \qquad u \in L^\infty \cap L^1. \tag{6.2}$$

Inequality (6.2) for $u \in C_0(X)$ has already been proved in the proof of Theorem 2(i). The proof for $u \in L^\infty \cap L^1$ is the same. The rest of Lemma 6.1 is clear.

Owing to Lemma 6.1(i), G_α on $L^2 \cap L^\infty$ can be uniquely extended to a linear operator $\bar{G}_\alpha$ on $L_0^\infty(X; m)$ (the closure of $L^2 \cap L^\infty$ in L^∞). $\{\bar{G}_\alpha, \alpha > 0\}$ is a *sub-Markov resolvent on L_0^∞*, that is,

($\bar{\text{G}}$.1) If $u \in L_0^\infty$ and $0 \leqq u \leqq 1$ m-a.e. then $0 \leqq \alpha\bar{G}_\alpha u \leqq 1$ m-a.e.

$$\bar{G}_\alpha - \bar{G}_\beta + (\alpha - \beta)\bar{G}_\alpha\bar{G}_\beta = 0, \qquad \alpha, \beta > 0. \tag{$\bar{\text{G}}$.2}$$

A closed subalgebra L of $L_0^\infty(X; m)$ is said *to satisfy condition* (R) if it enjoys the following two properties.

(R.1) $\bar{G}_\alpha(L) \subset L$ for every $\alpha > 0$.

(R.2) L is generated by a countable subset L_0 of $\mathscr{F} \cap L$ such that each $u \in L_0$ is nonnegative and satisfies $\alpha \bar{G}_{\alpha+\alpha_0} u \leqq u$, m-a.e., $\alpha > 0$.

THEOREM 3. (i) *There exists an L satisfying condition* (R) *as well as* (C). (ii) *Fix an L which satisfies* (C) *and* (R). *The regular representation of the given D-space with respect to this L turns out to be strongly regular.*

We need the next lemma for the proof of Theorem 3(i).

LEMMA 6.2. *Let S_0 be a set of countable nonnegative functions in $\mathscr{F} \cap L^\infty \cap L^1$. Then, there exists a set S possessing the following features.*

(S.1) *$S \supset S_0$ and S is a countably generated subalgebra of $\mathscr{F} \cap L^\infty \cap L^1$. Each function of S is expressed as a difference of nonnegative functions of S.*

(S.2) *For any $\alpha > 0$, $\bar{G}_\alpha$ makes the space $\bar{S}$ invariant, $\bar{S}$ being the closure of S in L^∞.*

Proof. According to F. Knight [11, Lemma 1], we construct S as follows. Starting with S_0, assume $S_1, \ldots, S_n$ are defined. Define S_{n+1} as an algebra generated by $\{S_n, G_{\alpha_1}(S_n), \ldots, G_{\alpha_n}(S_n), G_{\alpha_{n+1}}(S_n)\}$, where $\{a_k\}$ is the set of all positive rational numbers. Put $S = \bigcup_{n=0}^\infty S_n$, which satisfies condition (S.1) by virtue of Lemma 6.1 and of the fact that $\mathscr{F} \cap L^\infty \cap L^1$ is an algebra (Lemma 4.1). It is easy to see that condition (S.2) is met.

Proof of Theorem 3(i). Let D_0^+ be a countable subset of $C_0^+(X)$ such that the set $D_0 = \{u = u_1 - u_2;\, u_i \in D_0^+,\, i = 1, 2\}$ has the property in the proof of Theorem 2(i). Put $S_0 = G_{\alpha_0}(D_0^+)$, which satisfies the following.

(S_0.1) S_0 is a countable set of nonnegative functions in $\mathscr{F} \cap L^\infty \cap L^1$.

(S_0.2) The set $\{u = u_1 - u_2;\, u_i \in S_0,\, i = 1, 2\}$ is dense in $(\mathscr{F}, \mathscr{E}^{\alpha_0})$.

(S_0.3) $\alpha G_{\alpha+\alpha_0} u \leqq u$ m-a.e. for $u \in S_0$ and $\alpha > 0$.

For such an S_0, let S be a set which satisfies conditions (S.1) and (S.2) of Lemma 6.2. By (S.1), there exists a set $\tilde{S}$ of countable nonnegative functions in S whose linearization is just S. Let us put

$$L_0 = S_0 \cup G_{\alpha_0}(\tilde{S}), \tag{6.3}$$

$$L = \text{the closed subalgebra of } L^\infty \text{ generated by } L_0, \tag{6.4}$$

then the space L meets both conditions (C) and (R).

In order to check condition (C) of §5, denote by $\mathscr{R}_0$ the algebra generated by L_0. By (S_0.1), (S.1) and Lemma 6.1, L_0 and hence $\mathscr{R}_0$ are included in $\mathscr{F} \cap L^\infty \cap L^1$. Notice that $\mathscr{R}_0 \subset \mathscr{F} \cap L$ and that L is the closure of $\mathscr{R}_0$ in L^∞. Therefore both $\mathscr{F} \cap L$ and $L^1(X; m) \cap L$ are dense in L. Since $\mathscr{R}_0$ contains the set of (S_0.2), $\mathscr{F} \cap L$ is dense in $(\mathscr{F}, \mathscr{E}^{\alpha_0})$.

Coming to condition (R), it is clear that condition (R.2) is satisfied by L_0 of (6.3). Observe that L is the closed subalgebra of L^∞ generated by $S_0 \cup \bar{G}_{\alpha_0}(\bar{S})$

By conditions (S.1) and (S.2), this means $L \subset \bar{S}$ and hence $\bar{G}_\alpha(L) \subset \bar{G}_\alpha(\bar{S}) = \bar{G}_{\alpha_0}(\bar{S}) \subset L$ proving property (R.1) for L.

Proof of Theorem 3(ii). Let us fix an L which satisfies conditions (C) and (R) and let $(X', m', \mathscr{F}', \mathscr{E}')$ be the regular representation with respect to L according to Theorem 2(ii). We have to prove that $(\mathscr{F}', \mathscr{E}')$ is generated by a Ray resolvent kernel on X' and $\mathscr{F}' \cap C(X')$ contains a set C_1' attached to the Ray resolvent (Definition 2.5).

A Ray resolvent can be constructed by Φ of (4.1) which is an algebraic isomorph and isometry from L onto $C(X')$. Φ is a lattice isomorph and satisfies $\Phi(u \wedge 1) = (\Phi u) \wedge 1$ for $u \in L$. Indeed,

$$\bar{G}_\alpha' u' = \Phi \bar{G}_\alpha \Phi^{-1} u', \qquad u' \in C(X'), \quad \alpha > 0, \tag{6.5}$$

$$C_1' = \Phi(L_0) \tag{6.6}$$

define a Ray resolvent operator $\{\bar{G}_\alpha', \alpha > 0\}$ on $C(X')$ and a set C_1' attached to it.

$\bar{G}_\alpha'$ is a sub-Markov resolvent on $C(X')$ on account of (R.1) for L and ($\bar{\text{G}}$.1), ($\bar{\text{G}}$.2) for $\bar{G}_\alpha$ on L_0^∞. (R.2) implies that C_1' generates the closed algebra $C(X')$ and so that C_1' separates points of X' and, for any $x' \in X'$, there exists $u' \in C_1'$ nonvanishing at x'. The inequalities $u' \geqq 0$, $\alpha \bar{G}_{\alpha+\alpha_0} u' \leqq u'$ for $u' \in C_1'$ are obvious from (R.2).

We see that C_1' is included in $\mathscr{F}' \cap C(X')$ because of (5.8) and (R.2).

Finally, let us prove that $\{\bar{G}_\alpha', \alpha > 0\}$ generates the space $(\mathscr{F}', \mathscr{E}')$. It suffices to show

$$\bar{G}_\alpha' u = G_\alpha' u, \quad m'\text{-a.e.}, \qquad u \in L^2(X'; m') \cap C(X'), \tag{6.7}$$

where $\{G_\alpha', \alpha > 0\}$ is the L^2-resolvent associated with $(\mathscr{F}', \mathscr{E}')$.

Observe that G_α' is related to the L^2-resolvent G_α associated with $(\mathscr{F}, \mathscr{E})$ as follows.

$$G_\alpha' u' = \Phi_2 G_\alpha \Phi_2^{-1} u', \qquad u' \in L^2(X'; m'). \tag{6.8}$$

Here, Φ_2 denotes the unitary map from $L_0^2(X; m)$ onto $L^2(X'; m')$ as appeared in step (II) of the proof of Theorem 2(ii). We have indeed by (5.7), $\mathscr{E}'^\alpha(G_\alpha' u', v') = (u', v')_{X'} = (\Phi_2^{-1} u', \Phi_2^{-1} v')_X = \mathscr{E}^\alpha(G_\alpha \Phi_2^{-1} u', \Phi_2^{-1} v') = \mathscr{E}'^\alpha(\Phi_2 G_\alpha \Phi_2^{-1} u', v')$ for any $v' \in \mathscr{F}'$.

Since Φ and Φ_2 coincide on $\mathscr{F} \cap L$ and $\bar{G}_\alpha$ is equal to G_α on $\mathscr{F} \cap L$, (6.5) and (6.8) lead us to the equality (6.7) for $u' \in \mathscr{F}' \cap C(X')$. However $\mathscr{F}' \cap C(X')$ is dense in $C(X')$. Therefore, taking sub-Markovity of $\bar{G}_\alpha'$ and G_α' into account, we get (6.7) for $u' \in L^2(X'; m') \cap C(X')$.

The proof of Theorem 3 is complete.

The next lemma expresses the meaning of Theorem 3 for a special case.

LEMMA 6.1. *Suppose that m is everywhere dense. Suppose further that the next condition is satisfied.*

(G.3) $(\mathscr{F}, \mathscr{E})$ *is generated by a symmetric resolvent kernel* $\{\tilde{G}_\alpha, \alpha>0\}$ *on* X *such that* $\tilde{G}_\alpha$ *transforms* $C_b(X)$ *into* $C_b(X)$ *and* $\lim_{\alpha\to+\infty} \alpha G_\alpha u(x)=u(x)$ *for any* $x \in X$, $u \in C_b(X)$.

(i) *There exists then an algebra L which satisfies not only* (C) *and* (R) *but also the additional condition* (C.4) *of Lemma* 5.1.

(ii) *Let* $(X', m', \mathscr{F}', \mathscr{E}')$ *be the regular representation with respect to such an L. Then, this is strongly regular and X is embedded onto a dense subset of X' in such a way as Lemma* 5.1. *The associated Ray resolvent kernel* $\bar{G}'_\alpha$ *on* X' *is an extension of* $\tilde{G}_\alpha$ *of* (G.3) *in the following sense. For any Borel set A of X,*

$$\bar{G}'_\alpha(x, A) = \tilde{G}_\alpha(x, A), \qquad x \in X. \tag{6.9}$$

Proof. (i) By replacing L^2-resolvent $\{G_\alpha\}$ with the smooth resolvent $\{\tilde{G}_\alpha\}$ of (G.3), we can repeat the arguments of the proof of Theorem 3(i) to get an L in $C_b(X)$. Moreover, S_0 $(\subset L)$ separates points of X. In fact, assume that $\tilde{G}_{\alpha_0}u(x) =\tilde{G}_{\alpha_0}u(y)$ for every $u \in D_0^+$. Then, it is valid for $u \in C_b(X)$. Hence $\alpha\tilde{G}_\alpha u(x)=\alpha\tilde{G}_\alpha u(y)$ for all $\alpha>0$ and $u \in C_b(X)$. By letting α tend to infinity, we have $u(x)=u(y)$, $u \in C_b(X)$, which means $x=y$. In the same way, we see the existence of some function of S_0 nonvanishing at any preassigned point of X.

(ii) The identity (6.8) is equivalent to

$$\bar{G}'_\alpha u'(x) = \tilde{G}_\alpha u(x), \qquad u' \in C(X'), \quad x \in X, \tag{6.10}$$

where $u=u'|_X$ the restriction of u' to X. The right-hand side of (6.10) makes sense because $u \in C_b(X)$. Since (4.3) implies $u'|_X=\Phi^{-1}u'$ for any $u' \in C(X')$, we have

$$\bar{G}'_\alpha u'|_X = \Phi^{-1}\bar{G}'_\alpha u' = \Phi^{-1}\Phi\bar{G}_\alpha\Phi^{-1}u' = \bar{G}_\alpha u, \qquad u' \in C(X'), \quad \text{by (6.5)}.$$

However, $\bar{G}_\alpha$ and $\tilde{G}_\alpha$ are identical on L for they are on $L^2(X; m) \cap C(X)$.

Remark 6.1. We may consider that Theorem 3 treats the problem of finding strong Markov processes for a given resolvent operator. Theorem 3 solves this problem demanding that the construction procedure does not change the structure of certain associated function spaces. If we take off such a demand, we have much more possibilities of getting strong Markov processes. The proof of Theorem 3 indicates the following.

Suppose that we are given a sub-Markov resolvent operator $\{\bar{G}_\alpha, \alpha>0\}$ on a closed subalgebra A of $B(X)$ or $L^\infty(X; m)$. Here, $B(X)$ denotes the space of bounded functions with uniform norm. No kind of assumption of symmetry is imposed on $\bar{G}_\alpha$.

(I) If we are given a closed subalgebra L of A which satisfies condition (R)(7), then (6.5) defines a Ray resolvent (and consequently a strong Markov process of Ray in the sense of Remark 2.2(ii)) on the very character space X' of L.

(7) Here the term of condition (R) is used under a trivial modification that we do not require L_0 of (R.2) to be a subset of $\mathscr{F}$.

(II) Let D_0^+ be any countable subcollection of A^+. Then D_0^+ generates an L satisfying condition (R) quite in the same manner as in the proof of Theorem 3.

Our method to get L which satisfies (R) is due to H. Kunita and T. Watanabe [12]. The above mentioned facts tell the generality of their method and the scope of the Ray process.

REMARK 6.2. Consider a bounded domain D of R^N. The D-space of Example 1 of §3 meets the condition (G.3) of Lemma 6.1. According to Lemma 6.1, we get its strongly regular representation accompanied by a Ray process on an extension D' of D. On the other hand, we adopted in [5] the compactification D^* of D with respect to $G_1(D_0^+)$ to serve as a state space of an extended strong Markov process—a reflecting Brownian motion. This process is not necessarily a Ray's one in the strict sense of the word. However, it turns out that $(D^*, dx, \mathscr{E}_{L^2}^1, (\ ,\ )_{D,1})$ is a regular representation of the given D-space, for the algebra generated by $G_1(D_0^+)$ and 1 is obviously dense both in $C(D^*)$ and in $\mathscr{E}_{L^2}^1$.

The situation is quite the same for the D-space generated by each resolvent density of class $\boldsymbol{G}$ in [6].

Appendix. *Construction of D-spaces by means of completion.* Let X be a locally compact Hausdorff and separable space and m be a Radon measure on X. A pair $(\mathscr{A}, \mathscr{E})$ is said to satisfy condition $(\mathscr{A})$ if it enjoys the next three conditions.

$(\mathscr{A}.1)$ $\mathscr{A}$ is a linear subspace of $L^2(X; m)$ and $\mathscr{E}$ is a positive definite symmetric bilinear form on $\mathscr{A}$.

$(\mathscr{A}.2)$ If $u \in \mathscr{A}$, then $v = (0 \vee u) \wedge 1 \in \mathscr{A}$ and $\mathscr{E}(v, v) \leqq \mathscr{E}(u, u)$.

$(\mathscr{A}.3)$ If $u_n \in \mathscr{A}$ satisfies $(u_n, u_n)_X \to 0$ and $\mathscr{E}(u_n - u_m, u_n - u_m) \to 0$, then

$$\mathscr{E}(u_n, u_n) \to 0.$$

Condition $(\mathscr{A}.1)$ means that $\mathscr{A}$ is a real pre-Hilbert space with respect to inner product $\mathscr{E}^\alpha(u, v) = \mathscr{E}(u, v) + \alpha(u, v)_X$, $u, v \in \mathscr{A}$, for each $\alpha > 0$.

THEOREM. *Suppose that a pair $(\mathscr{A}, \mathscr{E})$ satisfies condition $(\mathscr{A})$. Let $\mathscr{F}$ be the completion of $\mathscr{A}$ with respect to a metric $\mathscr{E}^{\alpha_0}$ for a fixed $\alpha_0 > 0$. Then, $(X, m, \mathscr{F}, \mathscr{E})$ is a D-space.*

Proof. $(\mathscr{A}.1)$ and $(\mathscr{A}.3)$ imply that $\mathscr{F}$ is a linear subspace of $L^2(X; m)$ and that $(\mathscr{F}, \mathscr{E})$ satisfies the condition (D.2) of Definition 2.1. Therefore, for each $\alpha > 0$ and $u \in L^2(X; m)$, there exists $G_\alpha u \in \mathscr{F}$ such that $\mathscr{E}^\alpha(G_\alpha u, v) = (u, v)_X$ holds for any $v \in \mathscr{F}$. It suffices for us to show that $\{G_\alpha, \alpha > 0\}$ is an L^2-resolvent, because then $(\mathscr{F}, \mathscr{E})$ coincides with the D-space generated by $\{G_\alpha, \alpha > 0\}$. Obviously $\{G_\alpha, \alpha > 0\}$ satisfies the resolvent equation. To see its sub-Markov property, let us assume that $u \in L^2(X; m)$ and $0 \leqq u \leqq 1$ m-a.e.

If we put $\Phi(v) = \mathscr{E}(v, v) + \alpha(v - (1/\alpha)u, v - (1/\alpha)u)_X$ for $v \in \mathscr{F}$, then we have $\Phi(v) = \Phi(G_\alpha u) + \mathscr{E}^\alpha(G_\alpha u - v, G_\alpha u - v)$, which means that $G_\alpha u$ is a unique element of $\mathscr{F}$ minimizing the quadratic form Φ on $\mathscr{F}$. Further we see that $v_n \in \mathscr{F}$ converges to $G_\alpha u$ in $\mathscr{E}^\alpha$-norm if and only if v_n is a minimizing sequence for Φ: $\Phi(v_n) \to \Phi(G_\alpha u)$.

Since $\mathscr{A}$ is dense in $\mathscr{F}$ in $\mathscr{E}^\alpha$-norm, there exist $v_n \in \mathscr{A}$ which converges to $G_\alpha u$ in $\mathscr{E}^\alpha$-norm. Put $w_n = (0 \vee v_n) \wedge (1/\alpha)$. By condition ($\mathscr{A}$.2), $w_n \in \mathscr{A}$ and $\mathscr{E}(w_n, w_n) \leqq \mathscr{E}(v_n, v_n)$. Now it is easy to see that $\Phi(G_\alpha u) \leqq \Phi(w_n) \leqq \Phi(v_n)$ for each n. However, v_n is a minimizing sequence for Φ and so that w_n is. Hence, w_n converges to $G_\alpha u$ in $\mathscr{E}^\alpha$-norm and consequently a subsequence of w_n converges to $G_\alpha u$ m-a.e. Thus we get $0 \leqq G_\alpha u \leqq 1/\alpha$ m-a.e.

COROLLARY 1. *In addition to the condition in Theorem, we assume that m is everywhere dense on X and that $\mathscr{A}$ is a dense subset of $C(X)$. Then $(X, m, \mathscr{F}, \mathscr{E})$ of the theorem is a regular D-space.*

COROLLARY 2. *Suppose that we are given a D-space $(X, m, \mathscr{F}, \mathscr{E})$. Let $\mathscr{A}$ be a subspace of $\mathscr{F}$ such that $(0 \vee u) \wedge 1 \in \mathscr{A}$ whenever $u \in \mathscr{A}$. Denote by $\mathscr{F}_0$ the completion of $\mathscr{A}$ with respect to $\mathscr{E}^{\alpha_0}$-norm. Then, $(X, m, \mathscr{F}_0, \mathscr{E})$ is a D-space.*

COROLLARY 3. *Suppose that we are given a D-space $(X, m, \mathscr{F}, \mathscr{E})$ with everywhere dense m. We assume that $\mathscr{F} \cap C_0(X)$ is dense in $C_0(X)$. Denote by $\mathscr{F}_0$ the completion of $\mathscr{F} \cap C_0(X)$ with respect to $\mathscr{E}^{\alpha_0}$-norm. Then, $(X, m, \mathscr{F}_0, \mathscr{E})$ is a regular D-space.*

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DIRICHLET SPACES AND STRONG MARKOV PROCESSES

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Abstract. We show that there exists a suitable strong Markov process on the underlying space of each regular Dirichlet space. Potential theoretic concepts due to A. Beurling and J. Deny are then described in terms of the associated strong Markov process. The proof is carried out by developing potential theory for Dirichlet spaces and symmetric Ray processes and by using a method of transformation of underlying spaces.

Introduction. This paper is a continuation of [10]. We will use those notions and terminologies adopted in [10].

Let $(X, m, \mathscr{F}, \mathscr{E})$ be a D-space. We define (α_0-) *capacity* of an open set $A \subset X$ by

$$(0.1)\qquad \begin{aligned} \operatorname{Cap}(A) &= \inf_{u \in \mathscr{L}_A} \mathscr{E}^{\alpha_0}(u, u) \quad \text{if } \mathscr{L}_A \neq \varnothing, \\ &= +\infty \qquad\qquad\qquad \text{otherwise,} \end{aligned}$$

where α_0 is a fixed positive number and

$$(0.2)\qquad \mathscr{L}_A = \{u \in \mathscr{F}; u \geqq 1 \text{ } m\text{-a.e. on } A\}.$$

The capacity of an arbitrary set $A \subset X$ is defined by

$$(0.3)\qquad \operatorname{Cap}(A) = \inf_{A \subset B, B \text{ open}} \operatorname{Cap}(B).$$

We show in subsection 1.1 that this definition gives us a *Choquet capacity*[2]. A set $A \subset X$ is said to be *polar* if A has zero capacity. If A is polar, then $m(A) = 0$.

From subsection 1.2 to the end of this paper, we will concentrate our attention on regular D-spaces. According to Definition 2.3 of [10], a D-space is called *regular* if m is everywhere dense on X and the space $\mathscr{F} \cap C(X)$ is dense both in $\mathscr{F}$ with norm $\mathscr{E}^{\alpha_0}$ and in $C(X)$ with uniform norm, $C(X)$ being the space of all continuous functions vanishing at infinity on X. Our goal in this paper is to establish the following existence theorem of a strong Markov process.

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(2) This fact has been proved by J. Deny [3] under a kind of regularity condition for a function space.

THEOREM 4.1. *Let* $(X, m, \mathscr{F}, \mathscr{E})$ *be a regular D-space. There exist then a (possibly empty) Borel polar set* $B \subset X$ *and a right continuous strong Markov process* $M = (\Omega, \mathscr{M}, \mathscr{M}_t, X_t, P_x)$ *with state space* $X \cup \partial - B$ *such that the resolvent of the process* M *generates the given D-space* $(\mathscr{F}^*, \mathscr{E})$(3): *if we put*

$$R_\alpha f(x) = E_x\left(\int_0^{+\infty} e^{-\alpha t} f(X_t)\, dt\right), \qquad x \in X - B, \tag{0.4}$$

for $f \in L^2(X; m) \cap C(X)$ *under the convention that* $f(\partial) = 0$, *then the function* $R_\alpha f$ *belongs to the space* $\mathscr{F}^*$ *and the equation*

$$\mathscr{E}^\alpha(R_\alpha f, v) = (f, v)_X \tag{0.5}$$

holds for every $v \in \mathscr{F}$. *Furthermore the state space* $X \cup \partial - B$ *has no branch point and* M *is quasi-left continuous on* $[0, +\infty)$.

In §4 we will prove this theorem by constructing these objects $(\Omega, \mathscr{M}, \mathscr{M}_t, X_t, P_x)$ in a specific way and describing more detailed properties that they possess. It turns out that the process M is actually a *Hunt process*(4).

Here we give a brief account of our procedure.

§1 will provide some basic facts related to a regular D-space most of which are well known as the contents of Beurling-Deny's potential theory. We reproduce them because our definition of the regularity is slightly more general than Beurling-Deny's and further our approach to the potential theory is based on the concept of quasi-supermedian functions.

Theorem 2.1 of §2 will state that, if two regular D-spaces are equivalent in the sense of Definition 4.1 of [10], then their underlying spaces are related by a *capacity preserving quasi-homeomorphism*(5). We need the regular representation theorem [10] for the proof of Theorem 2.1.

In §3 we examine the relationship between two aspects of a strongly regular D-space—the potential theoretic one developed in §1 and the probability theoretic one corresponding to the associated Ray process. For instance, we prove in Theorem 3.12 that a set A is polar if and only if there is an m-negligible Borel set $B \supset A$ such that almost all sample paths of the Ray process starting at any point of $X - B$ will never contact with B.

The proof of Theorem 4.1 is accomplished in the following way. Let $(X, m, \mathscr{F}, \mathscr{E})$ be a regular D-space. Then by virtue of Theorem 3 of [10], there is a strongly regular D-space $(\tilde{X}, \tilde{m}, \tilde{\mathscr{F}}, \tilde{\mathscr{E}})$ which is equivalent to $(X, m, \mathscr{F}, \mathscr{E})$. Owing to Theorem 2.1, X is related to $\tilde{X}$ by a capacity-preserving quasi-homeomorphism q. q will transform the associated Ray process on $\tilde{X}$ into a process on X which turns out to have the properties of Theorem 4.1.

(3) $\mathscr{F}^*$ is the quasi-continuous modification of $\mathscr{F}$ (subsection 1.2).

(4) See P. A. Meyer [16, Chapitre XVI]. The state space of the process M is not necessarily a locally compact set but a Borel subset of the compactum $X \cup \partial$.

(5) We can find an analogous reasoning in M. Nakai [17].

Thus every regular D-space is endowed with a probabilistic structure and we can see that all theorems of §3 are generalized at once to the case of the regular D-space. Subsection 4.2 collects some of the generalizations—an identification of decomposition of the D-space and that of an associated Hunt process, an identification of quasi-continuity and q.e. fine continuity, etc. In particular our notion of polar sets turns out to be weaker in general than the usually adopted probabilistic one. They are identical, however, as we will see in subsections 3.6 and 4.2, if and only if the underlying measure m is a reference measure for the process.

Although we go no further at present, it may be asserted that sample paths governed by a Dirichlet space will run along "the roads" indicated by the 0-order Dirichlet form $\mathscr{E}$ and with "speed" indicated by the underlying measure m.

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1. **Potential theory for D-spaces.** Let $(X, m, \mathscr{F}, \mathscr{E})$ be a D-space. We do not assume any regularity condition in the first subsection. From subsection 1.2 throughout §1 we will assume that $(X, m, \mathscr{F}, \mathscr{E})$ is regular.

1.1. *Capacity.*

THEOREM 1.1. *The capacity defined by* (0.1) *and* (0.3) *for all subsets of X is a Choquet capacity, that is,*

(a) *it is increasing,*

(b) *for any increasing sequence of subsets A_n of X,*

$$\operatorname{Cap}\left(\bigcup_n A_n\right) = \sup_n \operatorname{Cap}(A_n),$$

and

(c) *for any decreasing sequence of compact subsets A_n of X,*

$$\operatorname{Cap}\left(\bigcap_n A_n\right) = \inf_n \operatorname{Cap}(A_n).$$

Furthermore it has the property that

(d) *it is nonnegative and countably subadditive.*

Our capacity is evidently nonnegative and increasing. Property (c) is also clear. In fact, for any $\varepsilon > 0$, there exists an open set $E \supset \bigcap_n A_n$ such as $\operatorname{Cap}(\bigcap_n A_n) \geqq \operatorname{Cap}(E) - \varepsilon$. However, $E \supset A_n$ for some n and we have $\operatorname{Cap}(\bigcap_n A_n) \geqq \inf_n \operatorname{Cap}(A_n)$.

According to P. A. Meyer [15, III, T23], the other assertions of Theorem 1.1 follow from the next lemma.

LEMMA 1.1. *The capacity defined by* (0.1) *for all open sets of X has the following properties. Denote by $\mathscr{U}$ the class of all open sets A for which $\mathscr{L}_A \neq \varnothing$.*

(i) *It is finite, nonnegative and increasing on $\mathscr{U}$.*

(ii) *It is strongly subadditive on $\mathscr{U}$: for any $A, B \in \mathscr{U}$,*

$$\operatorname{Cap}(A \cup B) + \operatorname{Cap}(A \cap B) \leqq \operatorname{Cap}(A) + \operatorname{Cap}(B).$$

(iii) *If $A_n \in \mathscr{U}$ is increasing and $\bigcup_n A_n \in \mathscr{U}$, then*

$$\mathrm{Cap}\left(\bigcup_n A_n\right) = \sup_n \mathrm{Cap}\,(A_n).$$

(iv) *For any open set A belonging to $\mathscr{U}_\sigma$,*

$$\mathrm{Cap}\,(A) = \sup_{B\subset A, B\in\mathscr{U}} \mathrm{Cap}\,(B).$$

Proof. For $A \in \mathscr{U}$, there exists a unique element $p_A \in \mathscr{L}_A$ minimizing the quadratic form $\mathscr{E}^{\alpha_0}(u, u)$ in $\mathscr{L}_A$, since $\mathscr{L}_A$ is a nonempty convex set of $\mathscr{F}$ closed with norm $\mathscr{E}^{\alpha_0}$. Evidently,

$$\mathrm{Cap}\,(A) = \mathscr{E}^{\alpha_0}(p_A, p_A). \tag{1.1}$$

Since $(0 \vee p_A) \wedge 1$, being a normal contraction of p_A, is identical with p_A, we have

$$0 \leqq p_A \leqq 1 \qquad m\text{-a.e. on } X, \tag{1.2}$$

$$p_A = 1 \qquad m\text{-a.e. on } A. \tag{1.3}$$

Further we have

$$\mathscr{E}^{\alpha_0}(p_A, v) \geqq 0 \tag{1.4}$$

for any $v \in \mathscr{F}$ which is nonnegative m-a.e. on A. This follows from

$$\mathscr{E}^{\alpha_0}(p_A+\varepsilon v, p_A+\varepsilon v) \geqq \mathscr{E}^{\alpha_0}(p_A, p_A), \qquad \varepsilon > 0.$$

It is easy to see that $p_A \in \mathscr{F}$ is characterized by two conditions (1.3) and (1.4). Keeping these in mind, let us prove Lemma 1.1.

(i) Trivial.

(ii) Since $|p_A - p_B|$ is a normal contraction of $p_A - p_B$, we have

$$\mathscr{E}^{\alpha_0}(p_A \vee p_B, p_A \vee p_B) + \mathscr{E}^{\alpha_0}(p_A \wedge p_B, p_A \wedge p_B) \leqq \mathscr{E}^{\alpha_0}(p_A, p_A) + \mathscr{E}^{\alpha_0}(p_B, p_B),$$

which implies the desired inequality.

(iii) For $n > m$,

$$\mathscr{E}^{\alpha_0}(p_{A_n} - p_{A_m}, p_{A_n} - p_{A_m}) = \mathrm{Cap}\,(A_n) - \mathrm{Cap}\,(A_m).$$

Since $\mathrm{Cap}\,(A_n)$ is bounded from above (by the capacity of $A = \bigcup_n A_n$), the preceding equality means that p_{A_n} converges to a $u_0 \in \mathscr{F}$ in norm $\mathscr{E}^{\alpha_0}$. $u_0 = 1$ m-a.e. on A because $p_{A_n} = 1$ m-a.e. on A_n. Moreover $\mathscr{E}^{\alpha_0}(u_0, v) = \lim_{n\to+\infty} \mathscr{E}^{\alpha_0}(p_{A_n}, v) \geqq 0$ for every $v \in \mathscr{F}$ which is nonnegative m-a.e. on A. Thus $u_0 = p_A$ and

$$\lim_{n\to+\infty} \mathrm{Cap}\,(A_n) = \lim_{n\to+\infty} \mathscr{E}^{\alpha_0}(p_{A_n}, p_{A_n}) = \mathscr{E}^{\alpha_0}(p_A, p_A) = \mathrm{Cap}\,(A).$$

(iv) Consider an element A of $\mathscr{U}_\sigma$ and put $c = \sup_{B\subset A, B\in\mathscr{U}} \mathrm{Cap}\,(B)$. There exists an increasing sequence of open sets $A_n \in \mathscr{U}$ such that $\bigcup_n A_n = A$. By making use of statement (iii), we easily obtain the equality $c = \lim_{n\to+\infty} \mathrm{Cap}\,(A_n) \leqq +\infty$. Now this equality combined with exactly the same argument as in the proof of (iii) leads us to the conclusion that c is finite if and only if $A \in \mathscr{U}$ and in this case $c = \text{Cap}(A)$. Hence, if $c = +\infty$, then $A \notin \mathscr{U}$ and $\text{Cap}(A) = +\infty$ by definition. In any case, we get the desired equality. The proof of Lemma 1.1 is complete.

Theorem 1.1 combined with Choquet's theorem implies that, for any analytic set $A \subset X$,

$$\text{Cap}(A) = \sup_{K \subset A, K \text{ compact}} \text{Cap}(K). \tag{1.5}$$

In subsection 1.5, we will give some characterizations of the capacity for compact sets in the case of the regular D-space.

A subset A of X is called *polar* if $\text{Cap}(A) = 0$. The expression "*quasi-everywhere*" or "*q.e.*" means "except for a polar set". Let E be an open set of X. A function u defined q.e. on E is called *quasi-continuous on* E if, for any $\varepsilon > 0$, there exists an open set $\omega \subset E$ such that $\text{Cap}(\omega) < \varepsilon$ and the restriction of u to $X - \omega$ is continuous there. Quasi-continuous functions on X are simply said to be quasi-continuous.

THEOREM 1.2. (i) *If A is polar, then $m(A) = 0$.*

(ii) *If u_1 and u_2 are quasi-continuous on an open set $E \subset X$ and $u_1 \geqq u_2$ m-a.e. on E, then $u_1 \geqq u_2$ q.e. on E.*

Proof. (i) This is evident in view of the inequality $\text{Cap}(A) \geqq \alpha_0 m(A)$ for the open set A, which is immediate from (0.1).

(ii) Fix an $\varepsilon > 0$. There exists then an open set $\omega \subset E$ with $\text{Cap}(\omega) < \varepsilon$ such that u_1 and u_2 are continuous on $E - \omega$. Put $\omega' = \{x \in E$; there exists a neighborhood $U(x)$ of x such that $U(x) \subset E$ and $m(U(x) - \omega) = 0\}$. It is easy to see that ω' is an open set, $\omega \subset \omega' \subset E$ and $m(\omega' - \omega) = 0$[6]. Hence $\mathscr{L}\omega' = \mathscr{L}\omega$ and, by (0.1), $\text{Cap}(\omega') = \text{Cap}(\omega) < \varepsilon$. Now let us show $A \subset \omega'$, where $A = \{x \in E; u_1(x) < u_2(x)\}$. Suppose that there is an element $x \in A \cap (E - \omega')$. Since $x \in A \cap (E - \omega)$, there exists a $U(x) \subset E$ such that $u_1 < u_2$ on $U(x) - \omega$. However, $m(U(x) - \omega) \neq 0$ because $x \in E - \omega'$. This contradicts the assumption that $u_1 \geqq u_2$ m-a.e. on E. Thus $A \subset \omega'$ and $\text{Cap}(A) < \varepsilon$, proving that A is polar.

1.2. *Quasi-continuous modification $\mathscr{F}^*$.* From now on we assume that the given D-space $(X, m, \mathscr{F}, \mathscr{E})$ is regular.

Theorem 1.2(ii) then implies that $\text{Cap}(A) > 0$ for every nonempty open set A. Moreover, if a subset $A \subset X$ has a compact closure, then $\text{Cap}(A)$ is finite. In fact, A is then included in an open set E with compact closure. $\mathscr{L}_E$ is not empty for such an E.

THEOREM 1.3. *For any $u \in \mathscr{F}$, there exists $u_n \in \mathscr{F} \cap C(X)$ and increasing closed subsets F_m such that $\mathscr{E}^{\alpha_0}(u_n - u, u_n - u) \to 0$, $\text{Cap}(\bigcap_{m=1}^{\infty} F_m^c) = 0$ and u_n converges*

(6) Since X is assumed to be separable, we can use the Lindelöf covering theorem to prove this point. Cf. Hilfssatz 5.9 in C. Constantinescu and A. Cornea, *Ideal ränder Riemannscher flächen*, Springer, 1963.

uniformly on each F_m. The limit function u^* of u_n is quasi-continuous and equal to u m-a.e.

Proof. By means of (0.1), we have

$$\text{Cap}\{x; |v(x)| > \varepsilon\} \leqq \mathscr{E}^{\alpha_0}(v, v)/\varepsilon^2 \tag{1.6}$$

for any $\varepsilon > 0$ and $v \in \mathscr{F} \cap C(X)$. Take $u \in \mathscr{F}$ and find $u_n \in \mathscr{F} \cap C(X)$ converging to u with $\mathscr{E}^{\alpha_0}$-norm. Subtracting a suitable subsequence if necessary, we can assume that Cap $(G_n) \leqq 1/2^n$ for the open set $G_n = \{x; |u_n(x) - u_{n+1}(x)| > 1/2^n\}$. The statement of Theorem 1.3 holds for $F_m = \bigcap_{n=m}^{+\infty} G_n^c$.

If a function u is defined m-a.e. on X and if u^* is quasi-continuous and equal to u m-a.e. on X, then u^* is called a *quasi-continuous modification of u*. Denote by $\mathscr{F}^*$ the set of all quasi-continuous modifications of functions of $\mathscr{F}$. We regard two functions of $\mathscr{F}^*$ to be equivalent if they are identical q.e. on X. On account of Theorems 1.2 and 1.3, the equivalence classes of $\mathscr{F}^*$ with inner product $\mathscr{E}^\alpha$ form a real Hilbert space which is just identical with the space $(\mathscr{F}, \mathscr{E}^\alpha)$, two functions of $\mathscr{F}$ being identified if they coincide m-a.e.

The next lemma can be proved exactly in the same manner as in J. Deny and J. Lions [5, II, Lemme 4.1 and Théorème 4.1].

LEMMA 1.2. (i) *The estimate* (1.6) *holds for any $\varepsilon > 0$ and $v \in \mathscr{F}^*$.*

(ii) *If u_n is a Cauchy sequence in $(\mathscr{F}^*, \mathscr{E}^{\alpha_0})$, then u_n converges to a function $u \in \mathscr{F}^*$ with $\mathscr{E}^{\alpha_0}$-norm. Further there exists a subsequence n_k such that* $\lim_{n_k \to +\infty} u_{n_k}(x) = u(x)$ *q.e. on X.*

1.3. *Quasi-supermedian functions and potentials.* Let $\{G_\alpha, \alpha > 0\}$ be the L^2-resolvent associated with the D-space $(\mathscr{F}, \mathscr{E})$. Each G_α is a linear operator from $L^2(X; m)$ into $\mathscr{F}$. From now on, however, we regard G_α as a linear operator from $L^2(X; m)$ into the space $\mathscr{F}^*$, as the preceding subsection 1.2 admits us to do.

We call a function $u \in L^2(X; m)$ (α_0-) quasi-supermedian if the following two conditions are satisfied.

(1.7) u is quasi-continuous and $u \geqq 0$ q.e.

(1.8) $\beta G_{\beta+\alpha_0} u \leqq u$ q.e., $\beta > 0$.

LEMMA 1.3. *A function $u \in \mathscr{F}^*$ is quasi-supermedian if and only if $\mathscr{E}^{\alpha_0}(u, v) \geqq 0$ for every $v \in \mathscr{F}^*$ such that $v \geqq 0$ q.e.*

Proof. If $u \in \mathscr{F}^*$ is quasi-supermedian then according to Lemma 2.1 of [10],

$$\mathscr{E}^{\alpha_0}(u, v) = \lim_{\beta \to +\infty} \beta(u - \beta G_{\beta+\alpha_0} u, v)_X \geqq 0 \quad \text{for } v \in \mathscr{F}$$

such that $v \geqq 0$ m-a.e.

Conversely assume that $u \in \mathscr{F}^*$ satisfies the inequality $\mathscr{E}^{\alpha_0}(u, v) \geqq 0$ for every $v \in \mathscr{F}^*$ which is nonnegative q.e. on X. Then u is the unique element minimizing the norm $\mathscr{E}^{\alpha_0}(w, w)$ in the convex set $\mathscr{L}_u = \{w \in \mathscr{F}^*; w \geqq u \text{ q.e.}\}$. $|u|$ is a normal contraction of u and belongs to $\mathcal{L}_u$. Thus $u=|u|\geqq 0$ q.e. proving (1.7). Furthermore we have, for any $v\in L^2(X;m)$ such as $v\geqq 0$ m-a.e.,

$$(u-\beta G_{\beta+\alpha_0}u, v)_X = \mathcal{E}^{\beta+\alpha_0}(u, G_{\beta+\alpha_0}u)-\beta(u, G_{\beta+\alpha_0}v)_X = \mathcal{E}^{\alpha_0}(u, G_{\beta+\alpha_0}v)$$

which is nonnegative because $G_{\beta+\alpha_0}v\in\mathcal{F}^*$ and $G_{\beta+\alpha_0}v\geqq 0$ q.e. (Theorem 1.2(ii)). This proves (1.8). The proof of Lemma 1.3 is complete.

Denote by M_0^+ the set of all nonnegative Borel measures μ on X satisfying the following two conditions:

(1.9) $$\mathcal{F}\cap C(X)\subset L^1(X;\mu).$$

(1.10) There exists a function $u\in\mathcal{F}^*$ such that

$$\mathcal{E}^{\alpha_0}(u, v) = \int_X v(x)\mu(dx) \quad \text{for any } v\in\mathcal{F}\cap C(X).$$

The function u of (1.10) is uniquely determined by $\mu\in M_0^+$. It is called the (α_0-) *potential of* μ and denoted by $U\mu$.

Every $\mu\in M_0^+$ is a Radon measure on X, namely, μ is finite for any compactum.

Any $u\in\mathcal{F}^*$ defines a linear functional l_u on $\mathcal{F}\cap C(X)$ by $l_u(v)=\mathcal{E}^{\alpha_0}(u, v)$, $v\in\mathcal{F}\cap C(X)$. Meanwhile, $\mathcal{F}\cap C(X)$ is closed under lattice operations and $v\wedge 1\in\mathcal{F}\cap C(X)$ for any $v\in\mathcal{F}\cap C(X)$ [10, Lemma 4.1]. Therefore by the general theory of Daniell integral [14, Chapter 3], l_u is an integral by means of the Baire measure with respect to the class $\mathcal{F}\cap C(X)$ if and only if l_u is a positive functional and continuous under monotone limits. Since the Baire family generated by $\mathcal{F}\cap C(X)$ is the set of Borel functions, we get the following

LEMMA 1.4. *$u\in\mathcal{F}^*$ is a potential if and only if*

(1.11) $\mathcal{E}^{\alpha_0}(u, v)\geqq 0$ *for any nonnegative* $v\in\mathcal{F}\cap C(X)$.

(1.12) $\mathcal{E}^{\alpha_0}(u, v_n)\downarrow 0$ *if* $v_n\in\mathcal{F}\cap C(X)$ *converges monotonically to zero.*

When X is compact, condition (1.12) is superfluous. If u is a potential, u determines the associated measure $\mu\in M_0^+$ uniquely.

Now we will state the relation of quasi-supermedian functions and potentials.

THEOREM 1.4. *A function $u\in\mathcal{F}^*$ is a potential if and only if u is a quasi-supermedian function satisfying condition* (1.12).

Proof. It suffices to show that condition (1.11) implies the stronger condition of Lemma 1.3. Suppose that $u\in\mathcal{F}^*$ satisfies (1.11). Let $v\in\mathcal{F}^*$ be nonnegative q.e. and $v_n\in\mathcal{F}\cap C(X)$ be a sequence converging to v in $\mathcal{E}^{\alpha_0}$-norm. Consider $v_n^+ = v_n\vee 0\in\mathcal{F}\cap C(X)$. Then $\mathcal{E}^{\alpha_0}(v_n^+, v_n^+)\leqq\mathcal{E}^{\alpha_0}(v_n, v_n)$ is bounded in n and

$$\mathcal{E}^{\alpha_0}(v_n^+, G_{\alpha_0}f) = (v_n^+, f)_X \to (v, f)_X = \mathcal{E}^{\alpha_0}(v, G_{\alpha_0}f) \quad \text{for any } f\in L^2(X;m).$$

Since $G_{\alpha_0}(L^2)$ is dense in $\mathcal{F}$, v_n^+ converges to v weakly. In particular,

$$\mathcal{E}^{\alpha_0}(u, v) = \lim_{n\to+\infty}\mathcal{E}^{\alpha_0}(u, v_n^+) \geqq 0.$$

1.4. *Basic properties of potentials.* Define the support $S\mu$ of $\mu \in M_0^+$ by $S\mu = \{x \in X; \mu(U_x) \neq 0$ for any neighborhood U_x of $x\}$. $S\mu$ is a closed set. Let us begin with an approximation lemma.

LEMMA 1.5. *Suppose that $\mu \in M_0^+$ has a compact support $S\mu$. Then for any open set E such that $E \supset S\mu$ and $\bar{E}$ is compact, there exist nonnegative functions $f_n \in L^2(X; m)$ which vanish m-a.e. on $X-E$ and satisfy*

(1.13) *$f_n \cdot m \to \mu$ vaguely as measures,*

(1.14) *$G_{\alpha_0} f_n \to U\mu$ weakly in $(\mathscr{F}, \mathscr{E}^{\alpha_0})$.*

Proof. By virtue of Theorem 1.4, the potential $U\mu$ is quasi-supermedian, and so

$$g_\beta = \beta(U\mu - \beta G_{\beta+\alpha_0}(U\mu)) \tag{1.15}$$

is nonnegative. Let us prove the equality

$$\lim_{\beta \to +\infty} \int_X v(x) g_\beta(x) m(dx) = \int_K v(x) \mu(dx) \tag{1.16}$$

for every continuous function v such as $|v| \leqq v_0$, where $K = S\mu$ and v_0 is an arbitrarily fixed function in $\mathscr{F} \cap C(X)$. According to Lemma 2.1 of [10] and (1.10), the equality (1.16) is true for every $v \in \mathscr{F} \cap C(X)$. Incidentally the measures $g_\beta \cdot m$ are uniformly bounded in β on any compactum. Turning to the case of general v, choose $v_k \in \mathscr{F} \cap C(X)$ such as $|v_k| \leqq v_0$ and $\|v_k - v\|_\infty \to 0$, $k \to +\infty$, and observe the following inequality:

$$\begin{aligned} &\left| \int_X v(x) g_\beta(x) m(dx) - \int_K v(x) \mu(dx) \right| \\ &\qquad \leqq \left| \int_X v_k(x) g_\beta(x) m(dx) - \int_K v_k(x) \mu(dx) \right| + \int_K |v(x) - v_k(x)| \mu(dx) \\ &\qquad\quad + \int_F |v(x) - v_k(x)| g_\beta(x) m(dx) + 2 \int_{X-F} v_0(x) g_\beta(x) m(dx). \end{aligned}$$

For any $\varepsilon > 0$, take a compactum F such that $v_0 < \varepsilon$ on $X-F$, then the superior limit in β of the last term of the right-hand side is less than $2 \int_K (v_0 \wedge \varepsilon) \mu(dx) \leqq 2\varepsilon \cdot \mu(K)$. Now by taking sufficiently large k, we can make the superior limit in β of the right-hand side arbitrarily small.

It is clear that (1.16) implies (1.13) with

$$f_n(x) = g_n(x) \chi_E(x), \qquad x \in X, \tag{1.17}$$

χ_E being the indicator function of the open set E. It follows from (1.13) that

$$\mathscr{E}^{\alpha_0}(G_{\alpha_0} f_n, v) = (f_n, v)_X \to \int_K v(x) \mu(dx) = \mathscr{E}^{\alpha_0}(U\mu, v)$$

for $v \in \mathscr{F} \cap C(X)$.

Since

$$\begin{aligned}\mathscr{E}^{\alpha_0}(G_{\alpha_0}f_n, G_{\alpha_0}f_n) &= (f_n, G_{\alpha_0}f_n)_X \leqq (g_n, G_{\alpha_0}g_n)_X \\ &= (g_n, nG_{n+\alpha_0}U\mu)_X \leqq (g_n, U\mu)_X\end{aligned}$$

is uniformly bounded by $\mathscr{E}^{\alpha_0}(U\mu, U\mu)$, we arrive at (1.14).

We will point out here that, for any $\mu \in M_0^+$ and compactum K, the measure μ_K defined by $\mu_K(\cdot)=\mu(K\cap\cdot)$ is also in M_0^+ and

$$\mathscr{E}^{\alpha_0}(U\mu_K, U\mu_K) \leqq \mathscr{E}^{\alpha_0}(U\mu, U\mu). \tag{1.18}$$

Indeed, the inequality

$$\begin{aligned}\left|\int_X v(x)\mu_K(dx)\right| &\leqq \int_X |v(x)|\mu(dx) = \mathscr{E}^{\alpha_0}(U\mu, |v|) \\ &\leqq \sqrt{(\mathscr{E}^{\alpha_0}(U\mu, U\mu))}\sqrt{(\mathscr{E}^{\alpha_0}(v, v))}, \qquad v \in \mathscr{F}\cap C(X),\end{aligned}$$

implies the existence of the potential $U\mu_K \in \mathscr{F}$ satisfying (1.10). Further we have $\mu-\mu_K \in M_0^+$. Therefore, by means of Lemma 1.3 and Theorem 1.4,

$$\mathscr{E}^{\alpha_0}(U(\mu-\mu_K), U\mu_K) \geqq 0,$$

which means (1.18). Keeping this in mind, let us proceed to

THEOREM 1.5(7). (i) *If A is polar, then $\mu(A)=0$ for every $\mu\in M_0^+$.*
(ii) *If $\mu \in M_0^+$, then $\mathscr{F}^*\subset L^1(X;\mu)$ and*

$$\mathscr{E}^{\alpha_0}(U\mu, u) = \int_X u(x)\mu(dx), \qquad u \in \mathscr{F}^*. \tag{1.19}$$

Proof. (i) Suppose that A is polar and $\mu\in M_0^+$. For any $\varepsilon>0$, there is an open set $E\supset A$ with Cap $(E)<\varepsilon$. Take any compactum K included in E and choose f_n satisfying conditions of the preceding lemma for μ_K and an open set $E_1\subset E$ with compact closure $\bar{E}_1$. Then,

$$\mathscr{E}^{\alpha_0}(U\mu_K, p_E) = \lim_{n\to+\infty} \mathscr{E}^{\alpha_0}(G_{\alpha_0}f_n, p_E) = \lim_{n\to+\infty}\int_{E_1} f_n(x)m(dx) = \mu(K).$$

Hence

$$\mu(K) \leqq \sqrt{(\mathscr{E}^{\alpha_0}(U\mu_K, U\mu_K))}\sqrt{(\mathscr{E}^{\alpha_0}(p_E, p_E))} \leqq \sqrt{(\mathscr{E}^{\alpha_0}(U\mu, U\mu))}\cdot\sqrt{(\text{Cap }(E))}$$

and

$$\mu(E) \leqq \sqrt{\varepsilon}\cdot\sqrt{(\mathscr{E}^{\alpha_0}(U\mu, U\mu))}.$$

Thus $\mu(A)=0$.

(7) This is a version of Theorem 4(iii) of [1] whose proof was recently given in [4].

(ii) Consider $\mu\in M_0^+$ and $u\in\mathscr{F}^*$. It is clear from assertion (i) that u is μ-measurable. Let $u_n\in\mathscr{F}\cap C(X)$, $F_k\subset X$ and u^* be those of Theorem 1.3 for the present u. It suffices to show that $u^*\in L^1(X;\mu)$ and

$$\mathscr{E}^{\alpha_0}(U\mu, u) = \int_X u^*(x)\mu(dx) \tag{1.20}$$

because $u=u^*$ q.e. and the right-hand sides of (1.19) and (1.20) are identical.

Let us prove (1.20). We may assume that each F_k is compact. Consider the sequence of measures $\mu_k=\mu_{F_k}$, then

$$\begin{aligned}\mathscr{E}^{\alpha_0}(U\mu_k, v) &= \int_{F_k} v(x)\mu(dx) \xrightarrow[k\to+\infty]{} \int_{\bigcup_k F_k} v(x)\mu(dx)\\ &= \int_X v(x)\mu(dx) = \mathscr{E}^{\alpha_0}(U\mu, v), \qquad v\in\mathscr{F}\cap C(X),\end{aligned}$$

which, combined with (1.18), implies that $U\mu_k$ converges to $U\mu$ weakly in $(\mathscr{F}, \mathscr{E}^{\alpha_0})$. On the other hand,

$$\begin{aligned}\mathscr{E}^{\alpha_0}(U\mu_k, u) &= \lim_{n\to+\infty} \mathscr{E}^{\alpha_0}(U\mu_k, u_n)\\ &= \lim_{n\to+\infty}\int_{F_k} u_n(x)\mu(dx) = \int_{F_k} u^*(x)\mu(dx).\end{aligned}$$

Therefore,

$$\mathscr{E}^{\alpha_0}(U\mu, u) = \lim_{k\to+\infty} \mathscr{E}^{\alpha_0}(U\mu_k, u) = \int_{\bigcup_k F_k} u^*(x)\mu(dx) = \int_X u^*(x)\mu(dx).$$

THEOREM 1.6. *Let K be a compact set. Then, for $u\in\mathscr{F}^*$, the next three conditions are mutually equivalent:*

(i) *u is a potential $U\mu$ with $S\mu\subset K$.*

(ii) *$\mathscr{E}^{\alpha_0}(u, v)\geqq 0$ for any $v\in\mathscr{F}\cap C(X)$ which is nonnegative on K.*

(iii) *$\mathscr{E}^{\alpha_0}(u, v)\geqq 0$ for any $v\in\mathscr{F}^*$ which is nonnegative q.e. on K.*

Proof. Owing to Theorem 1.5, (i) implies (iii). Trivially, (iii) implies (ii). All we have to do is to derive (i) from (ii). Suppose that $u\in\mathscr{F}^*$ satisfies condition (ii). We will first prove that u has the properties (1.11) and (1.12). (1.11) is trivial. Let w be a function of $\mathscr{F}\cap C(X)$ which is no less than 1 on K. If $v_n\in\mathscr{F}\cap C(X)$ is decreasing to zero, then v_n converges uniformly on X and $a_n=\sup_{x\in K} v_n(x)$ decreases to zero. Since $v_n\leqq a_n w$ on K,

$$\mathscr{E}^{\alpha_0}(u, v_n) \leqq a_n\mathscr{E}^{\alpha_0}(u, w)\to 0, \qquad n\to+\infty.$$

Thus, u satisfies (1.12). By means of Lemma 1.4, u is a potential of a measure $\mu\in M_0^+$. In view of equation (1.10) and condition (ii), we have $S\mu\subset K$.

1.5. *Equilibrium potential and capacity for the compact set.* We will first define equilibrium potentials for open sets in the class $\mathscr{U}$ of the subsection 1.1 and study their properties. Let A be in $\mathscr{U}$ and p_A be the function of $\mathscr{F}$ which is characterized by (1.3) and (1.4). Denote by e_A any quasi-continuous modification of p_A. We call e_A the (α_0-) *equilibrium potential for the open set* $A \in \mathscr{U}$. According to Theorem 1.2, e_A has the following properties:

(1.21) $\operatorname{Cap}(A)=\mathscr{E}^{\alpha_0}(e_A, e_A)$.

(1.22) $e_A=1$ q.e. on A.

(1.23) $\mathscr{E}^{\alpha_0}(e_A, v)\geqq 0$ for any $v \in \mathscr{F}^*$ which is nonnegative q.e. on A.

$e_A \in \mathscr{F}^*$ is characterized by (1.22) and (1.23) and indeed, it is a unique element which minimizes the norm $\mathscr{E}^{\alpha_0}(u, u)$ in the convex set $\{u \in \mathscr{F}^*; u \geqq 1 \text{ q.e. on } A\}$ of $\mathscr{F}^*$. Obviously e_A is a quasi-supermedian function.

In the particular case when the closure $\bar{A}$ of A is compact, we can see by Theorem 1.6 and (1.23) that e_A is a potential of a measure $\nu_A \in M_0^+$ with $S_{\nu_A} \subset \bar{A}$. We call ν_A the *equilibrium distribution for the open set* A. We have

$$(1.24) \qquad \operatorname{Cap}(A) = \nu_A(\bar{A}),$$

because there is a function $w \in \mathscr{F} \cap C(X)$ which is equal to 1 on $\bar{A}$ and we get $\operatorname{Cap}(A)=\mathscr{E}^{\alpha_0}(e_A, w)=\nu_A(\bar{A})$.

Now consider any compact set K of X and put $\mathscr{L}_K^*=\{u \in \mathscr{F}^*; u \geqq 1 \text{ q.e. on } K\}$. $\mathscr{L}_K^*$ is a nonempty convex set of $\mathscr{F}^*$ and closed in norm $\mathscr{E}^{\alpha_0}$ according to Lemma 1.2. Therefore there is a unique element e_K of $\mathscr{L}_K^*$ which minimizes the quadratic form $\mathscr{E}^{\alpha_0}(u, u)$ in $\mathscr{L}_K^*$. We call e_K the (α_0-) *equilibrium potential for the compactum K*. It is easy to see that e_K is characterized as an element of $\mathscr{F}^*$ which has the following two properties:

(1.25) $e_K=1$ q.e. on K.

(1.26) $\mathscr{E}^{\alpha_0}(e_K, v)\geqq 0$ for any $v \in \mathscr{F}^*$ which is nonnegative q.e. on K.

By virtue of Theorem 1.6 and (1.26), we see that e_K is a potential of a measure $\nu_K \in M_0^+$ with $S\nu_K \subset K$. We call ν_K the *equilibrium distribution for the compactum K*.

THEOREM 1.7. *Let K be compact.*

(i) *The equilibrium potential e_K is characterized as an element of $\mathscr{F}^*$ possessing properties* (1.25) *and*

$$(1.27) \qquad \mathscr{E}^{\alpha_0}(e_K, v) \geqq 0 \quad \textit{for any } v \in \mathscr{F} \cap C(X)$$

which is nonnegative on K.

(ii) *The next equalities hold*:

$$(1.28) \qquad \operatorname{Cap}(K) = \mathscr{E}^{\alpha_0}(e_K, e_K) = \nu_K(K).$$

$$(1.29) \qquad \operatorname{Cap}(K) = \inf_{u \in \mathscr{C}_K} \mathscr{E}^{\alpha_0}(u, u),$$

where $\mathscr{C}_K=\{u \in \mathscr{F} \cap C(X); u \geqq 1 \text{ on } K\}$.

Proof. (i) is evident, since (1.27) is equivalent to (1.26) by virtue of Theorem 1.6. The second equality of (1.28) is immediate from (1.19) and (1.25).

Let us prove the first equality of (1.28). For any $\varepsilon>0$, there is an open set $A\supset K$ such that Cap $(K)+\varepsilon>$ Cap (A). A is in $\mathscr{U}$. By (1.21), (1.22), (1.25) and (1.26), we have $\mathscr{E}^{\alpha_0}(e_A, e_K)=\mathscr{E}^{\alpha_0}(e_K, e_K)$ and $0\leqq\mathscr{E}^{\alpha_0}(e_A-e_K, e_A-e_K)=\text{Cap}\,(A)-\mathscr{E}^{\alpha_0}(e_K, e_K)$. Hence we get the inequality Cap $(K)\geqq\mathscr{E}^{\alpha_0}(e_K, e_K)$. In order to obtain the converse inequality, let us take a sequence of open sets A_n such that $\bar{A}_n$ is compact, $A_n\supset\bar{A}_{n+1}$ and $\bigcap_{n=1}^{\infty} A_n=K$. Let e_n and ν_n be the equilibrium potential and distribution for A_n respectively. Since $\mathscr{E}^{\alpha_0}(e_n-e_m, e_n-e_m)=\text{Cap}\,(A_n)-\text{Cap}\,(A_m)$, $n<m$, e_n converges to some $e_0\in\mathscr{F}^*$ in $\mathscr{E}^{\alpha_0}$-norm. Since $e_n=1$ q.e. on A_n, e_0 has the property (1.25). On the other hand, ν_n concentrates on $\bar{A}_n$ and $\nu_n(\bar{A}_n)=\text{Cap}\,(A_n)\leqq\text{Cap}\,(A_1)$ by (1.24). Therefore a subsequence of ν_n converges weakly to a measure ν_0 whose support is in K. Now the equality $\mathscr{E}^{\alpha_0}(e_n, v)=\int_{\bar{A}_n} v(x)\nu_n(dx)$ leads us to $\mathscr{E}^{\alpha_0}(e_0, v)=\int_K v(x)\nu_0(dx)$, $v\in\mathscr{F}\cap C(X)$, which enables us to conclude that e_0 has the property (1.27). Thus, by statement (i), we see that $e_0=e_K$ and $\mathscr{E}^{\alpha_0}(e_K, e_K)=\lim_{n\to+\infty}\mathscr{E}^{\alpha_0}(e_n, e_n)=\lim_{n\to+\infty}\text{Cap}\,(A_n)\geqq\text{Cap}\,(K)$.

Finally, we will show the equality (1.29). Put $c=\inf_{u\in\mathscr{C}_K}\mathscr{E}^{\alpha_0}(u, u)$ and take a minimizing sequence $u_n\in\mathscr{C}_K$: $\lim_{n\to+\infty}\mathscr{E}^{\alpha_0}(u_n, u_n)=c$. It is easy to see that u_n then forms a Cauchy sequence in norm $\mathscr{E}^{\alpha_0}$ and the limit function $u_0\in\mathscr{F}^*$ does not depend on the choice of the minimizing sequence u_n. Since $u_n\wedge 1\in\mathscr{C}_K$ forms a minimizing sequence as well, we have $u_0=1$ q.e. on K according to Lemma 1.2. Further the property (1.27) for u_0 can be derived from the inequality

$$\mathscr{E}^{\alpha_0}(u_n+\varepsilon v, u_n+\varepsilon v) \geqq \mathscr{E}^{\alpha_0}(u_0, u_0)$$

which holds for any $\varepsilon>0$ and $v\in\mathscr{F}\cap C(X)$ such as $v\geqq 0$ on K. Therefore, statement (i) means that $u_0=e_K$ and $c=\mathscr{E}^{\alpha_0}(u_0, u_0)=\text{Cap}\,(K)$. The proof of Theorem 1.7 is complete.

2. **Transformation of underlying spaces.** Consider two regular D-spaces $(X, m, \mathscr{F}, \mathscr{E})$ and $(\tilde{X}, \tilde{m}, \tilde{\mathscr{F}}, \tilde{\mathscr{E}})$. The concepts corresponding to the latter will be denoted with tilde ~.

DEFINITION 2.1. A mapping q defined q.e. on X taking values in $\tilde{X}$ is said to be a *quasi-homeomorphism* between X and $\tilde{X}$ if, for any $\varepsilon>0$, there exist closed sets $F\subset X$, $\tilde{F}\subset\tilde{X}$ such that Cap $(X-F)<\varepsilon$, Cap~ $(\tilde{X}-\tilde{F})<\varepsilon$ and the restriction of q to F is a homeomorphism onto $\tilde{F}$. X and $\tilde{X}$ are said to be quasi-homeomorphic if there exists a quasi-homeomorphism between X and $\tilde{X}$.

It is clear that q is a quasi-homeomorphism if and only if there exist increasing sequences of closed sets $F_k\subset X$ and $\tilde{F}_k\subset\tilde{X}$ with $\lim_{k\to+\infty}\text{Cap}\,(X-F_k)=0$, $\lim_{k\to+\infty}\text{Cap}^\sim\,(\tilde{X}-\tilde{F}_k)=0$ such that q is one-to-one from $X_0=\bigcup_{k=1}^{\infty}F_k$ onto $\tilde{X}_0=\bigcup_{k=1}^{\infty}\tilde{F}_k$ and its restriction to each F_k is a homeomorphism onto $\tilde{F}_k$. The domain of definition of a quasi-homeomorphism q will always be considered to be such an F_σ-set X_0. q and q^{-1} are then Borel measurable transformations between X_0 and $\tilde{X}_0$. Hence the images by q and q^{-1} of analytic sets are also analytic sets(8).

(8) Cf. [15, III, T11].

A quasi-homeomorphism q is said to be *capacity preserving* if, for any analytic set $A \subset X_0$,

$$\text{Cap}(A) = \text{Cap}^{\sim}(q(A))\text{(}^9\text{)}. \tag{2.1}$$

We will write as $X \cong \tilde{X}$ if there exists a capacity preserving quasi-homeomorphism between X and $\tilde{X}$.

LEMMA 2.1. *Consider the underlying spaces X, $\hat{X}$, $\tilde{X}$ of three regular D-spaces. If $X \cong \hat{X}$ and $\hat{X} \cong \tilde{X}$, then $X \cong \tilde{X}$.*

Proof. Suppose that X and $\hat{X}$ (resp. $\hat{X}$ and $\tilde{X}$) are related by the map q_1 (resp. q_2). For any $\varepsilon > 0$, there exist closed sets $F \subset X$, $\hat{F}_1 \subset \hat{X}$, $\hat{F}_2 \subset \hat{X}$ and $\tilde{F} \subset \tilde{X}$ satisfying the following: $\text{Cap}(X-F) < \varepsilon$, $\text{Cap}^{\wedge}(\hat{X}-\hat{F}_1) < \varepsilon$, $\text{Cap}^{\wedge}(\hat{X}-\hat{F}_2) < \varepsilon$, $\text{Cap}^{\sim}(\tilde{X}-\tilde{F}) < \varepsilon$ and q_1 (resp. q_2) is homeomorphic from F (resp. $\hat{F}_2$) onto $\hat{F}_1$ (resp. $\tilde{F}$). Put $F' = q_1^{-1}(\hat{F}_1 \cap \hat{F}_2)$ and $\tilde{F}' = q_2(\hat{F}_1 \cap \hat{F}_2)$. Then, $q = q_2 \cdot q_1$ is homeomorphic from F' onto $\tilde{F}'$ and

$$\begin{aligned} \text{Cap}(X-F') &\leqq \text{Cap}(X-F) + \text{Cap}(q_1^{-1}(\hat{F}_1 - \hat{F}_2)) \\ &= \text{Cap}(X-F) + \text{Cap}^{\wedge}(\hat{F}_1 - \hat{F}_2) < 2\varepsilon. \end{aligned}$$

In the same way, we have $\text{Cap}^{\sim}(\tilde{X}-\tilde{F}') < 2\varepsilon$. Thus, X and $\tilde{X}$ are quasi-homeomorphic by the map q. Evidently q is capacity preserving.

According to Definition 4.1 of [10], two D-spaces $(X, m, \mathcal{F}, \mathcal{E})$ and $(\tilde{X}, \tilde{m}, \tilde{\mathcal{F}}, \tilde{\mathcal{E}})$ are called equivalent if there exists an algebraic isomorphism Φ from $\mathcal{F} \cap L^{\infty}(X; m)$ onto $\tilde{\mathcal{F}} \cap L^{\infty}(\tilde{X}; \tilde{m})$ which preserves three kinds of metrics—L^{∞}-norm, L^2-norm and $\mathcal{E}$-norm. Notice that we always regard the normed algebra $\mathcal{F} \cap L^{\infty}(X; m)$ (resp. $\tilde{\mathcal{F}} \cap L^{\infty}(\tilde{X}; \tilde{m})$) as the set of equivalence classes in the sense that two functions of $\mathcal{F} \cap L^{\infty}(X; m)$ (resp. $\tilde{\mathcal{F}} \cap L^{\infty}(\tilde{X}; \tilde{m})$) are identified if they coincide m-a.e. ($\tilde{m}$-a.e.). The isomorphism Φ is viewed to transform each equivalence class to an equivalence class.

The isomorphism Φ can be uniquely extended to three kinds of transformations: a unitary map Φ_1 from $(\mathcal{F}, \mathcal{E}^{\alpha})$ onto $(\tilde{\mathcal{F}}, \tilde{\mathcal{E}}^{\alpha})$, a unitary map Φ_2 from $L_0^2(X)$ onto $L_0^2(\tilde{X})$ and an isometric isomorphism Φ_3 from $L_0^{\infty}(X)$ onto $L_0^{\infty}(\tilde{X})$, where $L_0^2(X)$ (resp. $L_0^{\infty}(X)$) is the closure of $\mathcal{F} \cap L^{\infty}(X)$ in the metric space $L^2(X)$ (resp. $L^{\infty}(X)$). $L_0^2(\tilde{X})$ and $L_0^{\infty}(\tilde{X})$ are defined in the same way. Suppose that two D-spaces are regular. Then Φ_1 is regarded as a unitary map from $(\mathcal{F}^*, \mathcal{E}^{\alpha})$ onto $(\tilde{\mathcal{F}}^*, \tilde{\mathcal{E}}^{\alpha})$, two functions being identified if they coincide q.e. Moreover we have in this case $L_0^2(X) = L^2(X)$ and $L_0^{\infty}(X) \supset C(X)$ because $\mathcal{F} \cap C(X)$ is dense in the metric space $L^2(X)$ (resp. $C(X)$) (see (5.4) of [10]). We also have $L_0^2(\tilde{X}) = L^2(\tilde{X})$ and $L_0^{\infty}(\tilde{X}) \supset C(\tilde{X})$.

Now we will state the theorem of this section.

(9) This definition does not depend on the choice of set X_0.

THEOREM 2.1. *Assume that two regular D-spaces* $(X, m, \mathscr{F}, \mathscr{E})$ *and* $(\tilde{X}, \tilde{m}, \tilde{\mathscr{F}}, \tilde{\mathscr{E}})$ *are equivalent under an isomorphism* Φ. *Then* $X \simeq \tilde{X}$ *under a capacity preserving quasi-homeomorphism q which has the following properties.*

(q.1) *q induces the extension of the given isomorphism* Φ: *put*

$$(\Phi^* u)(x) = u(q^{-1}\tilde{x}), \tag{2.2}$$

where u is a function on X and $\tilde{x}$ *is a point of* $\tilde{X}$ *for which* $u(q^{-1}\tilde{x})$ *makes sense, then* Φ^* *defines a transformation of functions which coincides on* $\mathscr{F}^*$ *with* Φ_1.

(q.2) *q is m-measure preserving*: $m(A)=m(q(A))$ *for any Borel set* $A \subset X_0$.

Before proceeding to the proof of Theorem 2.1, we need several notions related to a regular D-space $(X, m, \mathscr{F}, \mathscr{E})$. For a set $A \subset X$, we put

$$A' = \{x \in X; m(U(x) \cap A) \neq 0 \text{ for every neighbourhood } U(x) \text{ of } x\}. \tag{2.3}$$

Obviously A' is closed. We say a closed set F is *m-regular* if $F=F'$.

Consider any closed set F. Then F' is a closed set contained in F, $m(F-F')=0$ and $\text{Cap}(X-F')=\text{Cap}(X-F)$. We can see this in the same manner as in the proof of Theorem 1.2(ii). Furthermore F' is necessarily m-regular because $m(U(x) \cap F') \geqq m(U(x) \cap F)-m(F-F')>0$ for any neighborhood $U(x)$ of $x \in F'$.

Denote by X^Δ the compact space obtained from X by adjoining the point at infinity Δ. If X is already compact, we regard Δ to be isolated. For each set $A \subset X$, we put $A^\Delta = A \cup \Delta$ and consider this to be a topological subspace of X^Δ. A set $F \subset X$ is closed in X if and only if F^Δ is compact.

We further use the notion $|u|_A$ defined by $|u|_A = \sup_{x \in A} |u(x)|$ for a function u on $A \subset X$. Since m is everywhere dense, we have

$$\|u\|_\infty = |u|_X, \qquad u \in C(X). \tag{2.4}$$

Finally let $\{F_k\}$ be an increasing sequence of m-regular closed sets of X such that $\text{Cap}(X-F_k) \to 0$. Put $C(\{F_k\})=\{u;\ u$ is defined on $X_0 = \bigcup_{k=1}^\infty F_k$, $|u|_{X_0}$ is finite, the restriction of u to each F_k is continuous there and continuously extendable to F_k^Δ by setting $u(\Delta)=0\}$. Obviously $C(X) \subset C(\{F_k\}) \subset L^\infty(X; m)$. $C(\{F_k\})$ is a Banach algebra with norm $|\ |_{X_0}$. Further

$$\|u\|_\infty = |u|_{X_0}, \qquad u \in C(\{F_k\}). \tag{2.5}$$

This is clear from $m\text{-ess-sup}_{x \in F_k} |u(x)| = |u|_{F_k}$, $k=1, 2, \ldots$, which is due to the definition of m-regularity of F_k.

Each element u of $C(X)$ (resp. $C(\{F_k\})$) will always be regarded as a function on X^Δ (resp. X_0^Δ) by setting $u(\Delta)=0$.

LEMMA 2.2. *Let Q be any countable subcollection of* $\mathscr{F} \cap L^\infty$. *Then there exists an increasing sequence of m-regular closed sets* F_k *with* $\text{Cap}(X-F_k) \to 0$ *such that each element of Q has a unique modification belonging to* $C(\{F_k\})$.

Proof. For $Q=\{u_n\}$, $u_n \in \mathscr{F} \cap L^\infty$, $n=1, 2, \ldots$, we denote by u_n^* a quasi-continuous modification of u_n specified in Theorem 1.3. Thanks to the countable subadditivity of the capacity, we can select an increasing sequence of closed sets $F_k \subset X$ with $\text{Cap}\,(X-F_k) \to 0$ such that every function u_n^* has the following property: the restriction of u_n^* to each F_k is continuous there. By virtue of the special manner of the construction of u_n^* stated in Theorem 1.3, we may further assume that u_n^* is continuously extendable from F_k to F_k^Δ by setting $u_n^*(\Delta)=0$. In order to complete the proof of Lemma 2.2, we only have to replace F_k with its m-regularization F_k'. After the replacement, we can see by (2.5) that u_n^* becomes a unique element of $C(\{F_k\})$ which coincides with u_n m-a.e.

Now we will prove Theorem 2.1 by means of the next three lemmas.

LEMMA 2.3. *Under the assumption of Theorem* 2.1, *there exists an increasing sequence of m-regular closed sets $F_k \subset X$, $k=1, 2, \ldots$, with* $\lim_{k\to+\infty} \text{Cap}\,(X-F_k)=0$ *which satisfies the following. We put* $X_0=\bigcup_{k=1}^\infty F_k$.

(i) *There is an algebraic isomorphic and isometric transformation ψ from* $(C(\tilde{X}), |\ |_{\tilde{X}})$ *into* $(C(\{F_k\}), |\ |_{X_0})$. *ψ is just the restriction of the transform Φ_3^{-1} to* $C(\tilde{X})$.

(ii) *There is a mapping q from X_0^Δ into $\tilde{X}^{\tilde{\Delta}}$ such that $q(\Delta)=\tilde{\Delta}$ and the restriction of q to each F_k^Δ is continuous there. For each $x \in X^\Delta$, qx is characterized by*

$$\tilde{u}(qx) = (\psi\tilde{u})(x), \qquad \tilde{u} \in C(\tilde{X}). \tag{2.6}$$

Proof. (i) Since $\tilde{\mathscr{F}} \cap C(\tilde{X})$ is a dense subalgebra of $C(\tilde{X})$, we can find a countable subset $\tilde{C}_1 \subset \tilde{\mathscr{F}} \cap C(\tilde{X})$ such that the algebra $\mathscr{A}(\tilde{C}_1)$ generated by $\tilde{C}_1$ is dense in $C(\tilde{X})$ with maximum norm. Applying Lemma 2.2 to $\Phi^{-1}\tilde{C}_1 \subset \mathscr{F} \cap L^\infty(X; m)$, we get an increasing sequence $\{F_k\}$ of m-regular closed sets of X with $\text{Cap}\,(X-F_k) \to 0$ such that, for every $\tilde{u} \in \tilde{C}_1$, $\Phi^{-1}\tilde{u}$ has a unique modification belonging to $C(\{F_k\})$. Denote this modification by $\psi\tilde{u}$. The map ψ is extended to an algebraic isomorphism on $\mathscr{A}(\tilde{C}_1)$ which is consistent because of

$$|\tilde{u}|_{\tilde{X}} = |\psi\tilde{u}|_{X_0}, \qquad \tilde{u} \in \mathscr{A}(\tilde{C}_1), \tag{2.7}$$

where $X_0=\bigcup_{k=1}^\infty F_k$. The equality (2.7) follows from (2.4) and (2.5) as $|\tilde{u}|_{\tilde{X}} =\|\tilde{u}\|_{\tilde{\infty}}=\|\Phi_3^{-1}\tilde{u}\|_\infty=|\psi\tilde{u}|_{X_0}$. Now ψ is readily extended to a map from $C(\tilde{X})$ into $C(\{F_k\})$ satisfying conditions of the first statement of the present lemma.

(ii) For each $x \in X_0^\Delta$, $l_x(\tilde{u})=(\psi\tilde{u})(x)$, $\tilde{u} \in C(\tilde{X})$, is a character (a linear multiplicative functional) on $C(\tilde{X})$. Hence there exists a unique element $qx \in \tilde{X}^{\tilde{\Delta}}$ such as $l_x(\tilde{u})=\tilde{u}(qx)$, $\tilde{u} \in C(\tilde{X})$. Since $l_\Delta(\tilde{u})\equiv 0$, we have $q\Delta=\tilde{\Delta}$. Suppose that $x_n \in F_k^\Delta$ converges to $x \in F_k^\Delta$. Then $\tilde{u}(qx_n)=(\psi\tilde{u})(x_n)$ converges to $(\psi\tilde{u})(x)=\tilde{u}(qx)$, $\tilde{u} \in C(\tilde{X})$, which implies $qx_n \to qx$, $n \to \infty$, and hence the restriction of q to F_k^Δ is continuous there.

LEMMA 2.4. *In addition to the assumption of Theorem* 2.1, *we assume*

$$\Phi(\mathscr{F} \cap C(X)) \subset \tilde{\mathscr{F}} \cap C(\tilde{X}). \tag{2.8}$$

Then all the conclusions of Theorem 2.1 *are valid for the map q of Lemma* 2.3.

Proof. By assumption (2.8), there exists an algebraic isomorphic and isometric transformation φ from $C(X)$ into $C(\tilde{X})$: φ is just the restriction of the transform Φ_3 to $C(X)\subset L_0^\infty(X)$. Therefore there is a continuous map γ from $\tilde{X}^{\tilde{\Delta}}$ onto X^Δ such that, for each $\tilde{x}\in\tilde{X}^{\tilde{\Delta}}$, $\gamma\tilde{x}$ is characterized by

$$(2.9)\qquad u(\gamma\tilde{x}) = \varphi u(\tilde{x}), \qquad u\in C(X).$$

On the other hand, the map ψ of Lemma 2.3 is the inverse of φ in the sense that $\psi\varphi u(x)=u(x)$, $x\in X_0^\Delta$, for every $u\in C(X)$. Indeed $u\in C(X)$ and $\psi\varphi u$ $(=\Phi_3^{-1}\cdot\Phi_3 u)$ $\in C(\{F_k\})$ are in the same class of $L_0^\infty(X)$ and so they are identical on X_0^Δ by virtue of (2.5). Hence, in view of (2.6) and (2.9), the map γ is the inverse of q of Lemma 2.3:

$$(2.10)\qquad \gamma\cdot qx = x, \qquad x\in X_0^\Delta.$$

In particular $q(X_0)\subset\tilde{X}$ because $\gamma(\tilde{\Delta})=\Delta$. We put

$$(2.11)\qquad \tilde{F}_k = q(F_k), \quad k = 1, 2, \dots, \qquad \tilde{X}_0 = \bigcup_{k=1}^{\infty} \tilde{F}_k.$$

Since the restriction of q to the compactum $F_k^\Delta\subset X^\Delta$ is a continuous map, its image $q(F_k^\Delta)=\tilde{F}_k^{\tilde{\Delta}}$ is a compact set of $\tilde{X}^{\tilde{\Delta}}$. $\tilde{F}_k$ is therefore a closed subset of $\tilde{X}$.

From now on, let us restrict the domain of the definition of q (resp. γ) to X_0 (resp. $\tilde{X}$) and study the detailed properties they possess.

First of all we know from (2.10) that q is one-to-one from X_0 onto $\tilde{X}_0$ and its restriction to each F_k is a homeomorphism onto $\tilde{F}_k$, the inverse being γ.

We will prove that q is measure preserving between X_0 and $\tilde{X}_0$. It is enough to show

$$(2.12)\qquad m(q^{-1}(\tilde{K})) = \tilde{m}(\tilde{K})$$

for any compact set $\tilde{K}$ contained in some $\tilde{F}_k$. To see (2.12), choose a sequence $\tilde{u}_n\in\tilde{\mathscr{F}}\cap C(\tilde{X})$ converging to the indicator function of $\tilde{K}$ everywhere on $\tilde{X}$ as well as in $L^2(\tilde{X};\tilde{m})$-sense. This is possible because $\tilde{\mathscr{F}}\cap C(\tilde{X})$ is a lattice and a dense subset of $C(\tilde{X})$. Then $\psi\tilde{u}_n(x)=\tilde{u}_n(qx)$ converges to the indicator function of $q^{-1}(\tilde{K})$ $\subset F_k$ for each $x\in X_0$ and hence m-a.e. on X. Since ψ on $\tilde{\mathscr{F}}\cap C(\tilde{X})$ is a modification of Φ^{-1} which preserves L^2-norm, $\{\psi\tilde{u}_n\}$ also forms a Cauchy sequence in $L^2(X;m)$ and further

$$\tilde{m}(\tilde{K}) = \lim_{n\to+\infty}(\tilde{u}_n, \tilde{u}_n)_{\tilde{X}} = \lim_{n\to+\infty}(\psi\tilde{u}_n, \psi\tilde{u}_n)_X = m(q^{-1}(\tilde{K})),$$

getting (2.12).

Exactly in the same way as above, we can prove

$$(2.13)\qquad m(K) = \tilde{m}(\gamma^{-1}(K))$$

for any compact set $K \subset X$. Moreover, combining (2.12) and (2.13), we come to the conclusion that

$$\tilde{m}(\gamma^{-1}(F_k) - \tilde{F}_k) = 0, \qquad k = 1, 2, \ldots. \tag{2.14}$$

Indeed, fix a number k and take any compact set $\tilde{K} \subset \gamma^{-1}(F_k) - \tilde{F}_k$. Then put $K = \gamma(\tilde{K})$ and $\tilde{K}_1 = q(K)$. K and $\tilde{K}_1$ are compact sets in F_k and $\tilde{F}_k$ respectively. Since $\gamma^{-1}(K) \supset \tilde{K} \cup \tilde{K}_1$, we have

$$\tilde{m}(\tilde{K}_1) = m(q^{-1}(\tilde{K}_1)) = m(K) = \tilde{m}(\gamma^{-1}(K)) \geqq \tilde{m}(\tilde{K} \cup \tilde{K}_1)$$

from which follows $\tilde{m}(\tilde{K}) = 0$.

Next we have to show

$$\tilde{m}(\gamma^{-1}(\Delta)) = 0. \tag{2.15}$$

Observe that $\gamma^{-1}(\Delta) = \{\tilde{x} \in \tilde{X};\ \varphi u(\tilde{x}) = 0 \text{ for every } u \in C(X)\}$. Notice further that, since $\mathscr{F} \cap C(X)$ is dense in $L^2(X; m)$, the space $\varphi(\mathscr{F} \cap C(X))$ $(= \Phi(\mathscr{F} \cap C(X)))$ is dense in $L^2(\tilde{X}; \tilde{m})$ $(= \Phi_2(L^2(X; m)))$. Hence for any compactum $\tilde{K} \subset \gamma^{-1}(\Delta)$ there is a sequence $u_n \in \mathscr{F} \cap C(X)$ such that φu_n converges $\tilde{m}$-a.e. on $\tilde{X}$ to the indicator function of $\tilde{K}$. But $\varphi u_n(\tilde{x}) = 0$, $\tilde{x} \in \tilde{K}$, $n = 1, 2, \ldots$, and we have $\tilde{m}(\tilde{K}) = 0$ proving (2.15).

We are in a position to complete the proof of Lemma 2.4. Let us derive the inequality

$$\text{Cap}^{\sim}(\tilde{K}) \leqq \text{Cap}(K), \tag{2.16}$$

where $\tilde{K}$ is any compact subset of $\gamma^{-1}(X)$ and $K = \gamma(\tilde{K})$. Since γ is continuous, K is a compact set of X. Consider the sets $\mathscr{C}_K = \{u \in \mathscr{F} \cap C(X);\ u \geqq 1 \text{ on } K\}$ and $\tilde{\mathscr{C}}_{\tilde{K}} = \{\tilde{u} \in \tilde{\mathscr{F}} \cap C(\tilde{X});\ \tilde{u} \geqq 1 \text{ on } \tilde{K}\}$, and observe the inclusion $\varphi(\mathscr{C}_K) \subset \tilde{\mathscr{C}}_{\tilde{K}}$. Since φ coincides with Φ on $\mathscr{F} \cap C(X)$ and Φ preserves $\mathscr{E}^{\alpha_0}$-norm, we get from (1.29) that

$$\begin{aligned} \text{Cap}(K) &= \inf_{u \in \mathscr{C}_K} \mathscr{E}^{\alpha_0}(u, u) = \inf_{u \in \mathscr{C}_K} \tilde{\mathscr{E}}^{\alpha_0}(\varphi u, \varphi u) \\ &= \inf_{\tilde{u} \in \varphi(\mathscr{C}_K)} \tilde{\mathscr{E}}^{\alpha_0}(\tilde{u}, \tilde{u}) \geqq \text{Cap}^{\sim}(\tilde{K}). \end{aligned}$$

We can now show that q is capacity preserving on X_0. On account of Theorem 1.1(b) and (1.5), it suffices to prove for any compact subset $K \subset F_k$ with a fixed k,

$$\text{Cap}(K) = \text{Cap}^{\sim}(\tilde{K}), \tag{2.17}$$

where $\tilde{K} = q(K)$. Noting the inclusion

$$\psi(\tilde{\mathscr{C}}_{\tilde{K}}) \subset \{u \in \mathscr{F} \cap C(\{F_k\});\ u \geqq 1 \text{ on } K\} \subset \mathscr{L}_K^*,$$

we have

$$\begin{aligned} \text{Cap}^{\sim}(\tilde{K}) &= \inf_{\tilde{u} \in \tilde{\mathscr{C}}_K} \tilde{\mathscr{E}}^{\alpha_0}(\tilde{u}, \tilde{u}) = \inf_{\tilde{u} \in \tilde{\mathscr{C}}_{\tilde{K}}} \mathscr{E}^{\alpha_0}(\psi\tilde{u}, \psi\tilde{u}) \\ &= \inf_{u \in \psi(\tilde{\mathscr{C}}_{\tilde{K}})} \mathscr{E}^{\alpha_0}(u, u) \geqq \text{Cap}(K), \end{aligned}$$

which combined with (2.16), proves (2.17).

For the proof that q is a capacity preserving quasi-homeomorphism and measure preserving, it only remains to show

$$\text{(2.18)} \qquad \text{Cap}^{\sim}(\tilde{X}-\tilde{X}_0) = 0.$$

Choose any $\varepsilon>0$ and fix a number k such as $\text{Cap}(X-F_k)<\varepsilon$. We are going to show

$$\text{(2.19)} \qquad \text{Cap}^{\sim}(\tilde{X}-\tilde{F}_k) < \varepsilon.$$

Observe that $\tilde{X}-\tilde{F}_k$ is an open set of $\tilde{X}$ consisting of three disjoint parts: $\tilde{X}-\tilde{F}_k =\gamma^{-1}(X-F_k)+(\gamma^{-1}(F_k)-\tilde{F}_k)+\gamma^{-1}(\Delta)$. By (2.14) and (2.15), $\tilde{m}$-measures of the last two terms of the right-hand side are zero. $\gamma^{-1}(X-F_k)$ is open and contained in $\tilde{X}-\tilde{F}_k$. Hence by definition (0.1) of the capacity, we have

$$\text{(2.20)} \qquad \text{Cap}^{\sim}(\tilde{X}-\tilde{F}_k) = \text{Cap}^{\sim}(\gamma^{-1}(X-F_k)).$$

On the other hand (2.16) and (1.5) mean the following:

$$\text{Cap}^{\sim}(\gamma^{-1}(X-F_k)) = \sup_{\tilde{K}} \text{Cap}^{\sim}(\tilde{K}) \leqq \sup_{K=\gamma(\tilde{K})} \text{Cap}(K)$$
$$\leqq \text{Cap}(X-F_k) < \varepsilon,$$

the supremum being taken for all compact set $\tilde{K}\subset\gamma^{-1}(X-F_k)$. Thus we arrive at (2.19).

It is easy to see that our q possesses the property (q.1) of Theorem 2.1: (2.9) and (2.10) mean, for $u\in\mathscr{F}\cap C(X)$,

$$\text{(2.21)} \qquad \Phi^* u = \Phi_1 u \quad \text{q.e.},$$

which can be extended to $\mathscr{F}^*$ by virtue of Lemma 1.2. We have completed the proof of Lemma 2.4.

LEMMA 2.5. *Under the assumption of Theorem* 2.1, *there exists a regular D-space* $(\hat{X}, \hat{m}, \hat{\mathscr{F}}, \hat{\mathscr{E}})$ *satisfying the following*:

(1) *Both the given regular D-spaces are equivalent to* $(\hat{X}, \hat{m}, \hat{\mathscr{F}}, \hat{\mathscr{E}})$ *by isomorphisms, say,* Φ' *and* Φ''. Φ *is equal to* $(\Phi'')^{-1}\cdot\Phi'$.

(2) $\Phi'(\mathscr{F}\cap C(X))\subset\hat{\mathscr{F}}\cap C(\hat{X})$, $\Phi''(\tilde{\mathscr{F}}\cap C(\tilde{X}))\subset\hat{\mathscr{F}}\cap C(\hat{X})$.

Proof. This lemma is an application of the regular representation theorem of [10].

First of all we will establish the inclusion

$$\text{(2.22)} \qquad \Phi_3(C_0(X)) \subset L^1(\tilde{X};\tilde{m}), \qquad \Phi_3^{-1}(C_0(\tilde{X})) \subset L^1(X;m).$$

It is enough to prove the first. For any function $u\in C_0(X)$, there is a nonnegative function $v\in\mathscr{F}\cap C(X)$ such as $v\geqq\sqrt{|u|}$ on X. Since Φ_3 is a lattice isomorph as well as an algebraic isomorph and since $\Phi_3 v\in\tilde{\mathscr{F}}\subset L^2(\tilde{X};\tilde{m})$, we have $\Phi_3(\sqrt{|u|})\in L^2(\tilde{X};\tilde{m})$ and $|\Phi_3 u|=(\Phi_3(\sqrt{|u|}))^2\in L^1(\tilde{X};\tilde{m})$.

Now denote by L the closed subalgebra in $L_0^\infty(X)$ generated by $C(X)\cup\Phi_3^{-1}C(\tilde{X})$. Then L satisfies the condition (C) of [10, §5]. (C.1) and (C.2) are clear. By (2.22), $L^1(X;m)\cap L$ includes the algebra generated by $C_0(X)\cup\Phi_3^{-1}(C_0(\tilde{X}))$ which is dense in L, proving (C.3). Therefore we can take as $(\hat{X}, \hat{m}, \hat{\mathscr{F}}, \hat{\mathscr{E}})$ the regular representation of $(X, m, \mathscr{F}, \mathscr{E})$ with respect to L (Theorem 2 of [10]). The algebraic isomorphism Φ' associated with this representation is translating $\mathscr{F} \cap L$ onto $\hat{\mathscr{F}} \cap C(\hat{X})$ getting the first inclusion of (2). The second is also clear because $(\hat{X}, \hat{m}, \hat{\mathscr{F}}, \hat{\mathscr{E}})$ is the regular representation of $(\tilde{X}, \tilde{m}, \tilde{\mathscr{F}}, \tilde{\mathscr{E}})$ with respect to $\tilde{L}$ under the isomorphism $\Phi' \cdot \Phi^{-1}$, $\tilde{L}$ being the closed subalgebra of $L_0^\infty(\tilde{X}; \tilde{m})$ generated by $\Phi(C(X)) \cup C(\tilde{X})$.

Proof of Theorem 2.1. Lemmas 2.1, 2.4 and 2.5 admit us to conclude that $X \cong \tilde{X}$ under a capacity preserving quasi-homeomorphism q possessing the property (q.1). (q.2) is a consequence of (q.1) because Φ^* is L^2-norm preserving from $\mathscr{F}^*$ onto $\tilde{\mathscr{F}}^*$. The proof of Theorem 2.1 is complete.

If two D-spaces are equivalent and if one of them is regular, then it is said to be a regular representation of the other.

COROLLARY TO THEOREM 2.1. *The underlying space of a regular representation of a given D-space is unique up to a capacity preserving quasi-homeomorphism.*

3. **Potential theory for symmetric Ray processes.** Let $(X, m, \mathscr{F}, \mathscr{E})$ be a strongly regular D-space and $\{R_\alpha(x, E), \alpha > 0\}$ be its associated symmetric Ray resolvent kernel on X. For a function u on X, put

$$R_\alpha u(x) = \int_X R_\alpha(x, dy)u(y), \qquad x \in X, \tag{3.1}$$

whenever the right-hand side makes sense. The images by R_α of Borel (universally) measurable functions are also Borel (universally) measurable. By definition, $(\mathscr{F}, \mathscr{E})$ is generated by $\{R_\alpha(x, E), \alpha > 0\}$, that is, $R_\alpha(L^2(X; m) \cap C(X)) \subset \mathscr{F} \cap C(X)$ and $R_\alpha u$, $u \in L^2(X; m) \cap C(X)$, satisfies the equation

$$\mathscr{E}^\alpha(R_\alpha u, v) = (u, v)_X \tag{3.2}$$

for every $v \in \mathscr{F}$. Moreover $\mathscr{F} \cap C(X)$ includes a set C_1 attached to the Ray resolvent (Definition 2.5 of [10]).

3.1. *Supermedian and excessive functions.*

LEMMA 3.1. *For any nonnegative measurable function u of $L^2(X; m)$, the function $R_\alpha u$ defined by* (3.1) *belongs to the space $\mathscr{F}^*$ and satisfies the equation* (3.2) *for each* $\alpha > 0$.

We will prove this lemma by making use of the following proposition:

PROPOSITION. *Suppose that a set H of real-valued functions on X satisfies the next conditions.*

(H.1) *If $f_1, f_2 \in H$ and $c_1 f_1 + c_2 f_2 \geqq 0$ with some constants c_1, c_2, then $c_1 f_1 + c_2 f_2 \in H$.*

(H.2) *If $f_n \in H$ increases to $f \in L^2(X; m)$, then $f \in H$.*

(H.3) $C_0^+(X) \subset H$.

Then H contains all nonnegative Borel measurable functions of $L^2(X; m)$.

For the proof of the Proposition, it is enough to take any open set $E \subset X$ with compact closure and consider the class S of all Borel subsets of E whose indicator functions are in H. S contains all open subsets of E. Since S is a λ-system relative to E, it contains all Borel subsets of E (Lemma 0.1 of [7]). The rest of the proof is clear.

Proof of Lemma 3.1. Let H be the set of all nonnegative Borel measurable functions u of $L^2(X; m)$ such that $R_\alpha u$ belongs to $\mathscr{F}^*$ and satisfies equation (3.2) for each $\alpha > 0$. H satisfies (H.3) because $L^2(X; m) \cap C(X) \subset H$. Suppose that $u_n \in H$ increases to $u \in L^2(X; m)$. Then

$$\mathscr{E}^\alpha(R_\alpha u_n - R_\alpha u_m, R_\alpha u_n - R_\alpha u_m) = (u_n - u_m, R_\alpha(u_n - u_m))_X$$
$$\leqq (1/\alpha)(u_n - u_m, u_n - u_m)_X \to 0, \qquad n, m \to +\infty.$$

By virtue of Lemma 1.2, a subsequence of $R_\alpha u_n \in \mathscr{F}^*$ converges to a function $\mathscr{F}^*$ q.e. on X as well as in $\mathscr{E}^\alpha$-norm. However $R_\alpha u_n(x)$ converges to $R_\alpha u(x)$ for each $x \in X$. Therefore $R_\alpha u$ belongs to $\mathscr{F}^*$ and satisfies (3.2). Condition (H.2) is verified. Thus we see by the proposition that Lemma 3.1 is valid for any nonnegative Borel measurable function u of $L^2(X; m)$.

Finally let u be a nonnegative universally measurable function of $L^2(X; m)$. There exist nonnegative Borel measurable functions u_1 and u_2 such that $u_1 \leqq u \leqq u_2$ on X and $u_1 = u_2$ m-a.e. on X. We have $R_\alpha u_1 \leqq R_\alpha u \leqq R_\alpha u_2$ on X. Further, by the symmetry of R_α,

$$0 \leqq \int_X (R_\alpha u_2 - R_\alpha u_1)(x) m(dx) = \int_X R_\alpha 1(x)(u_2(x) - u_1(x)) m(dx)$$
$$\leqq \frac{1}{\alpha}\int_X (u_2(x) - u_1(x)) m(dx) = 0,$$

which implies $R_\alpha u_1 = R_\alpha u_2$ m-a.e. on X. Since $R_\alpha u_1$ and $R_\alpha u_2$ are quasi-continuous, we see by Theorem 1.2(ii) that $R_\alpha u_1 = R_\alpha u = R_\alpha u_2$ q.e. on X and consequently $R_\alpha u \in \mathscr{F}^*$. The equation (3.2) for u can be derived from that for u_1. The proof of Lemma 3.1 is complete.

DEFINITION 3.1. A function u on X is said to be (α_0-) supermedian if the following two conditions are satisfied:

(3.3) u is nonnegative and universally measurable,

(3.4) $\beta R_{\beta+\alpha_0} u(x) \leqq u(x)$, $x \in X$; $\beta > 0$.

A supermedian function u is said to be (α_0-) excessive if

(3.5) $\lim_{\beta \to +\infty} \beta R_{\beta+\alpha_0} u(x) = u(x)$, $x \in X$.

If u is nonnegative and universally measurable, then $R_{\alpha_0} u$ is excessive.

If u is a nonnegative universally measurable function and $\lim_{\beta \to \infty} \beta R_{\beta+\alpha_0} u(x) = \tilde{u}(x)$ exists for every $x \in X$, then the limit function $\tilde{u}$ is said to be the *regularization of* u. Every supermedian function has its regularization which turns out to be excessive.

THEOREM 3.1. *If a function u is nonnegative universally measurable, belongs to the space $\mathscr{F}$ and has its regularization $\tilde{u}$, then $\tilde{u}$ is a quasi-continuous modification of u. In particular any excessive function belonging to $\mathscr{F}$ is an element of $\mathscr{F}^*$.*

Proof. We see by Lemma 3.1, that $R_\alpha u \in \mathscr{F}^*$ and the operator R_α applied to u is identical with L^2-resolvent associated with $(\mathscr{F}, \mathscr{E})$. Hence by taking Lemma 2.1(iii) of [10] and Lemma 1.2 of the present paper into account, we see that a subsequence of $\beta R_{\beta+\alpha_0}u$ converges q.e. on X to a quasi-continuous modification of u. Thus we get Theorem 3.1.

REMARK 3.1. Every supermedian function belonging to the space $\mathscr{F}^*$ is quasi-supermedian in the sense of subsection 1.3. According to Theorem 3.1, every excessive function belonging to the space $\mathscr{F}$ is quasi-supermedian.

3.2. *The associated Ray process and the branch set.* Put $\bar{X} = X \cup \partial$ where ∂ is adjoined to X as the point at infinity if X is noncompact and as an isolated point if X is compact. Extend the kernel $\{R_\alpha(x, E), \alpha > 0\}$ to $\bar{X}$ in the manner of Remark 2.2(ii) of [10]. Then the extended kernel becomes a conservative Ray resolvent over the compactum $\bar{X}$ to which the original set C_1 is still attached if we extend each function u of C_1 to $\bar{X}$ by setting $u(\partial) = 0$.

Therefore the results of D. Ray [18, Theorem I, II and III] concerning resolvents on compact spaces and their improvements by H. Kunita and T. Watanabe [13, §2] can be brought over to our situation and we get the following conclusions.

The first conclusion is about the branch set. For each $x \in X$, the measure $\alpha R_\alpha(x, \cdot)$ on X converges to a unique substochastic measure $\mu(x, \cdot)$:

$$\lim_{\alpha \to +\infty} \alpha R_\alpha f(x) = \int_X \mu(x, dy) f(y) \quad \text{for any } f \in C(X).$$

A point $x \in X$ is said to be a branch point if the measure $\mu(x, \cdot)$ is not a unit distribution at x. The set X_b of all branch points of X is called the branch set. The measure $\mu(x, \cdot)$ is not supported by X_b for any $x \in X$. X_b is characterized as follows:

$$X_b = \bigcup_{g \in C_1'} \left\{ x; g(x) > \int_X g(y) \mu(x, dy) \right\}, \tag{3.6}$$

where $C_1' = \{g = f \wedge c; f \in C_1, c \text{ is any positive rational number}\}$.

The second is about the transition function. There is a unique sub-Markov transition function $P_t(x, E)$ on X such that

$$P_t f(x) = \int_X P_t(x, dy) f(y), \qquad f \in C(X), x \in X, \tag{3.7}$$

defines a right continuous function of $t > 0$ with

$$\int_0^{+\infty} e^{-\alpha t} P_t f(x)\, dt = R_\alpha f(x), \qquad \alpha > 0. \tag{3.8}$$

The third is the existence of a right continuous strong Markov process on X with transition function P_t. This is called the *Ray process* associated with $\{R_\alpha(x, E), \alpha>0\}$. We can adopt as the Ray process the *canonical realization* $M=(W, \mathscr{B}_t^0, P_x)$ of $\{R_\alpha(x, E), \alpha>0\}$ in the following sense(10). W consists of paths $\omega=\omega(t)$, $t \in [0, +\infty)$, taking values in $\bar{X}$ such that $\omega(t)$ is right continuous in $t \in [0, +\infty)$, has the left limit at any $t \in (0, +\infty)$ and stays at ∂ after its lifetime $\zeta(\omega)$. The tth coordinate $\omega(t)$ of ω is denoted by $X_t(\omega)$. $\zeta(\omega)$ is defined by $\inf\{t \geqq 0, X_t(\omega)=\partial\}$. $\mathscr{B}_t^0$ is the σ-field of subsets of W generated by $\{X_s \in E\}$ with $0 \leqq s \leqq t$ and Borel set $E \subset \bar{X}$. For each x, P_x is a unique probability measure on $\mathscr{B}^0=\bigvee_{t\geqq 0} \mathscr{B}_t^0$ which satisfies

$$
\begin{aligned}
&P_x(X_{t_1} \in E_1, X_{t_2} \in E_2, \ldots, X_{t_n} \in E_n) \\
(3.9) \qquad &= \int_{E_1}\int_{E_2}\cdots\int_{E_n} \bar{P}_{t_1}(x, dy_1)\bar{P}_{t_2-t_1}(y_1, dy_2)\cdots\bar{P}_{t_n-t_{n-1}}(y_{n-1}, dy_n)
\end{aligned}
$$

for $0<t_1<t_2<\cdots<t_n$ and Borel sets $E_1, E_2, \ldots, E_n$ of $\bar{X}$, where $\bar{P}_t(x, E) =P_t(x, E \cap X)+(1-P_t(x, X))\delta_{\{\partial\}}(E)$.

The Ray process $M=(W, \mathscr{B}_t^0, P_x)$ has the following properties:

(M.1) $P_x(X_0 \in E)=\mu(x, E)$ for $x \in X$ and Borel $E \subset X$. Define $\mathscr{B}$ to be the completion of $\mathscr{B}^0$ with respect to the family of measures $\{P\mu(\cdot)=\int_{\bar{X}} \mu(dx)P_x(\cdot)$; μ is a finite measure on $\bar{X}\}$ and $\mathscr{B}_t$ to be the completion of $\mathscr{B}_t^0$ in $\mathscr{B}$ with respect to the same family(11).

(M.2) Strong Markov property with respect to the augmented fields $\{\mathscr{B}_t\}$: for any stopping time T, $t>0$ and Borel $E \subset \bar{X}$, $P_x(X_{T+t} \in E|\mathscr{B}_T)=P_t(X_T, E)$, P_x-almost everywhere for each $x \in X$. Here, T is said to be a stopping time if $\{T \leqq t\} \in \mathscr{B}_t$ for any $t \geqq 0$ and $\mathscr{B}_T$ is defined as the collection of those sets $\Lambda \in \mathscr{B}$ such that $\{T \leqq t\} \cap \Lambda \in \mathscr{B}_t$ for all t.

(M.3) Quasi-left continuity in the restricted sense: if stopping times T_n increase to T, then $X_T=\lim_{n\to+\infty} X_{T_n}$ P_x-almost everywhere on the set

$$\left\{T < +\infty, \lim_{n\to+\infty} X_{T_n} \in \bar{X}-X_b\right\} \quad \text{for each } x \in X.$$

(M.4) $P_x(X_t \in \bar{X}-X_b \text{ for any } t \geqq 0)=1$, $x \in X$.

For a set $A \subset X$, we define the first entry time $\sigma_A(\omega)$ and the hitting time $\sigma'_A(\omega)$ by

$$
(3.10) \qquad
\begin{aligned}
\sigma_A(\omega) &= \inf\{t \geqq 0; X_t \in A\}, \\
\sigma'_A(\omega) &= \inf\{t > 0; X_t \in A\}.
\end{aligned}
$$

We define $\sigma_A(\omega)$ or $\sigma'_A(\omega)$ to be $\zeta(\omega)$ when the set in the braces is empty. If A is analytic, then random times σ_A, σ'_A and $\tau=\inf\{t>0; X_{t-} \in A\}$ are $\mathscr{B}_t$-stopping times. We can see this by [15, Chapter IV, T47 and 53].

---

(10) See [16, XIII] for the canonical realization of a Feller semigroup.

(11) See R. M. Blumenthal and R. K. Getoor [2, p. 26] for the terminology.

THEOREM 3.2. *The branch set X_b is polar in the sense of* §1.

Proof. By Lemma 4.1 of [10] and by the inclusion $C_1 \subset \mathscr{F} \cap C(X)$, we see that $\mathscr{F} \cap C(X)$ includes the countable collection C_1' which appeared in (3.6). The members of C_1' will be numbered as $g_1, g_2, \ldots, g_k, \ldots$. Put $X_{\alpha,n}^k = \{x; g_k(x) - \alpha R_{\alpha+\alpha_0} g_k(x) > 1/n\}$, which includes the set $\{x; g_k(x) > \int_X g_k(y)\mu(x, dy) + 1/n\}$ for every $\alpha > 0$. Lemma 2.1(iii) of [10] and the estimate (1.6) lead us to

$$\text{Cap}\,(X_{\alpha,n}^k) \leqq n^2 \mathscr{E}^{\alpha_0}(g_k - \alpha R_{\alpha+\alpha_0} g_k, g_k - \alpha R_{\alpha+\alpha_0} g_k) \to 0, \qquad \alpha \to +\infty.$$

For any $\varepsilon > 0$, take $\varepsilon_k > 0$ such as $\sum_{k=1}^{+\infty} \varepsilon_k < \varepsilon$. For each k and n, choose α such that the open set $Y_n^k = X_{\alpha,n}^k$ has the capacity less than $\varepsilon_k / 2^{n+1}$. Now X_b is included in the open set $\bigcup_k \bigcup_n Y_n^k$ whose capacity is less than ε, as was to be proved.

3.3. *Symmetry of the process.* Here we will observe how the behaviours of the associated Ray process reflect the symmetry of our Ray resolvent. It is clear that the symmetry of $\{R_\alpha(x, E), \alpha > 0\}$ implies the symmetry of the associated transition function $\{P_t(x, E), t > 0\}$: for any $t > 0$ and nonnegative Borel measurable functions f and g on X

$$\int_X P_t f(x) \cdot g(x) m(dx) = \int_X f(x) \cdot P_t g(x) m(dx) \leqq +\infty. \tag{3.11}$$

LEMMA 3.2. *For $0 < t_1 < \cdots < t_{n-1} < t_n$ and nonnegative Borel measurable functions $f_0, f_1, \ldots, f_{n-1}, f_n$ on X,*

$$\begin{aligned} &\int_X E_x(f_0(X_0) f_1(X_{t_1}) \cdots f_{n-1}(X_{t_{n-1}}) f_n(X_{t_n})) m(dx) \\ &\qquad = \int_X E_x(f_n(X_0) f_{n-1}(X_{t_n - t_{n-1}}) \cdots f_1(X_{t_n - t_1}) f_0(X_{t_n})) m(dx). \end{aligned} \tag{3.12}$$

Proof. Notice that $P_x(X_0 = x) = 1$ for m-a.e. $x \in X$ because of (**M**.1) and Theorem 3.2 of the preceding subsection and Theorem 1.2(i). We will prove this lemma by induction. Suppose that (3.12) holds for a given n. Then

$$\begin{aligned} &\int_X E_x(f_0(X_0) \cdots f_n(X_{t_n}) f_{n+1}(X_{t_{n+1}})) m(dx) \\ &\qquad = \int_X E_x(f_0(X_0) \cdots (f_n \cdot P_{t_{n+1} - t_n} f_{n+1})(X_{t_n})) m(dx) \\ &\qquad = \int_X P_{t_{n+1} - t_n} f_{n+1}(x) E_x(f_n(X_0) f_{n-1}(X_{t_n - t_{n-1}}) \cdots f_0(X_{t_n})) m(dx) \end{aligned}$$

which is equal to

$$\begin{aligned} &\int_X f_{n+1}(x) P_{t_{n+1} - t_n}(E \cdot (f_n(X_0) f_{n-1}(X_{t_n - t_{n-1}}) \cdots f_0(X_{t_n})))(x) m(dx) \\ &\qquad = \int_X E_x(f_{n+1}(X_0) f_n(X_{t_{n+1} - t_n}) \cdots f_0(X_{t_{n+1}})) m(dx) \end{aligned}$$

by virtue of (3.11) and the Markov property. Thus (3.12) holds for $n+1$, completing the proof of Lemma 3.2.

Since ∂ is not a branch point of M, (M.3) implies as in [2, (9.3)] that the left limits of sample paths must lie in X up to their lifetimes almost surely. In the following we assume without loss of generality that every $\omega \in W$ has the property that $X_{t-}(\omega) \in X$ for every $t < \zeta(\omega)$.

Fix a positive number $c > 0$. Denote by $\mathfrak{L}$ the set of all functions $\varphi(t)$ $(0 \leqq t \leqq c)$ taking values in X. The (time reversal) transformation q of the space $\mathfrak{L}$ is defined by $q\varphi(t) = \varphi(c-t)$, $0 \leqq t \leqq c$. For $\omega \in W$ such as $X_{c-}(\omega) \in X$, we define $\nu_r\omega$ and $\nu_l\omega \in \mathfrak{L}$ by

$$(\nu_r\omega)(t) = X_t(\omega), \quad 0 \leqq t < c, \qquad (\nu_l\omega)(t) = X_0(\omega), \quad t = 0,$$
$$= X_{c-}(\omega), \quad t = c; \qquad = X_{t-}(\omega), \quad 0 < t \leqq c.$$

Finally we put for $\Gamma \subset \{X_{c-} \in X\}$,

$$\gamma\Gamma = \nu_l^{-1} q \nu_r \Gamma \text{ [12]}. \tag{3.13}$$

Denote by $\mathscr{B}_{(0,c)}$ the restriction to $\{X_{c-} \in X\}$ of the σ-field $\bigvee_{t<c} \mathscr{B}_t^0$.

THEOREM 3.3[13]. *If* $\Gamma \subset \mathscr{B}_{(0,c)}$, *then* $\gamma\Gamma \in \mathscr{B}_{(0,c)}$ *and*

$$\int_X P_x(\gamma\Gamma)m(dx) = \int_X P_x(\Gamma)m(dx) \leqq +\infty. \tag{3.14}$$

Proof. It suffices to prove the theorem for the set

$$\Gamma = \{X_0 \in E_0, X_{t_1} \in E_1, \ldots, X_{t_{n-1}} \in E_{n-1}, X_{c-} \in E_n\}, \tag{3.15}$$

where $0 < t_1 < t_2 < \cdots < t_{n-1} < c$ and $E_0, \ldots, E_n$ are Borel subsets of X. Clearly

$$\gamma\Gamma = \{X_0 \in E_n, X_{(c-t_{n-1})-} \in E_{n-1}, \ldots, X_{(c-t_1)-} \in E_1, X_{c-} \in E_0\}.$$

By Lemma 3.2 we have for sufficiently small $\varepsilon > 0$ and $\delta > 0$,

$$\int_X E_x(f_0(X_0)f_1(X_{t_1+\varepsilon})\cdots f_{n-1}(X_{t_n+\varepsilon})f_n(X_{c-\delta}))m(dx)$$
$$= \int_X E_x(f_n(X_0)f_{n-1}(X_{c-t_{n-1}-\varepsilon-\delta})\cdots f_1(X_{c-t_1-\varepsilon-\delta})f_0(X_{c-\delta}))m(dx).$$

Assume that $f_0, f_1, \ldots, f_n \in C_0^+(X)$ and let ε and δ tend to zero. Then after a routine procedure we get the equality (3.14) for Γ of (3.15).

([12]) The operator γ was introduced by E. B. Dynkin [7, IV, §4] in connection with the multidimensional Brownian motion. The present author used a similar notion in the analysis of a reflecting Brownian motion [8]. However the notion γ defined in [8, p. 206] was insufficient for the situation there and he likes to correct it here: it must be replaced by the present definition (3.13).

([13]) Cf. Theorem 4.12 of [7].

Here we give two applications of Theorem 3.3.

According to the proof of IV, T52 of P. A. Meyer [15] we observe that, for any Borel set $B\subset X$ and $t>0$, the set $\{\sigma'_B<t\}$ is in the completion of the σ-field $\mathscr{B}_t^0$ relative to $P\mu$, μ being an arbitrary probability measure on X. This fact will be used in the proof of the following theorem:

THEOREM 3.4([14]). *For q.e.* $x\in X$,

$$P_x(X_{t-}\in X-X_b \textit{ for every } t\in(0,\zeta))=1. \tag{3.16}$$

COROLLARY. *If* T_n *are increasing stopping times with limit* T, *then*

$$P_x\Big(\lim_{n\to+\infty} X_{T_n}=X_T,\, T<\zeta\Big)=P_x(T<\zeta) \tag{3.17}$$

for q.e. $x\in X$.

This corollary is immediate from Theorem 3.4 and property (**M**.3). Here, the exceptional points $x\in X$ do not depend on the choice of $\{T_n\}$.

Proof of Theorem 3.4. Put $\Gamma_c=\{\sigma'_{X_b}<c,\, X_{c-}\in X\}$ and $\Lambda_c=\{X_{t-}\in X_b$ for some $t\in(0,c),\, X_{c-}\in X\}$. X_b is a Borel set (actually an F_σ-set) and $P_x(\Gamma_c)=0$, $x\in X$, according to (**M**.4). Hence there exists a set $\Gamma'_c\in B_{(0,c)}$ such that $\Gamma_c\subset\Gamma'_c$ and $\int_X P_x(\Gamma'_c)m(dx)=0$. Since $\Lambda_c=\gamma\Gamma_c\subset\gamma\Gamma'_c$, Theorem 3.3 implies that $P_x(\Lambda_c)=0$ for m-a.e. $x\in X$. Notice that $\Lambda=\{X_{t-}\in X_b$, for some $t\in(0,\zeta)\}=\bigcup_{c\in Q}\Lambda_c$, Q being the set of all positive rational numbers. We have therefore $P_x(\Lambda)=0$ for m-a.e. $x\in X$. On the other hand $u(x)=P_x(\Lambda)$ is an excessive function: u is universally measurable and

$$\exp(-\alpha_0 s)P_s u(x)=\exp(-\alpha_0 s)P_x\{X_{t-}\in X_b \text{ for some } t\in(s,\zeta)\}\uparrow u(x),\quad s\downarrow 0.$$

Thus $u(x)=0$ for q.e. $x\in X$ in view of Theorem 3.1 and Theorem 1.2.

Finally for an open or a closed set $A\subset X$ we define

$$P_t^0(x,E)=P_x(X_t\in E,\, t<\sigma_A),\qquad t\geqq 0. \tag{3.18}$$

$\{P_t^0(x,E),\, t\geqq 0\}$ is a sub-Markov transition probability on X. Since $P_t^0f(x)=\int_X P_t^0(x,dy)f(y)$ is right continuous in $t\geqq 0$ for $f\in C_0(X)$, $P_t^0(x,E)$ is measurable in $(t,x)\in[0,+\infty)\times X$ for each fixed Borel set $E\subset X$. Put

$$R_\alpha^0(x,E)=\int_0^{+\infty}e^{-\alpha t}P_t^0(x,E)\,dt. \tag{3.19}$$

$\{R_\alpha^0(x,E),\,\alpha>0\}$ is a sub-Markov resolvent on X. Obviously $R_\alpha^0f(x)=\int_X R_\alpha^0(x,dy)f(y)$ satisfies

$$R_\alpha^0f(x)=E_x\Big(\int_0^{\sigma_A}e^{-\alpha t}f(X_t)\,dt\Big),\qquad x\in X, \tag{3.20}$$

if f is Borel measurable and $R_\alpha^0f(x)$ is well defined.

([14]) Cf. Lemma 3.7(iv) of [8].

THEOREM 3.5[15]. *For an open or a closed set $A \subset X$, the transition probability and the resolvent kernel defined by* (3.18) *and* (3.19) *are m-symmetric:*

$$\int_X f(x)\cdot P_t^0 g(x)m(dx) = \int_X P_t^0 f(x)\cdot g(x)m(dx), \qquad t \geqq 0, \tag{3.21}$$

$$\int_X f(x)\cdot R_\alpha^0 g(x)m(dx) = \int_X R_\alpha^0 f(x)\cdot g(x)m(dx), \qquad \alpha > 0, \tag{3.22}$$

for $f, g \in C^+(X)$.

Proof. (3.22) is a direct consequence of (3.21). Let us show (3.21) when A is an open set. Fix a positive number c and Borel sets F, $G \subset X$. Consider the set $\Gamma = \{X_0 \in F, c \leqq \sigma_A, X_{c-} \in G\}$. Since A is open, $\Gamma \in \mathscr{B}_{(0,c)}$ and

$$\gamma\Gamma = \{X_0 \in G, c \leqq \sigma_A, X_{c-} \in F\}.$$

Noting that $P_x(X_0 = x) = 1$ m-a.e., we get by Theorem 3.3,

$$\int_F P_x(X_{c-} \in G, c \leqq \sigma_A)m(dx) = \int_G P_x(X_{c-} \in F, c \leqq \sigma_A)m(dx)$$

from which follows the equality

$$\int_X f(x)E_x(g(X_{c-}), c \leqq \sigma_A)m(dx) = \int_X E_x(f(X_{c-}), c \leqq \sigma_A)g(x)m(dx)$$

for $f, g \in C_0(X)$. By putting $c = t + 1/n$ and letting n tend to infinity in this equality, we obtain (3.21).

Next suppose that A is closed and choose a sequence of open sets A_n such as $A_n \supset \bar{A}_{n+1} \supset A$ and $A_n \downarrow A$. By virtue of the quasi-left continuity (Corollary to Theorem 3.4)

$$P_x(\sigma_{A_n} \uparrow \sigma_A) = 1, \qquad \text{q.e. } x \in X. \tag{3.23}$$

Hence (3.21) for the closed set A follows from those for open sets A_n.

3.4. *Probabilistic decomposition of* $(\mathscr{F}^*, \mathscr{E}^\alpha)$. Let A be an open or a closed set of X. We put

$$\begin{aligned} H_\alpha(x, E) &= E_x(e^{-\alpha\sigma_A}; X\sigma_A \in E), \\ H'_\alpha(x, E) &= E_x(e^{-\alpha\sigma'_A}; X\sigma'_A \in E). \end{aligned} \tag{3.24}$$

In this subsection we will consider the kernels H_α and H'_α on X as well as the localized resolvent R_α^0 defined by (3.19) and reveal the roles they play in the strongly regular D-space $(\mathscr{F}, \mathscr{E})$.

Any Borel measurable function on X is extended to $\bar{X} = X \cup \partial$ by defining its value at ∂ to be zero. It holds under this convention that $H_\alpha f(x) = E_x(e^{-\alpha\sigma_A} f(X_{\sigma_A}))$ and $R_\alpha f(x) = E_x(\int_0^\xi e^{-\alpha t} f(X_t)\,dt)$ for all $x \in \bar{X}$ when f is Borel measurable and

[15] Cf. Lemma 14.1 of [7].

$H_\alpha f(x) = \int_X H_\alpha(x, dy) f(y)$ and $R_\alpha f(x)$ are well defined for $x \in X$. $H'_\alpha f$ can be expressed in a similar way. If a Borel measurable function f is excessive, then $H_{\alpha_0} f$ is supermedian and $H'_{\alpha_0} f$ is excessive. When A is open, $\sigma_A = \sigma'_A$ and $H_\alpha = H'_\alpha$.

LEMMA 3.3. *If f is a bounded Borel measurable function on X, then for each $x \in X$*

$$R_\alpha f(x) = R^0_\alpha f(x) + H_\alpha R_\alpha f(x), \qquad \alpha > 0, \tag{3.25}$$

$$H_\alpha f(x) - H_\beta f(x) + (\alpha - \beta) R^0_\alpha H_\beta f(x) = 0, \qquad \beta > 0. \tag{3.26}$$

Proof. We can use the strong Markov property (**M**.2) to obtain the formula (3.25). (3.26) is a result of the Markov property.

By virtue of Theorem 3.5, the kernel $\{R^0_\alpha(x, E), \alpha > 0\}$ is an m-symmetric sub-Markov resolvent kernel on X. Let $(\mathscr{F}^{(0)}, \mathscr{E}^{(0)})$ be the Dirichlet space generated by $\{R^0_\alpha(x, E), \alpha > 0\}$: $R^0_\alpha(L^2(X; m) \cap C(X)) \subset \mathscr{F}^{(0)}$ and the function $R^0_\alpha f$, $f \in L^2(X; m) \cap C(X)$, satisfies

$$\mathscr{E}^{(0),\alpha}(R^0_\alpha f, v) = (f, v)_X \quad \text{for every } v \in \mathscr{F}^{(0)}. \tag{3.27}$$

THEOREM 3.6. *$\mathscr{F}^{(0)} \subset \mathscr{F}$ and $\mathscr{E}^{(0)}$ is the restriction of $\mathscr{E}$ to $\mathscr{F}^{(0)}$:*

$$\mathscr{E}^{(0)}(u, v) = \mathscr{E}(u, v), \qquad u, v \in \mathscr{F}^{(0)}. \tag{3.28}$$

Proof. Denote by $\mathfrak{B}$ the set of all bounded Borel measurable functions on X. Since $R^0_{\alpha_0}(L^2 \cap C)$ is dense in $\mathscr{F}^{(0)}$ with metric $\mathscr{E}^{(0),\alpha_0}$, it suffices to prove the following:

$$\text{For every } f \in \mathfrak{B} \cap L^2, \ R^0_{\alpha_0} f \in \mathscr{F} \text{ and } \mathscr{E}^{\alpha_0}(R^0_{\alpha_0} f, R^0_{\alpha_0} f) = (f, R^0_{\alpha_0} f)_X. \tag{3.29}$$

We can observe by Lemma 1 of [9] that $u \in L^2(X; m)$ is an element of $\mathscr{F}$ if and only if $\lim_{\beta \to +\infty} \beta(u - \beta R_{\beta+\alpha_0} u, u)_X$ is finite and in this case the limit is equal to $\mathscr{E}^{\alpha_0}(u, u)$. Thus the relation (3.29) is equivalent to

$$\lim_{\beta \to +\infty} \beta(u - \beta R_{\beta+\alpha_0} u, u)_X = (f, u)_X \tag{3.30}$$

where $u = R^0_{\alpha_0} f$, $f \in \mathfrak{B} \cap L^2$.

Let us show (3.30) for an open set A. By (3.25) and the resolvent equation for R^0_α,

$$\beta(u - \beta R_{\beta+\alpha_0} u, u)_X = \beta(R^0_{\beta+\alpha_0} f, u)_X - \beta^2 (H_{\beta+\alpha_0} R_{\beta+\alpha_0} u, u)_X.$$

Since R^0_α is symmetric and its L^2-norm is no greater than $1/\alpha$,

$$\beta(R^0_{\beta+\alpha_0} f, u)_X = (f, R^0_{\alpha_0} f - R^0_{\beta+\alpha_0} f)_X \to (f, u)_X, \qquad \beta \to +\infty.$$

We have to prove

$$\beta^2 (H_{\beta+\alpha_0} R_{\beta+\alpha_0} u, u)_X \to 0, \qquad \beta \to +\infty. \tag{3.31}$$

We may assume without loss of generality that f is nonnegative. By the symmetry of $R^0_{\alpha_0}$ and the formula (3.26), the left-hand side of (3.31) is no greater than

$\beta(H_{\alpha_0}R_{\beta+\alpha_0}u, f)_X$. Notice that u is the difference of two excessive functions: $u=R_{\alpha_0}f-H_{\alpha_0}(R_{\alpha_0}f)$. Hence $\lim_{\beta\to+\infty}\beta(H_{\alpha_0}R_{\beta+\alpha_0}u, f)_X=(H_{\alpha_0}u, f)_X$. However, since $\sigma_A(\theta_{\sigma_A}\omega)=0$ for $\omega\in\{\sigma_A<\zeta\}$, we have $H_{\alpha_0}\cdot H_{\alpha_0}(R_{\alpha_0}f)(x)=H_{\alpha_0}(R_{\alpha_0}f)(x)$, $x\in X$, which means $H_{\alpha_0}u(x)=0$, $x\in X$, yielding (3.31). Here θ denotes the usual translation operator on W (cf. [16]).

Thus (3.30) and hence (3.29) are established when A is open. (3.29) for a closed set A is now to be proved. Find open sets A_n such that $A_n\supset\bar{A}_{n+1}$ and $A_n\downarrow A$. Denote by ${}^nR^0_\alpha$ and ${}^nH_\alpha$ kernels corresponding to A_n. Put $u_n={}^nR^0_{\alpha_0}f$ for a non-negative $f\in\mathfrak{B}\cap L^2$. Then owing to (3.20) and (3.23), u_n increases to $R^0_{\alpha_0}f$ q.e. as $n\to+\infty$. Now observe the following equality: for $m\leqq n$,

$$\beta(u_n-\beta R_{\beta+\alpha_0}u_n, u_m)_X$$
$$=\beta({}^mR^0_{\beta+\alpha_0}f, u_m)_X+\beta({}^mH_{\beta+\alpha_0}\,{}^nR^0_{\beta+\alpha_0}f, u_m)_X-\beta^2({}^nH_{\beta+\alpha_0}R_{\beta+\alpha_0}u_n, u_m)_X.$$

Here we used the identity ${}^nR^0_\beta f={}^mR^0_\beta f+{}^mH_\beta\,{}^nR^0_\beta f$. Let β tend to infinity. Then the first term of the right-hand side of the equality tends to $(f, u_m)_X$ as was proved earlier. The second term is, in view of the symmetry of ${}^mR^0_{\alpha_0}$ and (3.26), no greater than $({}^mH_{\alpha_0}\,{}^nR^0_{\beta+\alpha_0}f, f)_X$, which decreases to zero. The absolute value of the last term is no greater than $\beta^2({}^nH_{\beta+\alpha_0}R_{\beta+\alpha_0}u_n, u_n)_X$, which also tends to zero by (3.31). What we have proved is $\mathscr{E}^{\alpha_0}(u_n, u_m)=(f, u_m)_X$, $m\leqq n$, which in turn tells us that u_n converges to $R^0_{\alpha_0}f$ in $\mathscr{E}^{\alpha_0}$-norm, arriving at (3.29) for the closed set A. The proof of Theorem 3.6 is complete.

On account of Theorem 3.6, $\mathscr{F}^{(0)}$ is a closed subspace of $\mathscr{F}$ with metric $\mathscr{E}^{\alpha_0}$. Let us denote by $\mathscr{H}_{\alpha_0}$ the orthogonal complement of $\mathscr{F}^{(0)}$ in the Hilbert space $(\mathscr{F}, \mathscr{E}^{\alpha_0})$.

LEMMA 3.4([16]). *If u is either an element of $\mathscr{F}\cap C(X)$ or of the form $R_{\alpha_0}h$, $h\in\mathfrak{B}\cap L^2$, then $H_{\alpha_0}u$ is quasi-continuous and*

$$H_{\alpha_0}u = P_{\mathscr{H}_{\alpha_0}}u, \tag{3.32}$$

$P_{\mathscr{H}_{\alpha_0}}$ being the projection on the space $\mathscr{H}_{\alpha_0}$.

Proof. Let us first show this for $u=R_{\alpha_0}h$, $h\in\mathfrak{B}\cap L^2$. By (3.25), (3.29) and Lemma 3.1, we have

$$\mathscr{E}^{\alpha_0}(H_{\alpha_0}R_{\alpha_0}h, R^0_{\alpha_0}g) = \mathscr{E}^{\alpha_0}(R_{\alpha_0}h-R^0_{\alpha_0}h, R^0_{\alpha_0}g)$$
$$= (h, R^0_{\alpha_0}g)_X-(h, R^0_{\alpha_0}g)_X = 0$$

for every $g\in\mathfrak{B}\cap L^2$. This means that the formula (3.25) represents the direct decomposition of $u=R_{\alpha_0}h$ into the sum of elements of $\mathscr{F}^{(0)}$ and $\mathscr{H}_{\alpha_0}$, getting (3.32). The quasi-continuity of $H_{\alpha_0}u$ is clear if A is open, because it is then the difference

([16]) We have further $H_{\alpha_0}u=H_{\alpha_0'}u$, q.e. for $u\in\mathscr{F}\cap C(X)$. Since $H_{\alpha_0'}R_{\alpha_0}u$ is the regularization of the quasi-continuous function $H_{\alpha_0}R_{\alpha_0}u$, these two are equal q.e. From this we can get the desired equality.

of excessive functions $H_{\alpha_0}(R_{\alpha_0}h^+)$ and $H_{\alpha_0}(R_{\alpha_0}h^-)$ of $\mathscr{F}$ to which we can apply Theorem 3.1. Coming to the case when A is closed, consider open sets A_n and corresponding kernels ${}^nR^0_{\alpha_0}$ and ${}^nH_{\alpha_0}$ just as in the second part of the proof of Theorem 3.6. Then the quasi-continuous functions ${}^nH_{\alpha_0}u=u-{}^nR^0_{\alpha_0}h$ converge to $u-R^0_{\alpha_0}h=H_{\alpha_0}u$ q.e. on X as well as in $\mathscr{E}^{\alpha_0}$-norm. Hence the latter must be quasi-continuous (Lemma 1.2).

Next take any $u\in\mathscr{F}\cap C(X)$. Since, for each $\beta>0$, $R_\beta u$ is equal to $R_{\alpha_0}h$ with some $h\in\mathfrak{B}\cap L^2$, we have $H_{\alpha_0}(\beta R_\beta u)=P_{\mathscr{H}_{\alpha_0}}(\beta R_\beta u)$.

By Lemma 2.1 of [10], $\beta R_\beta u\to u$ and hence

$$P_{\mathscr{H}_{\alpha_0}}(\beta R_\beta u)\to P_{\mathscr{H}_{\alpha_0}}u,\qquad \beta\to+\infty,$$

with respect to $\mathscr{E}^{\alpha_0}$-norm. On the other hand,

$$\begin{aligned}\lim_{\beta\to+\infty} H_{\alpha_0}(\beta R_\beta u)(x) &= \lim_{\beta\to+\infty} E_x(\exp(-\alpha_0\sigma_A)\beta G_\beta u(X_{\sigma_A});\ X_{\sigma_A}\notin X_b)\\ &= E_x(\exp(-\alpha_0\sigma_A)u(X_{\sigma_A})) = H_{\alpha_0}u(x),\qquad x\in X,\end{aligned}$$

by virtue of property (**M.4**). Thus we get (3.32). $H_{\alpha_0}u$ is quasi-continuous because it is the limit of quasi-continuous functions $H_{\alpha_0}(\beta R_\beta u)$ in $\mathscr{E}^{\alpha_0}$-norm as well as in the pointwise sense.

LEMMA 3.5. *Suppose that A is compact. Any quasi-supermedian function belonging to the space $\mathscr{H}_{\alpha_0}$ is a potential of a measure whose support is concentrated on A.*

Proof. Assume that u is quasi-supermedian and $u\in\mathscr{H}_{\alpha_0}$. Then $u\in\mathscr{F}^*$ and we have by Lemma 1.3 that $\mathscr{E}^{\alpha_0}(u,v)\geqq 0$ for all $v\in\mathscr{F}^*$ such as $v\geqq 0$ q.e. Let v be any function of $\mathscr{F}\cap C(X)$ which is nonnegative on A. By Lemma 3.4, $\mathscr{E}^{\alpha_0}(u,v)=\mathscr{E}^{\alpha_0}(u,H_{\alpha_0}v)$ which is nonnegative because $H_{\alpha_0}v(x)\geqq 0$, $x\in X$. According to Theorem 1.6, we arrive at Lemma 3.5.

The next two are the main theorems of this subsection.

THEOREM 3.7([17]). *Put*

$$\mathscr{F}^*_{X-A}=\{u\in\mathscr{F}^*;\ u=0\ q.e.\ on\ A\}. \tag{3.33}$$

Then $\mathscr{F}^*_{X-A}=(\mathscr{F}^{(0)})^*$, where $(\mathscr{F}^{(0)})^*$ denotes the set of all quasi-continuous modifications of functions in the space $\mathscr{F}^{(0)}$.

Proof. On account of Theorem 3.6 and Lemma 1.2, $(\mathscr{F}^{(0)})^*$ and $\mathscr{F}^*_{X-A}$ are closed subspaces of the Hilbert space $(\mathscr{F}^*,\mathscr{E}^{\alpha_0})$. If $f\in L^2\cap C$, then $R^0_{\alpha_0}f(x)=0$ on $A-X_b$ and $R^0_{\alpha_0}f=R^0_{\alpha_0}f-H_{\alpha_0}(R_{\alpha_0}f)$ is quasi-continuous in view of Lemma 3.4. Hence $\mathscr{F}^*_{X-A}$ contains $R^0_{\alpha_0}(L^2\cap C)$ which is dense in $(\mathscr{F}^{(0)})^*$. Thus, $\mathscr{F}^*_{X-A}\supset(\mathscr{F}^{(0)})^*$.

([17]) We can assert even more: for any nonnegative universally measurable function $f\in L^2(X;m)$, $R^0_\alpha f$ belongs to the space $\mathscr{F}^*_{X-A}$ and the equation $\mathscr{E}^\alpha(R^0_\alpha f, v)=(f,v)_X$ holds for every $v\in\mathscr{F}^*_{X-A}$. In view of the proof of Theorem 3.7, this is true for $f\in L^2\cap C(X)$. Now the general case can be obtained exactly in the same manner as in the proof of Lemma 3.1.

Let us prove the converse inclusion. Denote by $\mathscr{H}_{\alpha_0}^*$ the space of all quasi-continuous modifications of functions of the $\mathscr{H}_{\alpha_0}$. It suffices to show that $\mathscr{H}_{\alpha_0}^*$—the orthogonal complement of $(\mathscr{F}^{(0)})^*$ in $(\mathscr{F}^*, \mathscr{E}^{\alpha_0})$—is orthogonal to $\mathscr{F}_{X-A}^*$. Since $H_{\alpha_0}R_{\alpha_0}(L^2 \cap C)$ is in $\mathscr{H}_{\alpha_0}^*$ by Lemma 3.4 and dense there, it is enough to prove

$$\mathscr{E}^{\alpha_0}(H_{\alpha_0}R_{\alpha_0}f, v) = 0, \qquad f \in L^2 \cap C^+, v \in \mathscr{F}_{X-A}^*. \tag{3.34}$$

Assume first that A is compact. Since $H_{\alpha_0}R_{\alpha_0}f$ is supermedian, it is quasi-supermedian (Remark 3.1) and is a potential of a measure $\mu \in M_0^+$ with $S\mu \subset A$ by virtue of Lemma 3.5. Owing to Theorem 1.5, the left-hand side of (3.34) is equal to $\int_A v(x)\mu(dx) = 0$.

In the case when A is open or closed, we can find compact sets A_n such that $A_n \uparrow A$. Denote by ${}^nH_{\alpha_0}$ the kernel corresponding to A_n. It is easy to see that ${}^nH_{\alpha_0}R_{\alpha_0}f$ then converges to $H_{\alpha_0}R_{\alpha_0}f$ increasingly and in $\mathscr{E}^{\alpha_0}$-norm. If $v \in \mathscr{F}_{X-A}^*$, then $v \in \mathscr{F}_{X-A_n}^*$ and the left-hand side of (3.34) is equal to $\lim_n \mathscr{E}^{\alpha_0}({}^nH_{\alpha_0}R_{\alpha_0}f, v) = 0$. The proof of Theorem 3.7 is complete.

Owing to this theorem, we get an important conclusion about a local property of the space $\mathscr{H}_{\alpha_0}^*$: Any function in $\mathscr{H}_{\alpha_0}^*$ is determined by its restriction to the set A. In fact the space $(\mathscr{F}^*, \mathscr{E}^{\alpha_0})$ is expressed as a direct sum $\mathscr{F}^* = \mathscr{F}_{X-A}^* \oplus \mathscr{H}_{\alpha_0}^*$ and so any function of $\mathscr{H}_{\alpha_0}^*$ which vanishes q.e. on A should vanish q.e. on X. Keeping this in mind let us prove the next theorem.

THEOREM 3.8. *Suppose that A is a compact set or an open set belonging to the class $\mathscr{U}$. Then we have for q.e. $x \in X$,*

$$e_A(x) = E_x(\exp(-\alpha_0\sigma_A); \sigma_A < \zeta) = E_x(\exp(-\alpha_0\sigma_A'); \sigma_A' < \zeta). \tag{3.35}$$

Here e_A denotes the equilibrium potential of A defined in subsection 1.5.

Proof. Suppose that A is compact. By (1.25) and (1.26), e_A is an element of $\mathscr{H}_{\alpha_0}^*$ which is equal to 1 q.e. on A. The function $u(x) = E_x(\exp(-\alpha_0\sigma_A); \sigma_A < \zeta)$ has the same property. Indeed $u(x) = 1$, $x \in A - X_b$, and hence q.e. on A by Theorem 3.2. u can be expressed as $H_{\alpha_0}f(x)$ with any function $f \in \mathscr{F} \cap C(X)$ such as $f(x) = 1$ for $x \in A$. Hence $u \in \mathscr{H}_{\alpha_0}^*$ by Lemma 3.4. Thus we get the first equality of (3.35). The third term of (3.35) is the regularization of the second. Hence they are equal q.e. on X in view of Theorem 3.1.

When A is an open set of the class $\mathscr{U}$, (3.35) is also obtained by approximating A with a sequence of compact sets increasing to A and by noting (1.22) and (1.23).

3.5. *Regularity of quasi-continuous transformations along sample paths.* Let us begin with a lemma which states a probabilistic feature of polar sets.

LEMMA 3.6. *Let A be a Borel polar set and $\{A_n\}$ be a decreasing sequence of open sets such that $A_n \supset A$ and* $\lim_{n\to+\infty} \text{Cap}(A_n) = 0$. *Then the equalities*

$$P_x(X_t \text{ or } X_{t-} \in A \text{ for some } t \geqq 0) = 0, \tag{3.36}$$

$$(3.37) \qquad P_x\Big(\sigma_{A_n} = \zeta \textit{ for some } n \textit{ or } \lim_{n\to+\infty} \sigma_{A_n} = +\infty\Big) = 1$$

hold for q.e. $x \in X$.

Proof. (3.36) is a consequence of (3.37). In order to show (3.37), let us put $u_n(x) = E_x(\exp(-\alpha_0\sigma_{A_n});\ \sigma_{A_n} < \zeta)$. By Theorem 3.8 and (1.21) we have $\mathscr{E}^{\alpha_0}(u_n, u_n) = \mathrm{Cap}\,(A_n)$ which decreases to zero by the assumption. Since u_n is quasi-continuous, Lemma 1.2 implies that $\lim_{n\to+\infty} u_n(x) = 0$ q.e. on X. We arrive at (3.37) on account of the identity

$$\lim_{n\to+\infty} u_n(x) = E_x\Big(\exp\Big(-\alpha_0 \lim_n \sigma_{A_n}\Big);\ \bigcap_{n=1}^{+\infty} \{\sigma_{A_n} < \zeta\}\Big).$$

Since the branch set X_b is polar (Theorem 3.2), we can apply the above lemma to X_b to get

$$(3.38) \qquad P_x(X_{t-} \in X_b \text{ for some } t \geqq 0) = 0,$$

for q.e. $x \in X$. Notice that (3.38) is stronger than (3.16). We will further strengthen the assertions of Lemma 3.6 as follows:

THEOREM 3.9. *Under the same assumption as in Lemma* 3.6, *there exists a Borel polar set B including A such that the equalities* (3.37) *and*

$$(3.39) \qquad P_x(X_t \textit{ or } X_{t-} \in B \textit{ for some } t \geqq 0) = 0$$

are simultaneously valid for every $x \in X - B$.

Proof. By virtue of Lemma 3.6, we see that (3.36) and (3.37) are valid for every $x \in X$ except on a polar set N_1. By replacing N_1 with a G_δ-polar set including it if necessary, we may assume that N_1 is a Borel set. Apply again Lemma 3.6 to the Borel polar set $A \cup N_1$. We get

$$(3.40) \qquad P_x(X_t \text{ or } X_{t-} \in A \cup N_1 \text{ for some } t \geqq 0) = 0$$

for every $x \in X$ except on a Borel polar set N_2. Repeating the same argument, we have a sequence $\{N_k\}$ of Borel polar sets such that for each k the equality

$$(3.41) \qquad P_x(X_t \text{ or } X_{t-} \in A \cup N_1 \cup \cdots \cup N_k \text{ for some } t \geqq 0) = 0$$

holds for every $x \in X - N_{k+1}$. Put $B = A \cup (\bigcup_{k=1}^{+\infty} N_k)$. B is polar by Theorem 1.1. If $x \in X - B$, then (3.41) is valid for every k. Letting k tend to infinity, we get (3.39).

Turning to the main task of this subsection, let us consider a quasi-continuous function q on X taking values in some nice topological space. We fix a decreasing sequence $\{A_n\}$ of open subsets of X such that q is continuous on each $X - A_n$ and

$\lim_{n\to+\infty} \mathrm{Cap}(A_n)=0$. By virtue of Theorem 3.9, there is a Borel polar set B such that $B\supset\bigcap_{n=1}^{+\infty} A_n$ and equalities (3.37) and (3.39) hold for every $x\in\bar{X}-B$: if we put

$$(3.42)\qquad \begin{aligned} W_{11} &= \{\omega\in W;\ X_t(\omega) \text{ and } X_{t-}(\omega)\in\bar{X}-B \text{ for all } t\geqq 0\},\\ W_{12} &= \left\{\omega\in W;\ \sigma_{A_n}(\omega)=\zeta(\omega) \text{ for some } n \text{ or } \lim_{n\to+\infty}\sigma_{A_n}(\omega)=+\infty\right\},\end{aligned}$$

then

$$(3.43)\qquad P_x(W_1)=1,\qquad x\in\bar{X}-B,$$

where W_1 denotes the set $W_{11}\cap W_{12}$.

Now let us put

$$(3.44)\qquad \mathscr{B}^1=\mathscr{B}\cdot W_1,\quad \mathscr{B}_t^1=\mathscr{B}_t\cdot W_1,\qquad t\geqq 0,$$

and denote the restrictions of measures P_x, $x\in\bar{X}-B$, to $\mathscr{B}^1$ by P_x again. We also maintain the notion X_t to express its restriction to W_1. It is then clear that the process $M_1=\{W_1, \mathscr{B}^1, \mathscr{B}_t^1, X_t, P_x\}$ is a right continuous Markov process with state space $\bar{X}-B$.

THEOREM 3.10. (i) *The process M_1 is a strong Markov process with state space $\bar{X}-B$. The resolvent kernel of M_1 is the restriction to $X-B$ of the Ray resolvent kernel $\{R_\alpha(x, E), \alpha>0\}$ of the original process M. Further*

$$(3.45)\qquad R_\alpha(x, B)=0,\qquad x\in X-B.$$

(ii) *The σ-field $\mathscr{B}^1$ (resp. $\mathscr{B}_t^1$) is the completion of $\mathscr{B}^0\cdot W_1$ (resp. $\mathscr{B}_t^0\cdot W_1$ in $\mathscr{B}^1$) with respect to the family of measures*

$$\left\{P_\mu(\cdot)=\int_{\bar{X}-B}\mu(dx)P_x(\cdot);\ \mu \text{ is a finite measure on } \bar{X}-B\right\}.$$

(iii) *Assume an additional condition that*

$$(3.46)\qquad X_b\subset\bigcap_{n=1}^{+\infty} A_n.$$

Then the process M_1 is quasi-left continuous on $[0, +\infty)$: if $\{\mathscr{B}_t^1\}$-stopping times T_n increase to T, then $\lim_{n\to+\infty} X_{T_n}=X_T$ P_x-a.e. on $\{T<+\infty\}$ for every $x\in\bar{X}-B$.

(iv) *Assume an additional condition that*

(3.47) *q can be extended to $\bar{X}-\bigcap_{n=1}^{+\infty} A_n$ in such a way that the restriction of q to $\bar{X}-A_n$ is continuous there for every n.*

Then, for each $\omega\in W$, $q(X_t(\omega))$ and $q(X_{t-}(\omega))$ are well defined and $Y_t(\omega)=q(X_t(\omega))$ is right continuous in $t\geqq 0$. $Y_t(\omega)$ has the left limit at every $t>0$ with

$$(3.48)\qquad Y_{t-}(\omega)=q(X_{t-}(\omega)).$$

(v) *Assume that both the conditions (3.46) and (3.47) are valid. If $\{\mathscr{B}_t^1\}$-stopping times T_n increase to T, then $\lim_{n\to+\infty} Y_{T_n}=Y_T$ P_x-a.e. on $\{T<+\infty\}$ for every $x\in\bar{X}-B$.*

Proof. (i) The latter assertion together with (3.45) is evident. $\boldsymbol{M}_1$ is now a Markov process on $\bar{X}-B$ with right continuous sample paths and right continuous σ-fields $\{\mathscr{B}_t^1\}$. Obviously the right continuity of $R_\alpha f$, $f \in C(X)$, along the sample paths is preserved under the transfer from $\boldsymbol{M}$ to $\boldsymbol{M}_1$. Thus, $\boldsymbol{M}_1$ is a strong Markov process on $\bar{X}-B$.

(ii) Take a set $\Lambda \in \mathscr{B}^1$ and consider its property with respect to the original process $\boldsymbol{M}$. Property (**M**.1) implies that

$$P_x(\Lambda) \leqq P_x(W_{11}\cdot\{X_0 = x\}) = 0 \quad \text{if } x \in B - X_b$$

and

$$P_x(\Lambda) = \int_{\bar{X}-X_b} \mu(x, dy)P_y(\Lambda) = \int_{\bar{X}-B} \mu(x, dy)P_y(\Lambda) \quad \text{if } x \in X_b.$$

Therefore, for any finite measure μ on $\bar{X}$, $P_\mu(\Lambda)=P_{\mu_1}(\Lambda)$ with a finite measure μ_1 supported by the set $\bar{X}-B$. This means statement (ii).

(iii) By the hypothesis (3.46),

$$P_x(X_{t-} \in X_b \text{ for some } t \geqq 0) = 0, \qquad x \in \bar{X}-B.$$

Combining this with statement (i), we can prove the quasi-left continuity of $\boldsymbol{M}_1$ on $[0, +\infty)$ exactly in the same way as in [13, §2] (see [16, XIV, T15] for more information).

(iv) Fix an $\omega \in W_1$. If $\zeta(\omega) < +\infty$, then $\sigma_{A_n}(\omega)=\zeta(\omega)$ for some n and hence $X_t(\omega)$ and $X_{t-}(\omega)$ belong to the closed set $\bar{X}-A_n$ for all $t \geqq 0$. Hence we get the desired properties of Y_t by the hypothesis (3.47). If $\zeta(\omega)=+\infty$, then for any $t \geqq 0$ there exists an A_n such that $\sigma_{A_n}(\omega) > t$. Hence we get the desired conclusion in this case also.

(v) By the preceding two statements (iii) and (iv), we have $\lim_{n\to+\infty} Y_{T_n} = q(\lim_{n\to+\infty} X_{T_n}) = q(X_T) = Y_T$ P_x-a.e. on $\{T < +\infty\}$ for every $x \in \bar{X}-B$. The proof of Theorem 3.10 is complete.

REMARK 3.2. Here we give some remarks on the hypotheses (3.46) and (3.47) in Theorem 3.10. We can assume (3.46) without loss of generality because the branch set X_b is polar. Assertions (iv) and (v) are still valid up to the lifetime ζ without assuming (3.47). Condition (3.47) is satisfied by two important cases in which we have interest. Theorem 1.3 implies that each numerical function $u \in \mathscr{F}$ has a modification $\tilde{u}$ which is not only quasi-continuous but also satisfies (3.47) by setting $\tilde{u}(\partial)=0$. In case that q is a quasi-homeomorphism from X to the underlying space X' of some regular D-space, q satisfies (3.47) if we put $q(\partial)=\partial'$. We can see this immediately from the definition of quasi-homeomorphism.

Theorem 3.10 will be the key to prove Theorem 4.1. Here we state another application of Theorem 3.10. Consider a function u defined q.e. on X. Let us agree to say u to be *Borel* (resp. *universally*) *measurable* if there is a Borel (resp. universally) measurable function $\tilde{u}$ on X such as $u=\tilde{u}$ q.e. We call *u finely continuous q.e.* if there exists a nearly Borel polar set B satisfying the following: $B \supset X_b$, $X-B$ is a fine open set and u is finely continuous at each point $x \in X-B$, fine topology being defined in terms of M([18]). For instance take a quasi-continuous function u (not necessarily real valued). Then u is clearly Borel measurable in the above sense. Furthermore from the first and second remarks in Remark 3.2, we can see that u is finely continuous q.e. Thus we get the first part of the following theorem.

THEOREM 3.11. (i) *Every quasi-continuous function on X is finely continuous q.e. and Borel measurable.*

(ii) *Conversely if a function u of $\mathcal{F}$ is finely continuous q.e. and Borel (or more generally, universally) measurable, then u is quasi-continuous.*

Suppose that a function $u \in \mathcal{F}$ is finely continuous q.e. and universally measurable. Denote by $\tilde{u}$ a quasi-continuous modification of u. Then by the first part of Theorem 3.11, the m-negligible function $v=u-\tilde{u}$ is finely continuous q.e. Therefore the second part of Theorem 3.11 follows from the next lemma which is a counterpart of Theorem 1.2(ii).

LEMMA 3.7. *If a function v is finely continuous q.e. and universally measurable and if $v=0$ m-a.e., then $v=0$ q.e.*

Proof. By making use of Theorem 3.9, we see that there is a Borel polar set $B \supset X_b$ such that $X-B$ is finely open, v is finely continuous at each point of $X-B$ and v is universally measurable on $X-B$. The set $C=\{x \in X-B; v(x) \neq 0\}$ is then a fine open and universally measurable set which is consequently contained in the set $D=\{x \in X-B; R_{\alpha_0}(x, C)>0\}$. Since C is m-negligible and R_{α_0} is symmetric, D is m-negligible. Hence D becomes polar by virtue of Theorem 1.2(ii) and Lemma 3.1. v now vanishes except on the polar set $B \cup C$.

3.6. *Polar sets and absolute continuity conditions.* The first half of the preceding subsection gives us probabilistic interpretations of polar sets. Here we will complete them.

We say a Borel set $Y \subset X$ is *M-invariant* if the equality

$$P_x(X_t \text{ and } X_{t-} \in Y \cup \partial \text{ for all } t \geqq 0) = 1$$

holds for every $x \in Y$.

THEOREM 3.12. *The following statements are equivalent to each other:*

(i) *A Borel set A is polar.*

(ii) *For m-almost all $x \in X$,*

$$P_x(\sigma'_A < \zeta) = 0. \tag{3.49}$$

(iii) *There exists an m-negligible Borel set B including A such that $X-B$ is M-invariant.*

([18]) See [2, II] or [16, XV].

Proof. (i) implies (iii) according to Theorem 3.9. Statement (iii) means (ii). Suppose that statement (ii) is valid. Then $E_x(\exp(-\alpha_0\sigma_K'); \sigma_K' < \zeta) = 0$ m-a.e., for any compact set $K \subset A$. We have Cap $(K) = 0$ by Theorem 3.8 and (1.21). Owing to (1.5) we arrive at the statement (i).

It should be noticed that we cannot generally strengthen the above statement (ii) by replacing "m-almost all x" with "all x". The simplest example illustrative of this point is the case when $\mathscr{F} = L^2(X)$ and $\mathscr{E} \equiv 0$. In this case each m-negligible set is polar but every point of X is trap with respect to the corresponding Ray process. The next theorem will concern the conditions to eliminate such irregular situations.

Suppose that a Borel set A is of potential zero: $R_\alpha(x, A) = 0$ for all $x \in A$ and $\alpha > 0$. Then $m(A) = 0$. Indeed symmetry of the kernel implies $\alpha \int_A R_\alpha 1(x) m(dx) = \alpha \int_X R_\alpha(x, A) m(dx) = 0$. Letting α tend to infinity, we get $m(A) = m(A - X_b) = 0$.

THEOREM 3.13. *The following conditions are mutually equivalent:*

(i) *A Borel set A is polar if and only if* (3.49) *is satisfied for all $x \in X$.*

(ii) *m is a reference measure of $\mathbf{M}$: a set A is of potential zero if and only if $m(A) = 0$.*

(iii) *$R_\alpha(x, \cdot)$ is absolutely continuous with respect to m for each $x \in X$ and $\alpha > 0$.*

Proof. It suffices to show the equivalence of (i) and (iii). Suppose that condition (iii) is satisfied and consider a Borel polar set A. By Theorem 3.12, we have $u(x) = P_x(\sigma_A' < \zeta) = 0$, m-a.e., and consequently $u(x) = \lim_{\beta \to +\infty} \beta R_\beta u(x) = 0$ for every $x \in X$. Thus condition (i) is valid. Conversely assume that (i) is met. Let A be an m-negligible Borel set. Symmetry of the kernel and Lemma 3.1 then imply that $R_\alpha(x, A) = 0$ holds for every $x \in X$ except on a polar set, which is of potential zero under condition (i). Thus we have $R_\alpha(x, A) = \lim_{\beta \to +\infty} \beta R_{\beta+\alpha} R_\alpha(x, A) = 0$ for every $x \in X$ arriving at condition (iii).

REMARK 3.3. Suppose that condition (iii) of Theorem 3.13 is satisfied. Theorem 3.13 then tells us that we can adopt the set $(\bigcap_{n=1}^{\infty} A_n) \cup X_b$ as the set B in Theorems 3.9 and 3.10.

4. Regular Dirichlet spaces and strong Markov processes.

Let $(X, m, \mathscr{F}, \mathscr{E})$ be a regular D-space. We adjoin a point ∂ to X as the point at infinity if X is noncompact and as an isolated point if X is compact.

4.1. *Construction of a strong Markov process—proof of Theorem* 4.1. This subsection is devoted to the proof of Theorem 4.1 mentioned in the beginning of the present paper.

(I) By Theorem 3 of [10], there exists a strongly regular D-space $(\tilde{X}, \tilde{m}, \tilde{\mathscr{F}}, \tilde{\mathscr{E}})$ which is equivalent to $(X, m, \mathscr{F}, \mathscr{E})$. Every notion related to this strongly regular D-space will be written with tilde ~. We already have several related notions specified in §3—the Ray resolvent $\{\tilde{R}_\alpha(\tilde{x}, \tilde{E}), \alpha > 0\}$, the branch set $\tilde{X}_b$, the transition function $\{\tilde{P}_t(\tilde{x}, \tilde{E}), t > 0\}$ and the Ray process $(\tilde{W}, \tilde{\mathscr{B}}_t, \tilde{P}_{\tilde{x}})$ on the extended space $\tilde{X} \cup \tilde{\delta}$.

By Theorem 2.1, there exists a capacity preserving quasi-homeomorphism q from X to $\tilde{X}$: there are decreasing sequences of open sets $A_k \subset X$ and $\tilde{A}_k \subset \tilde{X}$ such that

$$\lim_{k\to+\infty} \mathrm{Cap}\,(A_k) = 0, \qquad \lim_{k\to+\infty} \mathrm{Cap}^{\sim}\,(\tilde{A}_k) = 0$$

and the restriction of q to $X-A_k$ is homeomorphic onto $\tilde{X}-\tilde{A}_k$ for each k. The equality (2.1) holds for every analytic set $A\subset X-\bigcap_{k=1}^{+\infty} A_k$. If we extend q by setting $q(\partial)=\tilde{\partial}$, then according to Remark 3.2,

(4.1) the restriction of q to $X\cup\partial-A_k$ is a homeomorphism onto $\tilde{X}\cup\tilde{\partial}-\tilde{A}_k$.

Moreover we can assume without loss of generality that

$$\tilde{X}_b \subset \bigcap_{k=1}^{+\infty} \tilde{A}_k, \tag{4.2}$$

because we can replace $\tilde{A}_k$ (resp. A_k) with

$$\tilde{A}_k \cup \tilde{D}_k = \tilde{A}_k \cup \left(\tilde{D}_k - \bigcap_{n=1}^{+\infty} \tilde{A}_n\right) \quad \left(\text{resp. } A_k \cup q^{-1}\left(\tilde{D}_k - \bigcap_{n=1}^{+\infty} \tilde{A}_n\right)\right)$$

if necessary. Here $\{\tilde{D}_k\}$ is a decreasing sequence of open sets of $\tilde{X}$ such that $\tilde{D}_k \supset \tilde{X}_b$ and $\lim_{k\to+\infty} \mathrm{Cap}^{\sim}\,(\tilde{D}_k)=0$.

By Theorem 3.9, there exists a Borel polar set $\tilde{B}\supset\bigcap_{k=1}^{+\infty} \tilde{A}_k$ which satisfies the following: if we put

$$\tag{4.3} \begin{aligned} \tilde{W}_{11} &= \{\tilde{\omega}\in\tilde{W};\ \tilde{X}_t(\tilde{\omega}) \text{ and } \tilde{X}_{t-}(\tilde{\omega})\in\tilde{X}\cup\tilde{\partial}-\tilde{B} \text{ for all } t\geqq 0\}, \\ \tilde{W}_{12} &= \left\{\tilde{\omega}\in\tilde{W};\ \sigma_{\tilde{A}_n}(\tilde{\omega}) = \tilde{\xi}(\tilde{\omega}) \text{ for some } n \text{ or } \lim_{n\to+\infty} \sigma_{\tilde{A}_n}(\tilde{\omega}) = +\infty\right\}, \end{aligned}$$

and $\tilde{W}_1=\tilde{W}_{11}\cap\tilde{W}_{12}$, then

$$\tilde{P}_{\tilde{x}}(\tilde{W}_1) = 1, \qquad \tilde{x}\in\tilde{X}\cup\tilde{\partial}-\tilde{B}. \tag{4.4}$$

According to Theorem 3.10, we have a right continuous strong Markov process $\tilde{M}_1=(\tilde{W}_1, \tilde{\mathscr{B}}^1_t, \tilde{X}_t, \tilde{P}_x)$ with state space $\tilde{X}\cup\tilde{\partial}-\tilde{B}$ which is quasi-left continuous on $[0, +\infty)$.

(II) *Definition of* $M=(\Omega, \mathscr{M}, \mathscr{M}_t, X_t, P_x)$. Let us define a set $B\subset X$ by

$$X-B = q^{-1}(\tilde{X}-\tilde{B}). \tag{4.5}$$

Since $B=(\bigcap_{k=1}^{\infty} A_k)\cup q^{-1}(\tilde{B}-\bigcap_{k=1}^{\infty} \tilde{A}_k)$, B is a Borel polar set including the set $\bigcap_{k=1}^{\infty} A_k$. We put

$$\Omega = \tilde{W}_1, \quad \mathscr{M} = \tilde{\mathscr{B}}^1, \quad \mathscr{M}_t = \tilde{\mathscr{B}}^1_t. \tag{4.6}$$

The element of Ω (resp. $\mathscr{M}$) is denoted by ω (resp. Λ) instead of $\tilde{\omega}$ (resp. $\tilde{\Lambda}$). Define X_t and P_x by

$$\tag{4.7} \begin{aligned} X_t(\omega) &= q^{-1}(\tilde{X}_t(\omega)), & \omega\in\Omega,\ t\geqq 0, \\ P_x(\Lambda) &= \tilde{P}_{qx}(\Lambda), & x\in X\cup\partial-B,\ \Lambda\in\mathscr{M}. \end{aligned}$$

X_t takes values in $X \cup \partial - B$. The field $\mathscr{M}_t^0 = \tilde{\mathscr{B}}_t^0 \cdot \tilde{W}_1$ is generated by functions X_s, $s \leqq t$, because it is generated by $\tilde{X}_s$, $s \leqq t$, and both q and q^{-1} are one-to-one Borel measurable between $X \cup \partial - B$ and $\tilde{X} \cup \tilde{\partial} - \tilde{B}$. The field $\mathscr{M}^0 = \tilde{\mathscr{B}}^0 \cdot \tilde{W}_1$ is of course generated by X_s, $s \geqq 0$.

By (4.4), P_x is a probability measure for each $x \in X \cup \partial - B$. $P_x(\Lambda)$ is for each $\Lambda \in \mathscr{M}^0$ a Borel measurable function of $x \in X \cup \partial - B$ because it is the composition of two Borel functions q and $\tilde{P}\cdot(\Lambda)$.

The field $\mathscr{M}$ is the completion of $\mathscr{M}^0$ with respect to the family of measures $\{P_\mu(\cdot) = \int_{X \cup \partial - B} \mu(dx) P_x(\cdot)$; μ is a finite measure on $X \cup \partial - B\}$. This follows from Theorem 3.10(ii) and from the following observation: there is a one-to-one correspondence between finite measures on $X \cup \partial - B$ and those on $\tilde{X} \cup \tilde{\partial} - \tilde{B}$ by the relation $\mu(E) = \tilde{\mu}(q(E))$ and in this case we have $P_\mu(\Lambda) = \tilde{P}_{\tilde{\mu}}(\Lambda)$, $\Lambda \in \mathscr{M}^0$. The field $\mathscr{M}_t$ is the completion of $\mathscr{M}_t^0$ in $\mathscr{M}$ with respect to the same family of measures.

(III) *M is a Hunt process on $X \cup \partial - B$.* M has namely the following properties (**M**.a)$\sim$(**M**.e).

(**M**.a) The sample path X_t is right continuous for $t \geqq 0$ and has the left limit in $X \cup \partial - B$ for $t > 0$, P_x-a.e. ($x \in X \cup \partial - B$). Further $X_t = \partial$ for $t \geqq \zeta$ P_x-a.e., where $\zeta(\omega) = \inf\{t \geqq 0;\ X_t(\omega) = \partial\}$.

(**M**.b) $P_x(X_0 = x) = 1$, $x \in X \cup \partial - B$.

(**M**.c) $\mathscr{M}_t = \mathscr{M}_{t+}$, $t > 0$. $\mathscr{M}_t$ is the completion in the sense of the preceding paragraph of the σ-field generated by $\{X_s,\ t \geqq s \geqq 0\}$.

(**M**.d) Strong Markov property.

(**M**.e) Quasi-left continuity.

In the present case, the statement (**M**.a) is valid for all $\omega \in \Omega$ in view of Theorem 3.10(iv) and (4.1). By (4.2) we see $\tilde{X}_0 \subset \tilde{B}$ and $P_x(X_0 = x) = \tilde{P}_{qx}$ $(\tilde{X}_0 = qx) = 1$, $x \in X \cup \partial - B$ yielding (**M**.b). The second property of (**M**.c) is evident by the observation of the preceding paragraph. The first is due to the right continuity of $\tilde{\mathscr{B}}_t$. (**M**.d) follows from Theorem 3.10(i). To see this, consider an $\mathscr{M}_t$-stopping time T and a set $\Lambda \in \mathscr{M}_T$. Then for any $t > 0$ and any Borel set $E \subset X \cup \partial - B$,

$$\begin{aligned} P_x(X_{T+t} \in E, \Lambda) &= \tilde{P}_{qx}(\tilde{X}_{T+t} \in qE, \Lambda) = \tilde{E}_{qx}(\tilde{P}_{\tilde{X}_T}(\tilde{X}_t \in qE), \Lambda) \\ &= \tilde{E}_{qx}(P_{q^{-1}(\tilde{X}_T)}(X_t \in E), \Lambda) = E_x(P_{X_T}(X_t \in E), \Lambda). \end{aligned}$$

(**M**.e) is due to Theorem 3.10(iv): if $\{T_n\}$ is an increasing sequence of $\mathscr{M}_t$-stopping times and if $T = \lim_{n \to +\infty} T_n$, then

$$\begin{aligned} P_x\Big(\lim_{n \to +\infty} X_{T_n} = X_T, T < \infty\Big) &= \tilde{P}_{qx}\Big(\lim_{n \to +\infty} X_{T_n} = X_T, T < \infty\Big) \\ &= \tilde{P}_{qx}(T < \infty) = P_x(T < \infty). \end{aligned}$$

(IV) *The resolvent of the process M generates $(\mathscr{F}^*, \mathscr{E})$.* If we define the resolvent kernel $\{R_\alpha(x, E), \alpha > 0\}$ on $x - B$ by

$$R_\alpha(x, E) = E_x\Big(\int_0^{+\infty} e^{-\alpha t} \chi_E(X_t)\, dt\Big), \tag{4.8}$$

then for any nonnegative universally measurable function f on $X-B$ which belongs to the space $L^2(X; m)$,

$$R_\alpha f \in \mathscr{F}^* \tag{4.9}$$

and $R_\alpha f$ satisfies the equation (0.5).

In order to prove this statement, put $\tilde{f}=\Phi^* f$ with Φ^* defined by (2.2): $\Phi^* f(\tilde{x}) = f(q^{-1}\tilde{x})$, $\tilde{x} \in \tilde{X}-\tilde{B}$. $\tilde{f}$ is then a nonnegative universally measurable function on $\tilde{X}-\tilde{B}$. If we extend $\tilde{f}$ to a function on $\tilde{X}$ by setting $\tilde{f}(\tilde{x})=0$, $\tilde{x} \in \tilde{B}$, then $\tilde{f}$ becomes universally measurable on $\tilde{X}$. Further by virtue of (q.2) of Theorem 2.1, $\tilde{f} \in L^2(\tilde{X}; \tilde{m})$. Hence we can see by Lemma 3.1 that $\tilde{R}_\alpha \tilde{f}$ belongs to $\tilde{\mathscr{F}}^*$ and satisfies the equation $\tilde{\mathscr{E}}^\alpha(\tilde{R}_\alpha \tilde{f}, \tilde{v}) = (\tilde{f}, \tilde{v})_{\tilde{X}}$ for all $\tilde{v} \in \tilde{\mathscr{F}}$. On the other hand we have $\tilde{R}_\alpha \tilde{f}(\tilde{x}) = \Phi^*(R_\alpha f)(\tilde{x})$, $\tilde{x} \in \tilde{X}-\tilde{B}$, because

$$\begin{aligned} R_\alpha f(x) &= E_x\left(\int_0^{+\infty} e^{-\alpha t} f(X_t)\, dt\right) \\ &= \tilde{E}_{qx}\left(\int_0^{+\infty} e^{-\alpha t} \tilde{f}(\tilde{X}_t)\, dt\right) = (\tilde{R}_\alpha \tilde{f})(qx), \qquad x \in X-B. \end{aligned}$$

Now (q.1) of Theorem 2.1 leads us to (4.9) and the equality $\mathscr{E}^\alpha(R_\alpha f, v) = \tilde{\mathscr{E}}^\alpha(\tilde{R}_\alpha \tilde{f}, \Phi^* v) = (\tilde{f}, \Phi^* v)_{\tilde{X}}$ which is equal to $(f, v)_X$ for $v \in \mathscr{F}^*$ according to the property (q.2).

4.2. *Generalizations of theorems of* §3. All results of §3 are still valid when the strongly regular D-space and the associated Ray process of §3 are replaced with the regular D-space $(X, m, \mathscr{F}, \mathscr{E})$ and its associated Markov process due to Theorem 4.1 respectively.

We consider a Borel polar set $B \subset X$ and a Markov process

$$M = (\Omega, \mathscr{M}, \mathscr{M}_t, X_t, P_x)$$

on $X \cup \partial - B$ which enjoys the properties (III) and (IV) of subsection 4.1([19]).

Observe that in the course of arguments of §3 the speciality of the Ray process that its resolvent leaves the space $C(X)$ invariant has been essentially used nowhere except in the proof of Lemma 3.1. Besides we now have the counterpart of Lemma 3.1, namely, property (IV) of 4.1. Thus all the arguments of §3 are immediately applicable to the present context to establish following generalizations.

THEOREM 4.2. *If a function u is nonnegative universally measurable on $X-B$, belongs to the space $\mathscr{F}$ and has its regularization $\tilde{u}$ on $X-B$, then $\tilde{u}$ is a quasi-continuous modification of u. In particular any excessive function on $X-B$ belonging to $\mathscr{F}$ is an element of $\mathscr{F}^*$.*

This corresponds to Theorem 3.1. The next is a generalization of Lemma 3.4, Theorems 3.7 and 3.8.

([19]) It is sufficient to assume property (IV) only for $f \in L^2 \cap C$.

THEOREM 4.3. *Let A be an open or a closed subset of X. Put*

$$\mathscr{F}^*_{X-A} = \{u \in \mathscr{F}^*;\ u = 0 \text{ q.e. on } A\}.$$

(i) *The D-space $(\mathscr{F}^*_{X-A}, \mathscr{E})$ is generated by the resolvent kernel $R_\alpha(x, E) = E_x(\int_0^{\sigma_A} e^{-\alpha t}\chi_E(X_t)\,dt)$, $\alpha > 0$, on $X-B$: for each nonnegative universally measurable function f on $X-B$ belonging to $L^2(X; m)$, the function $R^0_\alpha f(x) = \int_{X-B} R^0_\alpha(x, dy)f(y)$, $x \in X-B$, belongs to the space $\mathscr{F}^*_{X-A}$ and the equation $\mathscr{E}^\alpha(R^0_\alpha f, v) = (f, v)_X$ holds for every $v \in \mathscr{F}^*_{X-A}$.*

(ii) *Denote by $\mathscr{H}^*_\alpha$ the orthogonal complement of $\mathscr{F}^*_{X-A}$ in the Hilbert space $(\mathscr{F}^*, \mathscr{E}^\alpha)$ and define the kernel $H_\alpha(x, E)$ on $X-B$ by*

$$H_\alpha(x, E) = E_x(\exp(-\alpha\sigma_A);\ X_{\sigma_A} \in E).$$

Then the relation $P_{\mathscr{H}^*_\alpha}u = H_\alpha u$ holds for every $u \in \mathscr{F} \cap C(X)$ where $P_{\mathscr{H}^*_\alpha}$ stands for the projection on $\mathscr{H}^*_\alpha$.

(iii) *If A is an open set of the class $\mathscr{U}$ or a compact set, then the equality* (3.35) *holds q.e. on X.*

The first assertion of the above theorem generalizes Lemma 3.7(ii) of [8]. Finally we give generalized versions of Theorem 3.11, 12 and 13.

THEOREM 4.4. *A function $u \in \mathscr{F}$ is quasi-continuous if and only if u is finely continuous q.e. on X and universally measurable.*

This essentially generalizes a theorem of J. Deny and J. Lions [5, Chapitre II, Théorème 3.2] concerning BLD functions and Cartan's fine topology.

THEOREM 4.5. *The following statements are equivalent to each other:*

(i) *A Borel set A is polar.*

(ii) (3.49) *holds for m-almost all $x \in X$.*

(iii) *There exists an m-negligible Borel set $C \supset A \cup B$ such that $X-C$ is $\mathbf{M}$-invariant.*

THEOREM 4.6. *The following conditions are mutually equivalent:*

(i) *A Borel set A is polar if and only if* (3.49) *is satisfied for every $x \in X-B$.*

(ii) *m is a reference measure of the process $\mathbf{M}$.*

(iii) *$R_\alpha(x, \cdot)$ is absolutely continuous with respect to m for each $x \in X-B$ and* $\alpha > 0$.

(iv) *m is strictly positive on every nonempty finely open set.*

The last condition of Theorem 4.6 follows from condition (iii). The converse is also true because of symmetry of R_α.

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ON THE GENERATION OF MARKOV PROCESSES BY SYMMETRIC FORMS

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§1. Introduction

The advance of the theory of Markov processes during the past 15 years was principally led by the notion of the strong Markov property. Among others, Hunt built up a probabilistic potential theory based on a certain strong Markov process now called a Hunt process (cf. R.M. Blumenthal and R.K. Getoor [2]) and Dynkin established the theory of transformations of a slightly more general strong Markov process named a standard process (cf. E.B. Dynkin [5]).

As compared with such an intensive development of the theory, the existence theorems of strong Markov processes are not rich enough. In this article we will give a method of producing a wide class of symmetric Hunt processes from some concrete analytic data.

In order to illustrate our method, take a look at a formally self-adjoint second order elliptic partial differential operator

$$(1.1)\qquad Au(x) = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u(x)}{\partial x_j}\right) - c(x)u(x)$$

acting on functions u defined on an Euclidean domain $D \subset R^N$. Kolmogorov first discovered that the transition function of a given Markov process satisfies under certain regularity conditions a parabolic differential equation, which is of the form $\frac{\partial u(t,x)}{\partial t} = Au(t,x)$ in the symmetric case.

Given conversely coefficients $\{a_{ij}, c\}$, there have been two ways of constructing a strong Markov process whose transition function satisfies the given parabolic equation:

1°. an analytical method relying essentially upon the theory of partial differential equations,

2°. a probabilistic method of solving Ito's stochastic differential equations.

In the following, we will present another method:

3°. an analytical method relying essentially upon an analytic potential theory.

Instead of thinking about the elliptic operator (1.1) directly, we consider the bilinear form

$$(1.2) \quad \varepsilon(u, v) = (-Au, v)$$
$$= \int_D \sum_{i,j=1}^{N} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} a_{ij}(x)dx + \int_D u(x)v(x)c(x)dx$$

defined for $u, v \in C_0^\infty(D)$, $C_0^\infty(D)$ being the space of all infinitely differentiable functions with compact supports in D. Much more generally, we start with the integro-differential expression of the type

$$(1.3) \quad \varepsilon(u, v) = \sum_{i,j=1}^{N} \int_D \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \nu_{ij}(dx)$$
$$+ \int_{D \times D} (u(x)-u(y))(v(x)-v(y))\Phi(dx,dy) + \int_D u(x)v(x)n(dx),$$

$u, v \in C_0^\infty(D)$, and regard this as a symmetric bilinear form on the space $L^2(D\ ;\ m)$ based on another measure m.

We formulate our problem as follows; given a set of measures

$\{\nu_{ij}, \Phi, n, m\}$, how can we produce a strong Markov process on D governed by the form (1.3) in a certain sense? Our procedure consists of two steps. The first step is, given $\{\nu_{ij}, \Phi, n, m\}$, to construct a regular Dirichlet form (§2, 3, 6). The second step is, given a regular Dirichlet form, to construct a Hunt process (§4, 5). In 1959, A. Beurling and J. Deny [1] introduced for the first time the axioms of a Dirichlet space and developed an associated potential theory. It is this axiomatic potential theory that we are going to use in the second step.

The important notions in the above theory are, among others, the capacity of sets and the quasi-continuity of functions. Given a regular Dirichlet form, there corresponds uniquely a Markov semigroup $\{T_t\}$ on L^2-space (§3). By virtue of the potential theory, the functions $T_t u$ for sufficiently many u have the following properties : each $T_t u$ has a quasi-continuous modification and $\lim_{t_n \to 0} T_{t_n} u = u$ except on a set of zero capacity.

Accordingly we encounter in the second step a semigroup which is not very much worse than a strongly continuous Feller semigroup. Going along a similar line as in Blumenthal-Getoor [2 ; pp 46-50] but ignoring successively the sets of capacity zero on which things might go wrong, we can finally get a Hunt process outside some Borel set of capacity zero (§5). We regard two such Hunt processes as equivalent if they obey the same law outside some Borel set of capacity zero. Our Hunt process to be constructed should be considered as a representative of an equivalence class.

By allowing the state space to narrow this way, we are able to obtain a considerably wider class of symmetric Hunt processes than we ever know. We can observe this in §6 by several examples.

The idea for the first step appeared already in [9 ; Appendix], but we will exploit the method further by introducing the notion of Markov symmetric forms (§2).

The second step has been carried out in [10] for the first time but by taking a rather indirect course. Quite recently M. Silverstein [16] pointed out that a much more direct course is available. Our present purely potential theoretic approach is based on Silverstein's idea, although ours is different in many technical points.

It is conjectured that, if the measure Φ vanishes identically in the expression (1.3), then the constructed process is a diffusion : almost all sample paths are continuous. This condition for the form ε is closely related to its local property already defined by Beurling-Deny [1]. In this connection, we refer the reader to the article by N. Ikeda and S. Watanabe [11].

The third case (1°.c) in Example 1 of §6 is due to K. Sato who permits me to mention it in this paper.

§2. Markov symmetric forms

Let X be a locally compact separable Hausdorff space and m be a positive Radon measure on X. A symmetric form ε on the real L^2-space $L^2(X ; m)$ is, by definition, a non-negative definite symmetric bilinear form defined on $\mathcal{D}[\varepsilon] \times \mathcal{D}[\varepsilon]$, $\mathcal{D}[\varepsilon]$ being a dense linear subspace of $L^2(X ; m)$.

Given a symmetric form ε, we say that every unit contraction operates on ε if, for any $u \in \mathcal{D}[\varepsilon]$, the function $v = (0 \vee u) \wedge 1$ is again in $\mathcal{D}[\varepsilon]$ and $\varepsilon(v, v) \leq \varepsilon(u, u)$. We now define a notion of Markovity of ε which is more useful and general than the above one. They are equivalent however when ε is a closed form in the sense that will be specified later.

A symmetric form ε is called <u>Markov</u> if, for any $\delta > 0$, there exists a non-decreasing function $\phi_\delta(t)$, $-\infty < t < \infty$, satisfying the following conditions.

(2.1) $\phi_\delta(t) = t$ for $0 \leqq t \leqq 1$. Further $|\phi_\delta(t)| \leqq t$ and $-\delta \leqq \phi_\delta(t) \leqq 1 + \delta$ for all t.

(2.2) If a function u belongs to $\mathcal{D}[\varepsilon]$, then so is the composite function $\phi_\delta(u)$. Moreover $\varepsilon(\phi_\delta(u), \phi_\delta(u)) \leqq \varepsilon(u, u)$.

We are particularly interested in the following examples. Let D be a domain of the N-space R^N.

Example 1. The form

(2.3) $$\varepsilon(u, v) = \int_D \sum_{i,j=1}^{N} \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_j} a_{ij}(x)dx$$
$$+ \int_D \int_D (u(x) - u(y))(v(x) - v(y))\Phi(dx, dy) + \int_D u(x)v(x)n(dx)$$

(2.4) $$\mathcal{D}[\varepsilon] = C_0^\infty(D)$$

is a Markov symmetric form on $L^2(D ; m)$. Here $a_{ij}(x)$, $1 \leqq i,j \leqq N$, $x \in D$, is a symmetric non-negative definite matrix in i,j, and a locally integrable function in x. Φ is a positive symmetric measure on $D \times D$ such that $\int_K \int_K |x - y|^2 \Phi(dx, dy) < \infty$ for any compact set $K \subset D$. m and

n are positive Radon measures on D, m being everywhere dense. In order to verify the Markovity of ε, it suffices to take a C^∞-function $\phi_\delta(t)$ satisfying not only (2.1) but also the property $0 \leqq \phi'_\delta(t) \leqq 1$ for all t.

The next example is a case that the measure ν_{ij} in the expression (1.3) is not necessarily absolutely continuous with respect to the Lebesgue measure.

Example 2. On R^2, we define

$$(2.5) \qquad \varepsilon(u, v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial u(x)}{\partial x_1} \frac{\partial v(x)}{\partial x_1} dx_1 \mu(dx_2) + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial u(x)}{\partial x_2} \frac{\partial v(x)}{\partial x_2} \nu(dx_1) dx_2$$

$$(2.6) \qquad \mathcal{D}[\varepsilon] = C_0^\infty(R^2),$$

where μ and ν are positive Radon measures on R^1. This is a Markov symmetric form on $L^2(R^2) = L^2(R^2 ; dx)$. The proof is the same as in Example 1.

Example 3. The form

$$(2.7) \qquad \varepsilon(u, v) = \frac{1}{2}\int_D \sum_{i=1}^{N} \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_i} dx$$

with $\mathcal{D}[\varepsilon]$ given by (2.4) is a Markov symmetric form on $L^2(D) = L^2(D ; dx)$. This is a very special case of Example 1. But we can get Markov symmetric forms on $L^2(D)$ of quite different characters if other domains $\mathcal{D}[\varepsilon]$ rather than (2.4) are adopted. For instance

(2.9) $\mathcal{D}[\varepsilon] = H^1(D)$,

where $\hat{C}^\infty(D)$ is the restrictions to D of functions in $C_0^\infty(R^N)$ and $H^1(D)$ is the space of those functions of $L^2(D)$ whose distribution derivatives are also in $L^2(D)$. The Markovity of the form (2.7) with the domain (2.8) or (2.9) is verified in the same way as before.

§ 3. Generation of Dirichlet forms by a Markov one

A symmetric form ε on $L^2(X ; m)$ is called closed if $\mathcal{D}[\varepsilon]$ is complete with metric $\sqrt{\varepsilon(u, u) + (u, u)}$, where $(\, , \,)$ denotes the L^2-inner product. The space $\mathcal{D}[\varepsilon]$ is then a real Hilbert space with the inner product

(3.1) $\varepsilon_\alpha(u, v) = \varepsilon(u, v) + \alpha(u, v)$

for each $\alpha > 0$.

Let B be a non-negative definite self-adjoint linear operator on $L^2(X ; m)$ and $\{E_\lambda\}$ be the associated spectral family. For any non-negative continuous function $\phi(t)$ on $[0, \infty)$, the operator $\phi(B)$ defined by $\mathcal{D}(\phi(B)) = \{u \in L^2 ; \int_0^\infty \phi(\lambda)^2 d(E_\lambda u, u) < +\infty\}$, $(\phi(B)u, v) = \int_0^\infty \phi(\lambda) d(E_\lambda u, v)$, $u \in \mathcal{D}(\phi(B))$, $v \in L^2$, is again a non-negative definite self-adjoint operator. Now we proceed to

Theorem 3.1. All closed symmetric forms ε on $L^2(X ; m)$ and all-non-negative definite self-adjoint operators $-A$ on

$L^2(X\ ;\ m)$ stand in one to one correspondence. The correspondence is given by

(3.2) $\varepsilon(u, v) = (\sqrt{-A}\, u, \sqrt{-A}\, v)$

(3.3) $\mathcal{D}[\varepsilon] = \mathcal{D}(\sqrt{-A})$.

Let $-A$ be a non-negative definite self-adjoint operator, then so is $\sqrt{-A}$. Consequently $\sqrt{-A}$ is a closed linear operator, which means that the symmetric form ε defined by (3.2) and (3.3) is closed, proving the half of Theorem 3.1.

Before completing the proof of Theorem 3.1, we will give two lemmas. A family $\{G_\alpha, \alpha > 0\}$ of symmetric operators on $L^2(X\ ;\ m)$ is called a symmetric strong continuous contraction resolvent on L^2 if $G_\alpha - G_\beta + (\alpha - \beta)G_\alpha G_\beta = 0$, $(\alpha G_\alpha u, \alpha G_\alpha u) \leqq 1$, $(\alpha G_\alpha u - u, \alpha G_\alpha u - u) \longrightarrow 0$, $\alpha \longrightarrow \infty$, $u \in L^2$. Its generator A is defined by $A = \alpha I - G_\alpha^{-1}$, $\mathcal{D}(A) = \mathcal{R}(G_\alpha)$. A family $\{T_t, t > 0\}$ of symmetric operators on $L^2(X\ ;\ m)$ is called a symmetric strongly continuous contraction semigroup on L^2 if $T_t T_s = T_{t+s}$, $(T_t u, T_t u) \leqq 1$, $(T_t u - u, T_t u - u) \longrightarrow 0$, $t \longrightarrow 0$, $u \in L^2$. Its generator A is defined by $Au = \lim_{t \downarrow 0} \frac{T_t u - u}{t}$, $\mathcal{D}(A) = \{u \in L^2\ ;\ Au \in L^2\}$. In this case, A coincides with the generator of the resolvent $G_\alpha = \int_0^\infty e^{-\alpha t}\, T_t\, dt$.

Lemma 3.1. (i) For a given symmetric strongly continuous contraction semigroup or resolvent on L^2, let us denote its generator by A. Then $-A$ is a non-negative definite self-adjoint operator. (ii) Conversely, given a non-negative definite *self-adjoint operator* $-A$ *on* L^2, *then* $T_t = \exp(tA)$ (*resp.* $G_\alpha = (\alpha - A)^{-1}$) *becomes a symmetric strongly continuous contraction semigroup* (*resp. resolvent*) *on* L^2 *whose generator coincides with the given* A.

We only note the following : given any symmetric strongly continuous contraction resolvent $\{G_\alpha, \alpha > 0\}$, then $\frac{d}{d\alpha}(G_\alpha u, u) \leqq 0$ and $\lim_{\alpha\to\infty}(G_\alpha u, u) = 0$, $u \in L^2$. Hence G_α is non-negative definite. Let A be the generator of G_α. A is then self-adjoint and $-A$ is non-negative definite because $(-Au, u) + (\alpha u, u) = (G_\alpha^{-1}u, u) \geqq 0$, $u \in \mathcal{D}(A)$, for any $\alpha > 0$.

Lemma 3.2. *Let* $-A$ *be a non-negative definite self-adjoint operator on* L^2, ε *be the closed symmetric form generated by* $-A$ *according to* (3.2) *and* (3.3) *and finally* T_t (*resp.* G_α) *be the semigroup* (*resp. resolvent*) *generated by* A *according to* Lemma 3.1 (ii). *Then*

(i) $T_t(L^2) \subset \mathcal{D}[\varepsilon]$,

$$\varepsilon(T_t u, T_t u) \leqq \frac{1}{2t}((u, u) - (T_t u, T_t u)), \quad u \in L^2,$$

(ii) $G_\alpha(L^2) \subset \mathcal{D}[\varepsilon]$. *For any fixed* $u \in L^2$,

$$\varepsilon_\alpha(G_\alpha u, v) = (u, v), \quad v \in \mathcal{D}[\varepsilon].$$

(iii) *For any* $u \in \mathcal{D}[\varepsilon]$, $T_t u \longrightarrow u$ and

$$\frac{1}{t}(G_1 u - e^{-t} G_1 T_t u) \longrightarrow u \quad \text{as} \quad t \longrightarrow 0 \quad \text{in} \quad \varepsilon_1\text{-norm}.$$

Proof. All statements can be obtained simply by using the spectral family $\{E_\lambda\}$ associated with $-A$. For instance, integrating the inequality $\lambda e^{-2t\lambda} \leqq \frac{1}{2t}(1 - e^{-2t\lambda})$ with the measure $d(E_\lambda u, u)$, we arrive at (i).

We now return to the proof of the remaining half of Theorem 3.1. Given a symmetric form ε, there exists, for each $\alpha > 0$ and $u \in L^2$, a unique element $G_\alpha u \in \mathcal{D}[\varepsilon]$ such as $\varepsilon_\alpha(G_\alpha u, v) = (u, v)$, $v \in \mathcal{D}[\varepsilon]$, in view of Riesz representation theorem. It is quite easy to see that this $\{G_\alpha, \alpha > 0\}$ is a symmetric contraction resolvent on L^2. For $u \in \mathcal{D}[\varepsilon]$, $\beta(\beta G_\beta u - u, \beta G_\beta u - u) \leqq \varepsilon_\beta(\beta G_\beta u - u, \beta G_\beta u - u) = \beta^2(G_\beta u, u) - \beta(u, u) + \varepsilon(u, u) \leqq \varepsilon(u, u)$, and hence $(\beta G_\beta u - u, \beta G_\beta u - u) \leqq \frac{1}{\beta}\varepsilon(u, u) \longrightarrow 0$, $\beta \longrightarrow \infty$. Since $\mathcal{D}[\varepsilon]$ is dense in L^2 and $\{G_\alpha\}$ is contraction, we can see that $\{G_\alpha\}$ is also strongly continuous.

Let A be the generator of $\{G_\alpha\}$. $-A$ is then non-negative definite and self-adjoint by Lemma 3.1 (i). Let ε' be the closed symmetric form generated by $-A$ according to (3.2) and (3.3). By virtue of Lemma 3.2 (ii), $G_\alpha(L^2) \subset \mathcal{D}[\varepsilon']$ and $\varepsilon'_\alpha(G_\alpha u, G_\alpha u) = (G_\alpha u, u)$. Therefore $\varepsilon'_\alpha = \varepsilon_\alpha$ on the space $G_\alpha(L^2)$, which is however dense in $\mathcal{D}[\varepsilon]$ (resp. $\mathcal{D}[\varepsilon']$) with respect to the metric ε_α (resp. ε'_α), getting $\varepsilon' = \varepsilon$. Any closed symmetric form is thus generated by a non-negative definite self-adjoint operator, completing the proof of Theorem 3.1.

A bounded linear operator S on $L^2(X ; m)$ is called <u>Markov</u> if $0 \leqq Su \leqq 1$ m - a.e. whenever $u \in L^2$ and $0 \leqq u \leqq 1$ m - a.e.

Suppose that a symmetric form ε has the following property :

if $u \in \mathcal{D}[\varepsilon]$ and $v \in L^2$ are such that there exist their Borel modifications $\tilde{u}$ and $\tilde{v}$ satisfying the inequalities $|\tilde{v}(x)| \leqq |\tilde{u}(x)|$, $|\tilde{v}(x) - \tilde{v}(y)| \leqq |\tilde{u}(x) - \tilde{u}(y)|$ for every $x, y \in X$, then $v \in \mathcal{D}[\varepsilon]$ and $\varepsilon(v, v) \leqq \varepsilon(u, u)$. In this case we say that <u>every normal contraction operates on</u> ε.

<u>Theorem 3.2</u>. <u>Let</u> ε <u>be a closed symmetric form and</u> $\{T_t, t > 0\}$, $\{G_\alpha, \alpha > 0\}$ <u>be the associated semigroup and resolvent according to</u> <u>Theorem</u> 3.1 <u>and</u> <u>Lemma</u> 3.1. <u>Then the following five conditions are equivalent to each other</u>:

(a) ε <u>is Markov</u>,

(b) T_t <u>is Markov for each</u> $t > 0$,

(c) αG_α <u>is Markov for each</u> $\alpha > 0$,

(d) <u>every unit contraction operates on</u> ε,

(e) <u>every normal contraction operates on</u> ε.

<u>Proof</u>. (a) $\Longrightarrow$ (c). Take any function $u \in L^2$ such as $0 \leqq u \leqq 1$ m - a.e. For any $\delta > 0$ and $\alpha > 0$, let $\phi_\delta(t)$ be the function in the definition of the Markovity of ε (§2) and put $\phi_{\frac{1}{\alpha},\delta}(t) = \frac{1}{\alpha}\phi_{\alpha\delta}(\alpha t)$. Since $G_\alpha u \in \mathcal{D}[\varepsilon]$, the composite function $w = \phi_{\frac{1}{\alpha},\delta}(G_\alpha u)$ also belongs to $\mathcal{D}[\varepsilon]$, and $\varepsilon(w, w) \leqq \varepsilon(G_\alpha u, G_\alpha u)$.

Define a quadratic form $\Psi = \Psi_{\alpha,u}$ on $\mathcal{D}[\varepsilon]$ by

$$(3.4) \quad \Psi(v) = \varepsilon(v, v) + \alpha\left(v - \frac{u}{\alpha}, v - \frac{u}{\alpha}\right), \quad v \in \mathcal{D}[\varepsilon],$$

then

(3.5) $\Psi(G_\alpha u) + \varepsilon_\alpha(G_\alpha u - v, G_\alpha u - v) = \Psi(v)$,

namely, $G_\alpha u$ is a unique element in $\mathcal{D}[\varepsilon]$ minimizing the quadratic form Ψ. However it is easy to see that $(w - \frac{u}{\alpha}, w - \frac{u}{\alpha}) \leqq (G_\alpha u - \frac{u}{\alpha}, G_\alpha u - \frac{u}{\alpha})$. Hence $\Psi(w) \leq \Psi(G_\alpha u)$, from which follows $G_\alpha u = w$. Therefore $-\delta \leq G_\alpha u \leq \frac{1}{\alpha} + \delta$ for any $\delta > 0$, proving the Markovity of αG_α.

(c) $\Longrightarrow$ (b). This follows from

$$T_t u = \lim_{\beta\to\infty} e^{-t\beta} \sum_{n\geq 0} \frac{(t\beta)^n}{n!} (\beta G_\beta)^n u.$$

The implication (b) $\Longrightarrow$ (c) and (e) $\Longrightarrow$ (d) $\Longrightarrow$ (a) are trivial. As for the proof of the implication (c) $\Longrightarrow$ (e), we refer to J.Deny [3 ; pp 155].

Turning to the main task of this section, let us introduce an important notion.

We call a symmetric form ε closable if $\varepsilon(u_n, u_n) \longrightarrow 0$ whenever $u_n \in \mathcal{D}[\varepsilon]$ satisfies $\varepsilon(u_n - u_m, u_n - u_m) \longrightarrow 0$ and $(u_n, u_n) \longrightarrow 0$.

Given a symmetric form ε, any closed symmetric form $\tilde{\varepsilon}$ is called a closed extension of ε if $\mathcal{D}[\tilde{\varepsilon}] \supset \mathcal{D}[\varepsilon]$ and $\tilde{\varepsilon} = \varepsilon$ on $\mathcal{D}[\varepsilon] \times \mathcal{D}[\varepsilon]$.

The closability of a symmetric form is a necessary and sufficient condition for it to admit at least one closed extension. For a closable symmetric form ε, its smallest closed extension $\bar{\varepsilon}$ can be defined as follows: the domain $\mathcal{D}[\bar{\varepsilon}]$ of $\bar{\varepsilon}$ is just the abstract completion of $\mathcal{D}[\varepsilon]$ by means of the metric $\sqrt{\varepsilon(u, u) + (u, u)}$.

Theorem 3.3. *If a symmetric form* ε *is Markov and closable, then its smallest closed extension* $\bar{\varepsilon}$ *is also Markov.*

Proof. Let $\{G_\alpha, \alpha > 0\}$ be the resolvent associated with the closed symmetric form $\bar{\varepsilon}$ according to Theorem 3.1 and Lemma 3.1. On account of Theorem 3.2, it suffices to show that αG_α is Markov. Take any function $u \in L^2$ such as $0 \leqq u \leqq 1$ m - a.e. Then by making use of the identity (3.5) and following essentially the same line as in [9 ; Appendix], we can get $-\delta \leqq G_\alpha u \leqq \frac{1}{\alpha} + \delta$ m - a.e. for any $\delta > 0$.

A closed Markov symmetric form is called a *Dirichlet form*. Theorem 3.3 provides us with a method of generating a Dirichlet form starting with a form of the type in the preceding examples. Once we get a Dirichlet form, then we have a symmetric Markov semigroup on L^2 by virtue of Theorem 3.2. We will assert in § 5 that we can even get a Hunt process provided that the Dirichlet form is regular. In the final section, the examples of § 2 will be examined to see whether they generate regular Dirichlet forms.

Incidentally we mention some more about closed extensions. Suppose that a symmetric form ε satisfies the following:

(3.6) if $u_n \in \mathcal{D}[\varepsilon]$ converges to zero in L^2, then $\varepsilon(u_n, v) \longrightarrow 0$ for any $v \in \mathcal{D}[\varepsilon]$.

Then ε is readily seen to be closable. In particular this criterion applies to the case when a symmetric form is expressible by some *symmetric operator*.

Assume that S is a symmetric linear operator densely defined on $L^2(X ; m)$ such as $(-Su, u) \geqq 0$ for all $u \in \mathcal{D}(S)$. Then

(3.7) $\varepsilon_S(u, v) = (-Su, v)$, $\mathcal{D}[\varepsilon_S] = \mathcal{D}(S)$, is a closable symmetric form. Let $-A_F$ be the non-negative definite self-adjoint operator associated with the smallest closed extension of ε_S. A_F turns out to be a self-adjoint extension of S. A_F is called Friedrichs extension of S.

To any non-negative definite self-adjoint extension $-A$ of $-S$, there corresponds a closed symmetric extension ε_A of ε_S. Among them, there is one, say A_K, which is maximum in the sense that $\mathcal{D}[\varepsilon_{A_K}] \supset \mathcal{D}[\varepsilon_A]$, $\varepsilon_{A_K}(u, u) \leqq \varepsilon_A(u, u)$, $u \in \mathcal{D}[\varepsilon_A]$, for every A's. We call A_K Krein extension of S ([13]).

Suppose that the given ε_S is Markov. Theorem 3.2 and 3.3 tell us that A_F then generates a Markov semigroup. However the same statement does not hold for A_K in general. In a sense ε_{A_K} is too big to be Markov. As for a description of all possible closed Markov extensions ε_A of ε_S, see the papers by the auther [8] and by J. Elliott [6]. Among those extensions, there is the maximum one, which is related to the reflecting barrier Markov process (c.f. § 6, Example 3).

§ 4. Potential theoretic preparations

From now on, we assume that $m(A) > 0$ for any non-empty open set $A \subset X$. Let us consider a Dirichlet form ε on $L^2(X ; m)$ which is regular in the following sense : the space $\mathcal{D}[\varepsilon] \cap C(X)$ is dense both in $\mathcal{D}[\varepsilon]$ with metric ε_1 and in $C(X)$ with the uniform norm. Here $C(X)$ is the space of all continuous functions on X vanishing at infinity.

Denote by $\mathcal{O}$ the class of all open subsets of X. (1-)capacity of a set $A \in \mathcal{O}$ is defined by

$$(4.1) \quad \mathrm{Cap}(A) = \begin{cases} \inf_{u \in \mathcal{L}_A} \varepsilon_1(u, u) & \mathcal{L}_A \neq \phi \\ \infty & \mathcal{L}_A = \phi, \end{cases}$$

where $\mathcal{L}_A = \{u \in \mathcal{D}[\varepsilon] \; ; \; u \geq 1 \quad m - \text{a.e.} \quad \text{on} \quad A\}$. The capacity of any subset $B \subset X$ is defined by $\mathrm{Cap}(B) = \inf_{B \subset A,\ A \in \mathcal{L}_A} \mathrm{Cap}(A)$. This gives rise to a strongly subadditive Choquet capacity [10 ; Theorem 1.1].

Let us put $\Theta_0 = \{A \in \Theta \; ; \; \mathcal{L}_A \neq \phi\}$. For $A \in \Theta_0$, there is a unique element $e_A \in \mathcal{D}[\varepsilon]$ called the (1-)<u>equilibrium potential</u> which minimizes $\varepsilon_1(u, u)$ on $\mathcal{L}_A[\varepsilon]$. It has the following properties.

(4.2) $0 \leqq e_A \leqq 1 \quad m - \text{a.e.} \quad \text{on} \quad X$

$e_A = 1 \quad m - \text{a.e.} \quad \text{on} \quad A.$

(4.3) $\varepsilon_1(e_A, v) \geqq 0$ for any $v \in \mathcal{D}[\varepsilon]$ such as $v \geqq 0$ m - a.e. on A.

(4.4) $e^{-t} T_t e_A \leqq e_A \qquad m - \text{a.e.} \quad \text{on} \quad X.$

Here T_t is the Markov semigroup associated with ε. In fact, for any $v \in L^2$ such as $v \geqq 0$ m - a.e., $(e_A - e^{-t} T_t e_A, v) = \varepsilon_1(e_A - e^{-t} T_t e_A, G_1 v) = \varepsilon_1(e_A, G_1 v - e^{-t} T_t G_1 v)$ which is non-negative in view of (4.3) and the inequality $G_1 v \geqq e^{-t} T_t G_1 v$.

(4.5) If $A, B \in \Theta_0$ and $A \subset B$, then $e_A \leqq e_B$ m - a.e.

This follows from $\varepsilon_1(e_A - e_A \wedge e_B, e_A - e_A \wedge e_B)$
$= \varepsilon_1(-e_A \wedge e_B, (e_A - e_B)^+)$

$= \varepsilon_1((e_A - e_B)^-, (e_A - e_B)^+) - \varepsilon_1(e_B, (e_A - e_B)^+) \leqq 0.$

Now let us introduce several notions. A set A is called <u>almost polar</u> if $\mathrm{Cap}(A) = 0$. "<u>Quasi-everywhere</u>" or "<u>q.e.</u>" means "except on an almost polar set". Denote by $X \cup \partial$ the one-point compactification of X. ∂ is adjoined as an isolated point if X is compact already. A function u defined q.e. on X is called <u>quasi-continuous</u> (<u>in a restricted sense</u>) if, for any $\delta > 0$, there is an open set G with $\mathrm{Cap}(G) < \delta$ such that the restriction of u to $X - G$ is continuous and continuously extendable to $X \cup \partial - G$ by setting $u(\partial) = 0$.

An increasing family $\{F_k\}$ of closed sets such as $\mathrm{Cap}(X - F_k) \longrightarrow 0$ is called a <u>nest</u>. A closed set F is said to be <u>m-regular</u> if $m(U(x) \cap F) \neq 0$ for any $x \in F$ and any its neighbourhood $U(x)$. A <u>regular nest</u> $\{F_k\}$ is a nest such that each F_k is m-regular.

The notion of m-regularity has been introduced in [10]. See [10 ; pp 198-199] for the proof of the next theorem.

<u>Theorem 4.1</u>. (i) <u>Let</u> Q <u>be a countable family of quasi-continuous functions</u>. <u>Then there exists a nest</u> $\{F_k\}$ <u>such that</u> $Q \subset C(\{F_k\})$, <u>where</u>

(4.6) $C(\{F_k\}) = \{u$; <u>the restriction of</u> u <u>to each</u> F_k <u>is continuous and continuously extendable to</u> $F_k \cup \partial$ <u>by setting</u> $u(\partial) = 0\}$.

(ii) <u>Let</u> $\{F_k\}$ <u>be a nest</u>. <u>Then</u> $F_k' = \{x \in F_k ; m(F_k \cap U(x)) \neq 0$ <u>for any neighbourhood</u> $U(x)$ <u>of</u> $x\}$ <u>defines a regular nest</u> $\{F_k'\}$.

(iii) <u>Let</u> $\{F_k\}$ <u>be a regular nest</u>. <u>If</u> $u \in C(\{F_k\})$ <u>and</u> $u \geq 0$

m - a.e., <u>then</u> $u(x) \geq 0$ <u>for every</u> $x \in \bigcup_{k=1}^{\infty} F_k$.

This theorem particularily implies the following : if a quasi-continuous function u is non-negative m - a.e., then $u \geq 0$ q.e. We note that this statement can be localized to any open set (see [10 ; Theorem 1.2]).

The next theorem due to Deny is fundamental for regular Dirichlet forms (see [4], [10]).

<u>Theorem 4.2</u>. (i) <u>Any function</u> $u \in \mathcal{D}[\varepsilon]$ <u>admits a quasi-continuous version</u> $\tilde{u}$: $\tilde{u}$ <u>is quasi-continuous and</u> $\tilde{u} = u$ m - a.e.
(ii) <u>If quasi-continuous functions</u> $\tilde{u}_n \in \mathcal{D}[\varepsilon]$ <u>form a Cauchy sequence with respect to the norm</u> ε_1, <u>then there is a subsequence</u> n_k <u>such that</u> $\tilde{u}_{n_k}$ <u>converges q.e. to a quasi-continuous function</u> $\tilde{u}$. <u>Furthermore</u> u_n <u>converges to</u> $\tilde{u}$ <u>in</u> ε_1-<u>norm</u>.
(iii) <u>If functions</u> $u_n \in \mathcal{D}[\varepsilon]$ <u>form a Cauchy sequence in norm</u> ε_1 <u>and if their suitable quasi-continuous versions</u> $\tilde{u}_n$ <u>converges to a function</u> $\tilde{u}$ q.e., <u>then</u> $\tilde{u}$ <u>is quasi-continuous and</u> u_n <u>converges to</u> $\tilde{u}$ <u>in</u> ε_1-<u>norm</u>.

§ 5. Generation of a Hunt process by a regular Dirichlet form

We follow Blumenthal-Getoor [2 ; Chap. I] about the definitions of a Markov process and a Hunt process except that we allow the state space of a Hunt process to be an arbitrary Borel subset of X. See P.A. Meyer [15 ; Chap. XIV] where the definition of a Hunt process is relaxed in this respect.

For a Borel set $A \subset X$, we denote by $\mathcal{B}(A)$ the topological σ-field of subsets of A and by $\underset{\sim}{B}(A)$ the space of all bounded $\mathcal{B}(A)$-measurable functions on A. $\mathcal{B}(X)$ and $\underset{\sim}{B}(X)$ are simply denoted by $\mathcal{B}$ and $\underset{\sim}{B}$ respectively.

Suppose that we are given a regular Dirichlet form ε on $L^2(X\,;\,m)$ and a Markov process $\underset{\sim}{M} = (\Omega, \mathcal{M}, \mathcal{M}_t, X_t, \theta_t, P_x)$ with state space $(Y, \mathcal{B}(Y))$, Y being some Borel subset of X. We adjoin a "death" point ∂ to Y regarding $Y \cup \partial$ as the topological subspace of the one-point compactification $X \cup \partial$ of X. The Markov semigroup on $L^2(X\,;\,m)$ generated by the form ε will be denoted by $\{T_t, t > 0\}$, while the transition semigroup of the process $\underset{\sim}{M}$ will be denoted by $\{P_t, t > 0\}$: $P_t f(x) = E_x(f(X_t))$, $x \in Y$, $f \in \underset{\sim}{B}(Y)$. Let us agree to say that <u>the Markov process</u> $\underset{\sim}{M}$ <u>is properly associated with the Dirichlet form</u> ε if

(5.1) $\quad \mathrm{Cap}(X - Y) = 0,$

(5.2) $\quad P_t f$ is a quasi-continuous version of $T_t f$ for each $f \in L^2 \cap \underset{\sim}{B}$ and $t > 0$.

Our main theorem is the following.

<u>Theorem 5.1</u>. <u>For any regular Dirichlet form</u> ε <u>on</u> $L^2(X\,;\,m)$, <u>there exists a Hunt process properly associated with</u> ε.

This theorem can be reduced to the next proposition. For a Borel set $Y \subset X$ and a nest $\{F_k\}$ such as $\bigcup_{k=1}^{\infty} F_k \supset Y$, let $C(\{F_k\}, Y)$ be the restrictions to Y of those functions in $C(\{F_k\})$. Q^+ will stand for the set of all positive rational numbers.

<u>Proposition 5.1</u>. <u>For any regular Dirichlet form</u> ε <u>on</u> $L^2(X\,;\,m)$, <u>there exists a normal Markov process</u> $\underset{\sim}{M} = (\Omega, \mathcal{M}, \mathcal{M}_t,$

$X_t, \theta_t, P_x)$ with state space $(Y, \mathcal{B}(Y))$ satisfying the following conditions.

(i) $\mathrm{Cap}(X - Y) = 0$.

(ii) For each $w \in \Omega$, the sample path $X_t(w)$ is right continuous $(t \geqq 0)$ and has the left limits $(t > 0)$ on $Y \cup \partial$.

(iii) $P_t f$ is a version of $T_t f$ for each $f \in L^2 \cap C(X)$ and $t \in Q^+$.

(iv) There is a regular nest $\{F_k\}$ such that

(a) $\bigcup_{k=1}^{\infty} F_k \supset Y$,

(b) $P_t(C(X)) \subset C(\{F_k\} ; Y)$ for each $t \in Q^+$,

(c) $\lim_{n\to\infty} \sigma_k(w) = \infty$ for each $w \in \Omega$, where $\sigma_k(w) = \inf\{t > 0$ $X_t(w) \in Y - F_k\}$.

It is easy to see that the Markov process $\underset{\sim}{M}$ described in Proposition 5.1 is properly associated with ε. To prove this, take any $t > 0$ and $f \in L^2 \cap C(X)$. Then $P_{t_n} f(x) \longrightarrow P_t f(x)$, $x \in Y$, as $t_n \in Q^+$ decreases to t. On the other hand $T_{t_n} f = T_{t_n - t} T_t f$ converges to $T_t f$ in ε_1-norm by virtue of Lemma 3.2. Hence $P_t f$ is a quasi-continuous version of $T_t f$ on account of Theorem 4.2. The same statement can be proved for any $f \in L^2 \cap \underset{\sim}{B}$ in the same way as in Lemma 5.1 below.

Proposition 5.1 implies Theorem 5.1. To see this, it now suffices to show that the Markov process $\underset{\sim}{M}$ of Proposition 5.1 gives rise to a Hunt process. Take the canonical modification of $\underset{\sim}{M}$ exactly in the same manner as in Blumenthal-Getoor

[2 ; pp 49-50]. It is then a Hunt process satisfying Proposition 5.1 with (iv)(c) replaced by a.e. statement. Its strong Markovity and quasi-left continuity follow easily from the following observation for $\underset{\sim}{M}$: $P_s f(X_t(w))$ is right continuous in $t \geqq 0$ and $\lim_{t' \uparrow t} P_s f(X_{t'}(w)) = P_s f(X_{t-}(w))$, $t > 0$, for each fixed $w \in \Omega$, $s \in Q^+$ and $f \in C(X)$.

From now on, we will concentrate our attention on the proof of Proposition 5.1. Suppose that we are given a regular Dirichlet form ε on $L^2(X ; m)$ with the associated semigroup $\{T_t, t > 0\}$, resolvent $\{G_\alpha, \alpha > 0\}$ and equilibrium potentials $\{e_A ; A \in \mathcal{O}_0\}$. We will produce a Markov process $\underset{\sim}{M}$ of Proposition 5.1 by six steps (I) $\sim$ (VI).

(I) Integral operators $\widetilde{P_t}$, $t \in Q^+$, and $\widetilde{G_1}$.

Since the form ε is regular, we can find a countable subcollection $\underset{\sim}{B}_0$ of $\mathcal{D}[\varepsilon] \cap C(X)$ such that $\underset{\sim}{B}_0$ is linear over rationals, uniformly dense in $C(X)$ and closed under the operation of taking the absolute value. We then put $H_0 = (\bigcup_{t \in Q^+} T_t(\underset{\sim}{B}_0)) \cup G_1(\underset{\sim}{B}_0)$. H_0 is a countable subset of $\mathcal{D}[\varepsilon]$ by Lemma 3.2, and consequently each element $u \in H_0$ admits a quasi-continuous version $\tilde{u}$ according to Theorem 4.2. Applying Theorem 4.1 to $\tilde{H}_0 = \{\tilde{u} ; u \in H_0\}$, find a regular nest $\{F_k{}^0\}$ such that $\tilde{H}_0 \subset C(\{F_k{}^0\})$. Let us put $Y_0 = \bigcup_{k=1}^{\infty} F_k{}^0$.

By virtue of Theorem 4.1, $\widetilde{T_t(u+v)}(x) = \widetilde{T_t u}(x) + \widetilde{T_t v}(x)$, $\widetilde{T_t(au)}(x) = a\,\widetilde{T_t u}(x)$ for every $u, v \in \underset{\sim}{B}_0$, rational a and $x \in Y_0$. Further $0 \leqq u \leqq 1$, $u \in \underset{\sim}{B}_0$, implies $0 \leqq \widetilde{T_t u}(x) \leqq 1$,

$x \in Y_0$. Therefore there exist unique stochastic measures $\{\widetilde{P_t}(x, \cdot), t \in Q^+, x \in Y_0\}$ on $\mathcal{B}(X)$ such that $\widetilde{T_t u}(x) = \int_X \widetilde{P_t}(x, dy) u(y)$, $u \in \underset{\sim}{B}_0$, $t \in Q^+$, $x \in Y_0$. In the same way, we can introduce unique substochastic measures $\{\widetilde{G_1}(x, \cdot), x \in Y_0\}$ such that $\widetilde{G_1 u}(x) = \int_X \widetilde{G_1}(x, dy) u(y)$, $u \in \underset{\sim}{B}_0$, $x \in Y_0$.

Let us define

$$(5.3) \qquad \widetilde{P_t} u(x) = \begin{cases} \int \widetilde{P_t}(x, dy) u(y), & x \in Y_0 \\ 0, & x \in X - Y_0. \end{cases}$$

$\widetilde{G_1} u$ is similarily defined. It is easy to see that $\widetilde{P_t}$ and $\widetilde{G_1}$ are then linear operators from $\underset{\sim}{B}$ into $\underset{\sim}{B}$. Furthermore

Lemma 5.1. *For each $u \in L^2 \cap \underset{\sim}{B}$, the functions $\widetilde{P_t} u$ and $\widetilde{G_1} u$ are quasi-continuous versions of $T_t u$ and $G_1 u$ respectively $(t \in Q^+)$.*

Since $\widetilde{P_t}(C^+(X))$ and $\widetilde{G_1}(C^+(X))$ are subsets of $C(\{F_k{}^0\})$, this lemma can be shown just in the same manner as in [10; § 3] by making use of Lemma 3.2.

(II) A regular nest $\{F_k\}$.

Let $\{A_n\}$ be a countable open base of X such that each A_n is relatively compact. Put $\Theta_1 = \{A$; A is a finite union of A_n's$\}$. Obviously $\Theta_1 \subset \Theta_0$. For each $A \in \Theta_1$, choose a quasi-continuous version $\widetilde{e_A}$ of e_A. Let $\widetilde{\mathcal{D}[\varepsilon]}$ be the collection of all quasi-continuous versions of elements in $\mathcal{D}[\varepsilon]$. Define $\widetilde{H}$ as the smallest subfamily of $\widetilde{\mathcal{D}[\varepsilon]}$ satisfying the following. $\widetilde{H}$ is then countable.

$(\widetilde{H}.1)$ $\widetilde{H} \supset \underset{\sim}{B}_0$, $\{\widetilde{e_A}\ ;\ A \in \Theta_1\}$.

$(\widetilde{H}.2)$ $\widetilde{P_t}(\widetilde{H}) \subset \widetilde{H}$, $t \in Q^+$, and $\widetilde{G_1}(\widetilde{H}) \subset \widetilde{H}$.

$(\widetilde{H}.3)$ $\widetilde{H}$ is an algebra over rationals.

Lemma 5.2. There exists a regular nest $\{F_k\}$ satisfying the following. Put $Y_1 = \bigcup_{k=1}^{\infty} F_k$.

(5.4) $\widetilde{H} \subsetneq C(\{F_k\})$, $F_k \subset F_k{}^0$, $k = 1, 2, \cdots$.

(5.5) $\widetilde{e_A}(x) = 1$, $x \in A \cap Y_1$, $A \in \Theta_1$.

(5.6) There exists a sequence of rationals $t_k \downarrow 0$ such that, as $t_k \downarrow 0$, $\widetilde{P_{t_k}}u(x) \longrightarrow u(x)$ and $\frac{1}{t_k}(\widetilde{G_1}u(x) - e^{-t_k}\widetilde{G_1}\widetilde{P_{t_k}}u(x)) \longrightarrow$ $u(x)$ for every $x \in Y_1$ and $u \in \widetilde{H}$.

(5.7) $\widetilde{P_t}\widetilde{P_s}u(x) = \widetilde{P_{t+s}}u(x)$, $x \in Y_1$, $s,t \in Q^+$, $u \in \widetilde{H}$.

(5.8) $e^{-t}\widetilde{P_t}\widetilde{G_1}u(x) \leq \widetilde{G_1}u(x)$, $x \in Y_1$, $t \in Q^+$, $u \in \widetilde{H}$.

(5.9) $e^{-t}\widetilde{P_t}\widetilde{e_A}(x) \leq \widetilde{e_A}(x)$, $x \in Y_1$, $t \in Q^+$, $A \in \Theta_1$.

(5.10) $0 \leq \widetilde{e_A}(x) \leq 1$, $x \in Y_1$, $A \in \Theta_1$.

(5.11) $\widetilde{e_A}(x) \leq \widetilde{e_B}(x)$, $x \in Y_1$, $A,B \in \Theta_1$, $A \subset B$.

Proof. Use Theorem 4.1 (i), the equality of (4.2), the remark preceding to Theorem 4.2, Lemma 3.2 and Theorem 4.2 (ii) to find a nest $\{F_k\}$ satisfying (5.4) $\sim$ (5.6) and then pass to its m-regularization (Theorem 4.1 (ii)). Theorem 4.1 (iii) implies the remaining properties of this lemma. We can use (4.4), (4.2) and (4.5) for (5.9), (5.10) and (5.11).

(III) A Markov process with time parameter Q^+

Lemma 5.3. There exists a Borel set $Y_2 \subset Y_1$ such that

$\mathrm{Cap}(X - Y_2) = 0$ and $\tilde{P_t}(x, X - Y_2) = 0$ for every $x \in Y_2$ and $t \in Q^+$.

Proof. Since $m(X - Y_1) = 0$, $\tilde{P_t}(x, X - Y_1) = 0$ for m - a.e. $x \in X$. Lemma 5.1 implies that there is a Borel set $Y_1^{(1)} \subset Y_1$ such that $\mathrm{Cap}(X - Y_1^{(1)}) = 0$ and $\tilde{P_t}(x, X - Y_1) = 0$ for all $x \in Y_1^{(1)}$ and $t \in Q^+$. Apply the same argument to $X - Y_1^{(1)}$ to get a Borel set $Y_1^{(2)} \subset Y_1^{(1)}$. Finally $Y_2 = \bigcap_{k=1}^{\infty} Y_1^{(k)}$ works.

Let us define

$$(5.12) \qquad P_t(x, B) = \begin{cases} \tilde{P_t}(x, B) & x \in Y_2, \quad B \in \mathcal{B} \\ 0 & x \in X - Y_2, \quad B \in \mathcal{B}, \end{cases}$$

and put $P_t u(x) = \int_X P_t(x, dy) u(y)$. Then

$$(5.13) \qquad P_t u(x) = \tilde{P_t} u(x), \quad x \in Y_2, \quad u \in \underset{\sim}{B},$$

$$(5.14) \qquad P_t P_s u(x) = P_{t+s} u(x), \quad x \in X, \quad u \in \underset{\sim}{B}.$$

This follows from (5.7) and the above lemma. Extending P_t to $(X \cup \partial, \mathcal{B}(X \cup \partial))$ by $P_t(x, \{\partial\}) = 1 - P_t(x, X)$ and $P_t(\partial, \{\partial\}) = 1$, we now have a transition probability over $X \cup \partial$.

We put $\Omega_0 = (X \cup \partial)^{Q^+}$, $X_t^0(w) = w(t)$, $w \in \Omega_0$, $t \in Q^+$, $M_t^0 = \sigma\{X_s^0 \ ; \ 0 < s \leq t, \ s \in Q^+\}$, $t \in Q^+$, and $\mathcal{M} = \bigvee_{t \in Q^+} \mathcal{M}_t^0$. Then there are uniquely probability measures P_x, $x \in X \cup \partial$, over $(\Omega_0, \mathcal{M})$ such that

$$(5.15) \qquad E_x(f(X_t^0)) = P_t f(x)$$

$$(5.16) \qquad E_x(f(X_{t+s}^0) / \mathcal{M}_t^0) = E_{X_t^0}(f(X_s^0))$$

$$x \in X \cup \partial, \quad t, s \in Q^+, \quad f \in \underset{\sim}{B}(X \cup \partial).$$

Y_2 is obviously an invariant set for the Markov process $\underset{\sim}{M}^0 = (\Omega_0, \mathcal{M}, \mathcal{M}_t{}^0, X_t{}^0, P_x)$, $t \in Q^+$:

(5.17) $\quad P_x(X_t{}^0 \in Y_2 \cup \partial \text{ for every } t \in Q^+) = 1, \quad x \in Y_2.$

(IV) <u>Supermartingale</u> $Y_t{}^0 = e^{-t}\widetilde{e_A}(X_t{}^0)$, $t \in Q^+$.

Take $A \in \Theta_1$ and fix a point $x \in Y_2$. Then $(Y_t{}^0, \mathcal{M}_t{}^0, P_x)$, $t \in Q^+$, is a positive bounded supermartingale because of (5.9), (5.16) and (5.17). Therefore, for almost all $w \in \Omega_0$, the right limits $Y_t = \lim_{s \in Q^+,\, s \downarrow t} Y_s{}^0$ exist for all $t \geqq 0$ and $(Y_t, \mathcal{M}_t{}', P_x)$, $t \geqq 0$, is again a positive bounded supermartin gale.

Here

(5.18) $\quad \mathcal{M}_t = \bigcap_{s \in Q^+,\, s > t} \mathcal{M}_s{}^0, \qquad \mathcal{M}_t{}' = \mathcal{M}_t \vee \mathcal{N},$

$\mathcal{N}$ being the collection of those sets $\Gamma \in \mathcal{M}$ such as $P_x(\Gamma) = 0$ for every $x \in Y_2$ (c.f. Meyer [14 ; VI]).

<u>Lemma 5.4</u>. $\quad P_x(Y_t = Y_t{}^0 \text{ for all } t \in Q^+) = 1.$

<u>Proof</u>. For any $f \in C[0, 1]$ and any polynomial g with rational coefficients, we have $E_x(f(Y_t{}^0)g(Y_{t+t_k}^0)) = E_x(f(Y_t{}^0) P_{t_k}g(Y_t{}^0))$, $t \in Q^+$. Since $g \circ \widetilde{e_A}$ is an element of $\widetilde{H}$, $P_{t_k}g(Y_t{}^0) = \widetilde{P}_{t_k}(g \circ \widetilde{e_A})(X_t{}^0)$ converges to $g \circ \widetilde{e_A}(X_t{}^0) = g(Y_t{}^0)$ in view of (5.6). Hence $E_x(f(Y_t{}^0)g(Y_t)) = E_x(f(Y_t{}^0)g(Y_t{}^0))$, completing the proof.

Let us define, for an open set $G \subset X$, (5.19) $\sigma_G^0 = \inf\{t \in Q^+ ; X_t^0 \in G\}$.

σ_G^0 is an $\mathcal{M}_t$-stopping time. We are ready to prove the following.

Lemma 5.5. (i) *For each* $A \in \mathcal{O}_1$ *and* $x \in Y_2$, $E_x(e^{-\sigma_A^0}) \leq \tilde{e}_A(x)$.

(ii) *For any* $G \in \mathcal{O}_0$, e_G *admits a quasi-continuous version* $\tilde{e}_G$ *such that*

$E_x(e^{-\sigma_G^0}) \leq \tilde{e}_G(x) \quad , \quad x \in Y_2$.

(iii) *For any decreasing sequence* $G_n \in \mathcal{O}_0$ *such as* $\mathrm{Cap}(G_n) \longrightarrow 0$ $n \longrightarrow \infty$, *we have*

$P_x(\lim_{n\to\infty} \sigma_{G_n}^0 = \infty) = 1$ *for* *q.e.* $x \in Y_2$.

Proof. (i) When either σ_A^0 is irrational or σ_A^0 is rational but $X^0_{\sigma_A^0} \notin A$, then (5.5) implies $Y_{\sigma_A^0} = e^{-\sigma_A^0}$. When σ_A^0 is rational and $X^0_{\sigma_A^0} \in A$, then Lemma 5.4 means that $Y_{\sigma_A^0} = Y^0_{\sigma_A^0} = e^{-\sigma_A^0}$. Hence, by the supermartingale inequality ([14 ; VI]), $E_x(e^{-\sigma_A^0}) = E_x(Y_{\sigma_A^0}) \leq E_x(Y_0)$ $= \lim_{s\downarrow 0} e^{-s} P_s \tilde{e}_A(x) \leq \tilde{e}_A(x)$.

(ii) It suffices to choose $A_n \in \mathcal{O}_1$ such as $A_n \uparrow G$ and put $\tilde{e}_G(x) = \lim_{n\to\infty} \tilde{e}_{A_n}(x)$, $x \in Y_2$. The limit exists in view of (5.11). Moreover $\varepsilon_1(e_G - \tilde{e}_{A_n}, e_G - \tilde{e}_{A_n}) = \mathrm{Cap}(G) - \mathrm{Cap}(A_n) \longrightarrow 0$.

(iii) Since $\varepsilon_1(\tilde{e}_{G_n}, \tilde{e}_{G_n}) = \mathrm{Cap}(G_n) \longrightarrow 0$, a subsequence of $\tilde{e}_{G_n}$ converges to zero q.e. on X. Hence (ii) implies (iii).

(V) Regularity of sample paths

Lemma 5.6. There exists a Borel set $Y_3 \subset Y_2$ with $\mathrm{Cap}(X - Y_3) = 0$ and the following statements hold.

(i) Put $\Omega_{01} = \{w \in \Omega_0 ; \lim_{k\to\infty} \sigma^0_{X-F_k} = \infty\}$, $\Omega_{02} = \{w \in \Omega_0 ; X^0_s(w)$ has the right and left limits in $Y_1 \cup \partial$ at every $t \geqq 0$ through $Q^+\}$, $\Omega_{03} = \{w \in \Omega_0 ; w([0,t] \cap Q^+)$ is bounded in X if $X^0_t(w) \in X$, $t \in Q^+\}$ and $\Omega_1 = \Omega_{01} \cap \Omega_{02} \cap \Omega_{03}$. Then $P_x(\Omega_1) = 1$ for every $x \in Y_3$.

(ii) For $w \in \Omega_1$ and $t \geqq 0$, we put

(5.20) $$X_t(w) = \lim_{s \in Q^+,\ s \downarrow t} X^0_s(w).$$

Then $P_x(X_t = X^0_t$ for every $t \in Q^+) = 1$, $x \in Y_3$.

(iii) $P_x(X_0 = x) = 1$, $x \in Y_3$.

Proof. (i) Lemma 5.5 (iii) implies that there is a Borel set $Y_3 \subset Y_2$ such that $\mathrm{Cap}(X - Y_3) = 0$ and $P_x(\Omega_{01}) = 1$ for $x \in Y_3$. Next (5.4) and (5.6) imply that $\tilde{G}_1(H^+)$ is contained in $C(\{F_k\})$ and separates points of Y_1. Therefore we have $\Omega_{01} - \Omega_{02} \subset \bigcup_{u \in \tilde{H}^+} \{w \in \Omega_0 ; \tilde{G}_1 u(X^0_s), s \in Q^+,$ has an oscillatory discontinuity at some $t \geqq 0\}$. However $\{e^{-s}\tilde{G}_1 u(X^0_s), \mathcal{M}^0_s, P_x\}$ $s \in Q^+$, is a bounded positive supermartingale for each $u \in \tilde{H}^+$ and $x \in Y_2$ on account of (5.8). Hence $P_x(\Omega_{01} - \Omega_{02}) = 0$, $x \in Y_2$. Finally there is, by Lemma 5.2, a function $v \in C^+$ such that $\tilde{G}_1 v \in C(\{F_k\})$ is strictly positive on Y_1 and satisfies (5.8). We have then $\Omega_{01} - \Omega_{03} = \bigcup_{t \in Q^+} \{w \in \Omega_{01} ; \tilde{G}_1 v(X^0_t) > 0, \inf_{s \in Q^+, s \leqq t} \tilde{G}_1 v(X^0_s) = 0\}$, which has zero P_x-measure $(x \in Y_2)$ in view of [2; Chap. 0, (1.6)]. (ii) For $u, v \in B_0$, $t \in Q^+$ and $x \in Y_2$, $E_x(u(X^0_t)v(X_t)) = \lim_{t_k \downarrow 0} E_x(u(X^0_t)P_{t_k}v(X^0_t)) = E_x(u(X^0_t)v(X^0_t))$ by virtue of (5.6). (iii) $E_x(u(X_0)) = \lim_{t_k \downarrow 0} P_{t_k}u(x) = u(x)$,

$u \in B_0$, $x \in Y_2$.

Lemma 5.7. There exists a Borel set $Y \subset Y_3$ with $\mathrm{Cap}(X - Y) = 0$ satisfing the following condition: the set $\Gamma = \{w \in \Omega_1 ; X_t(w)$ or $X_{t-}(w)$ $X - Y$ for some $t \geqq 0\}$ is contained in a set $\Gamma_0 \in \mathcal{M}$ such that $P_x(\Gamma_0) = 0$ for every $x \in Y$.

Proof. Choose a decreasing sequence of open sets $G_n \supset X - Y_3$ such as $\mathrm{Cap}(G_n) \to 0$. Put $\Gamma_3 = \{w \in \Omega_1 ; X_t$ or $X_{t-} \in X - Y_3$ for some $t \geqq 0\}$, $\Gamma_3^0 = \{w \in \Omega_0 ; \lim_{n\to\infty} \sigma^0_{G_n} < +\infty\}$. Then $\Gamma_3 \subset \Gamma_3^0$ and Lemma 5.5 (iii) further implies $P_x(\Gamma_3^0) = 0$ if x is in some Borel set $Y_4 \subset Y_3$ with $\mathrm{Cap}(X - Y_4) = 0$. Apply the same argument to Y_4. We thus get sequences $Y_3 \supset Y_4 \supset \cdots$, $\Gamma_3 \subset \Gamma_4$ $\cdots$ and $\Gamma_3^0 \subset \Gamma_4^0 \subset \cdots$. Put $Y = \bigcap_{k=3}^{\infty} Y_k$, then obviously $\Gamma = \bigcup_{k=3}^{\infty} \Gamma_k$. Now $\Gamma_0 = \bigcup_{k=3}^{\infty} \Gamma_k^0$ works.

(VI) Extended Markov property

Let us put $\Omega = \Omega_1 - \Gamma_0$. The restrictions to Ω of $X_t^0, \mathcal{M}_t^0$ $(t \in Q^+)$, X_t, $\mathcal{M}_t$ $(t \geqq 0)$, $\mathcal{M}$ and P_x $(x \in Y \cup \partial)$ are again denoted by the same notations.

Lemma 5.8. $\underset{\sim}{M} = (\Omega, \mathcal{M}, \mathcal{M}_t, X_t, P_x)$ is a normal Markov process with state space $(Y, \mathcal{B}(Y))$ satisfying

(5.21) $\quad P_x(X_t \in B) = P_t(x, B), \quad t \in Q^+, \quad x \in Y, \quad B \in \mathcal{B}(Y).$

Moreover $\underset{\sim}{M}$ possesses all the properties of Proposition 5.1.

Proof. (5.21) and the normality of $\underset{\sim}{M}$ follow from Lemma 5.6 (ii) and (iii) respectively. Other properties of $\underset{\sim}{M}$ are now evident except for its Markov property.

Take $t, s \geq 0$, $f \in C(X)$ and $\Lambda \in \mathcal{M}_t$. We have, by (5.16) $E_x(f(X^0_{t'+s'}) ; \Lambda) = E_x(E_{X^0_{t'}}(f(X^0_{s'})) ; \Lambda)$ for any $t' > t$ and $s' > s$, $t', s' \in Q^+$, because $\Lambda \in \mathcal{M}^0_{t'}$. Note that the function $v(x) = E_x(f(X^0_{s'})) = P_{s'}f(x)$ is an element of $C(\{F_k\} ; Y)$. Hence, by letting t' decrease to t and then s' to s, we arrive at $E_x(f(X_{t+s}) ; \Lambda) = E_x(E_{X_t}(f(X_s)) ; \Lambda)$ the Markovity of $\underset{\sim}{M}$.

§ 6. Examples

We now return to the examples of Markov symmetric forms in § 2 to see if they give rise to regular Dirichlet forms. According to Theorem 3.3, it suffices to check the closability of a given Markov symmetric form and then the regularity of its smallest closed extension.

1°. Consider Example 1. Let us confine ourselves to the case when $\Phi \equiv 0$, $n \equiv 0$ and m is the Lebesgue measure on D. If either

(1°. a) the first order (Schwartz) distribution derivatives of $a_{ij}(x)$ are locally integrable functions

or

(1°. b) $a_{ij}(x)$ is uniformly elliptic : there is a positive constant δ such that for any N-vector ξ,

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \delta|\xi|^2 , \quad x \in D,$$

then ε is closable and the smallest closed extension $\bar{\varepsilon}$ is clearly a regular Dirichlet form. An associated Hunt process (according to Theorem 5.1) is the well-known absorbing barrier diffusion process on D if $A_{ij}(x)$ are sufficiently smooth.

In case of (1°. a), ε can be expressed as (3.7) with the symmetric operator $Su = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}(a_{ij}(x) \frac{u(x)}{x_j})$, $\mathcal{D}(S) = C_0^\infty(D)$. Hence ε is closable. Next assume that (1°. b) is satisfied. Consider $u_n \in C_0^\infty(D)$ such that $\varepsilon(u_n - u_m, u_n - u_m) \longrightarrow 0$ and $u_n \longrightarrow 0$ in $L^2(D)$. Then $\{u_n\}$ forms a Cauchy sequence with respect to the usual Dirichlet integral $\underset{\sim}{D}$. Since $\underset{\sim}{D}$ is a special form satisfying (1°.a), $\underset{\sim}{D}(u_n, u_n) \longrightarrow 0$, which in turn implies that a subsequence n_k exists and $\frac{\partial u_{n_k}}{\partial x_i} \to 0$ a.e. on D, $i = 1, 2, \cdots, N$. By Fatou's lemma,

$$\varepsilon(u_m, u_m) = \int_D \varliminf_{n_k \to \infty} \left(\sum_{i,j=1}^{N} \frac{\partial(u_{nk} - u_m)}{\partial x_i}(x) \frac{\partial(u_{n_k} - u_m)}{\partial x_j}(x) a_{ij}(x) \right) dx$$

$$\leqq \varliminf_{n_k \to \infty} \varepsilon(u_{n_k} - u_m, u_{n_k} - u_m),$$ which is small if m is sufficiently large. Hence ε is closable.

Here is an example of K.Sato. Assume that $D = R^N$ and

(1°.c) $\frac{\partial}{\partial x_k} a_{ij}(x)$ is locally integrable for $1 \leqq k \leqq N-1$ but $\frac{\partial}{\partial x_N} a_{ij}(x)$ is defined and continuous only on $\{x \in R^N; x_N \neq 0\}$,

then ε is closable (and $\bar{\varepsilon}$ is a regular Dirichlet form).

To prove this, let us put $\mathcal{F}^0 = \{u \in C_0^\infty(R^N)\ ;\ u(x) = u(x_1, x_2, \cdots, x_{N-1})$ if $x_N \in (-\delta, \delta)$ for some $\delta > 0\}$. We can see that our form ε with domain restricted to $\mathcal{F}^0$ satisfies (3.6). Hence this is closable. Moreover we can find, for any $u \in C_0^\infty(R^N)$, a sequence $u_n \in \mathcal{F}^0$ such that $\varepsilon(u_n - u, u_n - u) \to 0$ and $(u_n - u, u_n - u) \to 0$, proving that ε is closable. For instance, it suffices to take $u_n \in \mathcal{F}^0$ such that $u_n(x) \to u(x)$,

$\frac{\partial u_n(x)}{\partial x_i} \longrightarrow \frac{\partial u(x)}{\partial x_i}$ boundedly on $R^N - F$ and all u_n have a common compact support.

2°. Consider Example 2. We assume that there is a positive constant δ such that $\mu(E) \geqq \delta|E|$, $\nu(E) \geqq \delta|E|$ for any linear Borel set E, $|E|$ being the Lebesgue measure. ε is then closable and the smallest closed extension $\bar{\varepsilon}$ is a regular Dirichlet form. An associated Hunt process on R^2 is one of the diffusions that are constructed more concretely by N. Ikeda and S. Watanabe [11].

Suppose $u_n \in C_0^\infty(R^2)$ satisfies $\varepsilon(u_n - u_m, u_n - u_m) \longrightarrow 0$ and $u_n \longrightarrow 0$ in $L^2(R^2)$, then $\frac{1}{2}\underset{\sim}{D}(u_n, u_n) + (u_n, u_n) \longrightarrow 0$, which means, by virtue of Theorem 4.2 and [10 ; Theorem 4.5], that a subsequence $u_{n_k}(x)$ converges to zero on R^2 except on a Borel polar set $\mathcal{N}$ of the two-dimensional standard Brownian motion. In particular the linear Lebesgue measure of $\mathcal{N} \cap \ell$ vanishes for any straight line ℓ on R^2.

There is a Borel function $\phi(x_1, x_2)$ on R^2 such that $\frac{\partial u_n}{\partial x_1}$ converges to ϕ in $L^2(R^2 ; dx_1 \times d\mu)$. This implies

$$\lim_{n_k\to\infty} \int_{-\infty}^{\infty} \left(\frac{\partial u_n(x_1,x_2)}{\partial x_1} - \phi(x_1, x_2)\right)^2 dx_1 = 0 \quad \text{for } \mu\text{-almost all}$$

$x_2 \in R^1$. Hence, by making use of the above observation and the equality

$$\int_a^x \phi(x_1, x_2)dx_1 = u_{n_k}(x_1, x_2) - u_{n_k}(a, x_2) - \int_a^x \left(\frac{\partial u_{n_k}}{\partial x_1} - \phi\right)dx_1,$$

we can see that $\phi(x_1, x_2) = 0$ for a.e. $x_1 \in R^1$.

Thus $\frac{\partial u_n}{\partial x_1}$ converges to zero in $L^2(R^2 ; d\nu \times dx_2)$. In this way we get $\varepsilon(u_n, u_n) \longrightarrow 0$ the closability of ε.

3°. Consider Example 3. The form (2.7) with the domain $H^1(D)$ is closed already. Hence this is a Dirichlet form, which is not regular however unless $R^N - D$ has zero Newtonian outer capacity (in case $N \geq 3$).

More generally consider a linear space $\mathcal{D}$ such that $C_0^\infty(D) \subset \mathcal{D} \subset H^1(D)$ and denote by $\varepsilon_{\mathcal{D}}$ the form (2.7) with the domain $\mathcal{D}$. We suppose that $\varepsilon_{\mathcal{D}}$ is Markov. Then $\varepsilon_{\mathcal{D}}$ is closable and its smallest closed extension $\bar{\varepsilon}_{\mathcal{D}}$ is a Dirichlet form. $\bar{\varepsilon}_{\mathcal{D}}$ is not regular in general, nevertheless we can regard this as a regular Dirichlet form by a suitable enlargement of the underlying space D.

A locally compact separable Hausdorff space D^* is called an admissible enlargement of D relative to $\mathcal{D}$ if D is continuously embedded onto a dense subset of D^* and if the intersection $\mathcal{D} \cap C(D^*)$ is dense in the domain of $\bar{\varepsilon}_{\mathcal{D}}$ and is uniformly dense in $C(D^*)$.

Given $\mathcal{D}$ and D^* as above, let m be the measure on D^* induced by the Lebesgue measure on $D : m(E) = |E \cap D|$. Identify the space $L^2(D)$ with $L^2(D^*) = L^2(D^* ; m)$. Then the form $\bar{\varepsilon}_{\mathcal{D}}$ turns out to be a regular Dirichlet form on $L^2(D^*)$ (in the terminology of [9], $(D^*, m, \bar{\varepsilon}_{\mathcal{D}})$ is a regular representation of $\bar{\varepsilon}_{\mathcal{D}}$). In this way, we can get a strong Markov process on D^* which may be considered as an extension of the absorbing barrier Brownian motion on D.

Let us examine three cases.

(3°. a) $\mathcal{D} = C_0^\infty(D)$. D itself is an admissible enlargement of D relative to $C_0^\infty(D)$. A process associated with $\bar{\varepsilon}_{\mathcal{D}}$ is the absorbing barrier Brownian motion on D.

(3°. b) $\mathcal{D} = \hat{C}^\infty(D)$. The closure $\bar{D}$ of D in R^N is an admissible enlargement of D relative to $\hat{C}^\infty(D)$ in view of Tietze extension theorem of a continuous function. The associated process on $\bar{D}$ (minus an exceptional set) may be considered as a reflecting barrier Brownian motion if the domain of $\bar{\varepsilon}_{\mathcal{D}}$ coincides with $H^1(D)$. This is the case if the boundary $\partial D = \bar{D} - D$ is sufficiently smooth.

(3°. c) $\mathcal{D} = H^1(D)$. As we have just mentioned, $\bar{D}$ is an admissible enlargement of D relative to $H^1(D)$ if ∂D is sufficiently smooth. In the general case, we can take as an admissible enlargement D^* the space constructed in [9 ; §6]. The associated process on D^* is, by definition, a reflecting barrier Brownian motion. If D_1^* and D_2^* are admissible enlargements of D relative to the same space $H^1(D)$, then D_1^* and D_2^* are related to each other by a capacity preserving quasi-homeomorphism [10 ; §2] which is the identity on D. This transformation interrelates the reflecting barrier processes on the respective spaces.

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Almost polar sets and an ergodic theorem

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§ 1. Introduction.

We will develop a potential theory for two Markov processes which are in duality and apply it to an extension of the Chacon-Ornstein ergodic theorem.

Let X be a locally compact separable Hausdorff space and m be a positive Radon measure on X. Consider standard Markov processes $\boldsymbol{M}=(X_t, P_x)$ and $\hat{\boldsymbol{M}}=(\hat{X}_t, \hat{P}_x)$ which are in duality with respect to m in the sense that the equality

$$(f, T_t g)=(\hat{T}_t f, g), \qquad t>0, \tag{1.1}$$

holds for any non-negative Borel functions f and g on X. Here T_t (resp. $\hat{T}_t$) is the semi-group associated with $\boldsymbol{M}$ (resp. $\hat{\boldsymbol{M}}$) and (f, g) is the integral $\int_X f(x)g(x)m(dx)$. The quantities relative to the dual process $\hat{\boldsymbol{M}}$ are denoted with ^ and designated by the prefix co-. Notice that the present duality is much weaker than that of Blumenthal-Getoor [2; VI] and we do not assume the absolute continuity of resolvents or transition probabilities.

A set $A\subset X$ is said to be *almost polar* if there is a Borel set B such that $A\subset B$ and

$$P_x(\sigma_B<+\infty)=0 \qquad \text{for } m\text{-a. e. } x\in X, \tag{1.2}$$

where σ_B is the hitting time $\inf\{t>0;\ X_t\in B\}$. "Quasi-everywhere" or "q. e." will mean "except on an almost polar set".

Recently the notion of almost polarity was employed independently by S. Port and C. Stone [12] for additive processes with m being the Haar measure and by the author [8] for general m-symmetric Markov processes whose associated Dirichlet spaces are regular[1]. In both cases, almost polar sets were identified with the sets of λ-capacity zero, the λ-capacity being defined suitably according to the respective situations. When $\boldsymbol{M}$ is the Brownian motion on R^3, the almost polar set is just the set of the Newtonian outer capacity zero.

1) Almost polar sets are called "essentially polar" in [12] and "polar" in [8].

In § 2, we will study almost polar sets together with *q. e. finely continuous functions* and present some fundamental properties that they possess. Assertions (**i**)$\sim$(**ix**) of § 2 are the generalizations of those established in [**8**; § 3, 4], while (**x**)$\sim$(**xiv**) are our versions of those in Blumenthal-Getoor [**2**; VI]. The second assertion states that each almost polar set is m-negligible. But the converse is not necessarily true. Proposition (**viii**) asserts that the resolvent of $\boldsymbol{M}$ is absolutely continuous with respect to m if and only if each almost polar set is semipolar. The final assertion states that the next two conditions (C_1) and (C_1') are equivalent.

(C_1) Each semipolar set is almost polar,

(C_1') A function is q. e. finely continuous if and only if it is q. e. cofinely continuous.

In particular, condition (C_1) is met when $\boldsymbol{M}$ is m-symmetric ($\boldsymbol{M}=\hat{\boldsymbol{M}}$).

In § 3, we will prove under the assumption (C_1) that, for any bounded Borel f, $g\in L^1(X; m)$, $g\geqq 0$, the ratio

$$\frac{\int_0^t T_s f(x)ds}{\int_0^t T_s g(x)ds} \tag{1.3}$$

converges, as $t\to+\infty$, to a finite limit q. e. on the set where the denominator becomes eventually positive. The novelty of this generalization of the Chacon-Ornstein theorem is that the original statement of a. e. convergence is strengthened to q. e. convergence. Since almost polarity and m-negligibility are equivalent, however, in the cases of discrete time Markov processes, the two notions have not been distinguished in ergodic theory so far. Our key step is to generalize Brunel's ergodic inequality following the line of A. Garsia [**10**]. Brunel's inequality combined with a potential theoretic result of § 2 will immediately implies q. e. convergence of the ratio (1.3).

We note that, in other fields of analysis, especially in Fourier analysis and the boundary limit theorem, we often encounter this kind of phenomena of the transfer from a. e. to q. e. (cf. Carleson's monograph [4]).

Our ratio limit theorem can be applied to an ergodic decomposition of the state space. In § 4 we will show that the conservative (resp. dissipative) part of our decomposition consists, after a suitable modification by a polar set, just of finely recurrent (resp. transient) points in the sense of Azema, Kaplan-Duflo and Revuz [**1**], assuring further that the conservative part is not only T_t-invariant but also sample path-invariant. An additional absolute continuity condition will be imposed in § 4, for we will have to use a theorem of J. L. Doob [**6**] concerning the quasi Lindelöf property of the fine topology.

§ 2. Potential theory for M and $\hat{M}$.

Our assertions will be listed up.

(**i**) *Let E be universally measurable. The next three conditions are equivalent.*

(α) $m(E)=0$.

(β) $p(t, x, E)=0$ *m-a.e. $x\in X$, for each* $t>0$.

(γ) $G_\alpha(x, E)=0$ *m-a.e. $x\in X$, for every $\alpha>0$ (equivalently for some $\alpha>0$). Here $p(t, x, E)$ (resp. $G_\alpha(x, E)$) is the transition probability (resp. the resolvent kernel) of the process* $\boldsymbol{M}$.

PROOF. We only show the implication $(\beta)\Rightarrow(\alpha)$:

$$0=\lim_{t\downarrow 0}\int_X T_t I_E(x)m(dx)=\lim_{t\downarrow 0}\int_X \hat{T}_t 1(x)I_E(x)m(dx)\geqq\int_X I_E(x)m(dx)$$

where I_E denotes the indicator of E.

(**ii**) *If N is almost polar, then* $m(N)=0$.

PROOF. There is a Borel almost polar set $E\supset N$. Then E satisfies (β). Hence $m(E)=0$.

(**iii**) *Let A be a Borel set and put*

$$T_t^0 f(x)=E_x(f(X_t);\ t<\sigma_A),$$

$$H_A^\alpha f(x)=E_x(e^{-\alpha\sigma_A}f(X_{\sigma_A}))\,.$$

Then we have

$$(f,\ T_t^0 g)=(\hat{T}_t^0 f,\ g),\qquad t>0, \tag{2.1}$$

$$(f,\ H_A^\alpha G_\alpha g)=(\hat{H}_A^\alpha \hat{G}_\alpha f,\ g),\qquad \alpha>0, \tag{2.2}$$

for any non-negative Borel functions f and g. Here G_α is the resolvent of T_t.

PROOF. (2.2) is equivalent to (2.1). (2.1) was proved by Dynkin [**7**; Lemma 14.1] for the Brownian motion by making use of a method of time reversion. The same method has been extended to m-symmetric Markov processes in [**8**; Theorem 3.5]. The argument there is independent of the symmetry of T_t and only the relation of duality (1.1) is enough to get (2.1) for open A[2]. Next (2.1) for any Borel A can be obtained just as in [**2**; pp. 262] by noticing that any semi-polar set (resp. cosemi-polar set) is of potential zero (resp. copotential zero) and hence m-negligible according to (**i**).

For a nearly Borel set $E\subset X$, we will write

$$\begin{cases} e_E^\alpha(x)=H_E^\alpha 1(x) \\ e_E(x)=H_E^{0+}1(x)=P_x(\sigma_E<+\infty)\,. \end{cases} \tag{2.3}$$

2) Recently T. Watanabe [**14**] gave a quite different proof of the relation (2.2) for open sets by making use of a method of the balayage of excessive measures.

Here are two consequences of the relation (2.2).

(**iv**) *Almost polarity and almost copolarity are equivalent.*

PROOF. Let E be Borel and almost polar. Then, for any $g \in \boldsymbol{C}_0(X)$ (the space of continuous functions with compact supports), $0=(1, H_E^\alpha G_\alpha g)=(\hat{H}_E^\alpha \hat{G}_\alpha 1, g)$. Hence $0=\hat{H}_E^\alpha \hat{G}_\alpha 1 \geqq \hat{H}_E^\alpha \hat{G}_\beta 1$ m-a.e. for all $\beta \geqq \alpha$. But then $\hat{e}_E^\alpha(x)=\lim_{\beta\to+\infty} \beta \hat{H}_E^\alpha \hat{G}_\beta 1(x)=0$ for m-a.e. $x \in X$, proving that E is almost copolar.

(**v**) *Let A be nearly Borel and finely open. Suppose that a nearly Borel subset $E \subset A$ has the property that*

$$\hat{e}_E(x)=0 \quad m\text{-a.e. on } A.$$

Then E is almost polar.

PROOF. If E is nearly Borel, then there are, for a strictly positive function $h \in L^1(X, m)$, some Borel sets E' and E'' such that $E' \subset E \subset E''$ and $P_{h\cdot m}(X_t \in E''-E'$ for some $t \geqq 0)=0$, which means that $E''-E'$ is almost polar. Hence it suffices to show the proposition (**v**) for a Borel set E.

Take any compactum $K \subset E$. Since $\hat{e}_K=0$ m-a.e. on A, we have $0=(\hat{H}_K^\alpha \hat{G}_\alpha f, I_A)=(f, H_K^\alpha G_\alpha I_A)$ for any $f \in C_0(X)$. Therefore $H_K^\alpha G_\beta I_A=0$ m-a.e. for all $\beta > \alpha$. H_K^α is supported by K but $\lim_{\beta\to\infty} \beta G_\beta I_A(y)=1$ for $y \in K(\subset A)$ because A is finely open. We get $e_K^\alpha(x)=0$ m-a.e. Now find, for a strictly positive $h \in L^1(X; m)$, an increasing sequence of compact sets $K_n \subset E$ such that $(h, e_E^\alpha)=\lim_{n\to\infty}(h, e_{K_n}^\alpha)$ to complete the proof.

DEFINITION. A function f defined q.e. on X is called *q.e. finely continuous* if the following conditions are satisfied: there exists a nearly Borel almost polar set B such that $X-B$ is finely open and f is nearly Borel measurable and finely continuous on $X-B$.

(**vi**) *If f is q.e. finely continuous and if $f \geqq 0$ m-a.e. on X, then $f \geqq 0$ q.e. on X.*

PROOF. Let B be the set appeared in the above definition of q.e. fine continuity of f. Then the set $A=(X-B) \cap \{x; f(x)<0\}$ is nearly Borel and finely open. Since $m(A)=0$, $\hat{e}_A=0$ m-a.e. on A trivially and A is almost polar by (**v**).

The following characterization of almost polar sets already appeared in [8; Theorem 3.12]. We say that a set E is $\boldsymbol{M}$-invariant if $P_x(X_t \in E$ for every $t \in [0, \zeta))=1$ for every $x \in E$[3].

(**vii**) *A set N is almost polar if and only if there exists a Borel set $B \supset N$ such that $m(B)=0$ and $X-B$ is $\boldsymbol{M}$-invariant.*

PROOF. Let N be almost polar then there is a Borel set $B_0 \supset N$ such that $e_{B_0}(x)=0$ m-a.e. Since e_{B_0} is excessive, it is nearly Borel and finely continuous.

3) ζ is the life time of the process $\boldsymbol{M}$.

Hence, by the previous assertion, $e_{B_0}(x)=0$ q. e., that is, except on some Borel almost polar set B_1. Apply the same argument to the function $e_{B_0\cup B_1}$. In this way, we get a sequence $B_0, B_1, \cdots, B_k, \cdots$ of Borel almost polar sets. It suffices to put $B=\bigcup_{k=0}^{\infty} B_k$.

Now we will give some criteria for the absolute continuity of the resolvent in terms of the relationship among almost polarity, polarity and semipolarity.

(**viii**) *The following four conditions are mutually equivalent.*

(α) *A set is almost polar if and only if it is polar.*

(β) *Any almost polar set is semipolar.*

(γ) *m is a reference measure for $\boldsymbol{M}$: a set is of potential zero if and only if it is m-negligible.*

(δ) *$G_\alpha(x, \cdot)$ is absolutely continuous with respect to m for each $\alpha>0$ and $x\in X$.*

PROOF. (γ) and (δ) are equivalent in view of the first assertion (i). (α) implies (β). Suppose that the condition (β) is satisfied. Let E be an m-negligible Borel set. Then $G_\alpha(x, E)=0$ m-a. e. $x\in X$, by virtue of (**i**). But $G_\alpha(\cdot, E)$ is α-excessive and finely continuous. Hence $G_\alpha(x, E)=0$ q. e. by (**vi**) and moreover except on a semipolar set by the present assumption. Since any semipolar set is of potential zero, we have $G_\alpha(x, E)=\lim_{\beta\to\infty}\beta G_{\beta+\alpha}G_\alpha I_E(x)=0$, $x\in X$, arriving at (δ). Evidently (δ) implies (α). The proof is finished.

REMARK **1°**. Assertion (**viii**) is a generalization of [**8**; Theorem 3.13]. Combining (**vii**) and (**viii**), we get the following criterion: $G_\alpha(x, \cdot)$ is not absolutely continuous with respect to m for some $\alpha>0$ and $x\in X$ if and only if there exists an m-negligible Borel set E such that $X-E$ is $\boldsymbol{M}$-invariant but E is not thin.

2°. In the case that $\boldsymbol{M}=\hat{\boldsymbol{M}}$, those conditions in (**viii**) are also equivalent to the following one (ε) [**9**].

(ε) The transition probability $p(t, x, \cdot)$ is absolutely continuous with respect to m for each $t>0$ and $x\in X$.

The next proposition says that we can reduce the nearly Borel measurability of q. e. finely continuous functions to the Borel measurability.

(**ix**) *A function f is q. e. finely continuous if and only if there exists a Borel almost polar set B such that $X-B$ is $\boldsymbol{M}$-invariant and f is Borel measurable and finely continuous on $X-B$.*

PROOF. Let f be q. e. finely continuous. Then by the definition and (**vii**), there is a Borel almost polar set B_0 such that $X-B_0$ is $\boldsymbol{M}$-invariant and f is nearly Borel and finely continuous on $X-B_0$. For a fixed natural number M we define the trancated function f^M of f by $f^M=(f\wedge M)\vee(-M)$ on $X-B_0$. We extend f^M by setting its value to be zero on B_0. By the fine continuity of f^M on $X-B_0$, we have

$$\lim_{n\to\infty} nG_n f^M(x) = f^M(x)\,, \qquad x \in X - B_0\,.$$

On the other hand, there are Borel functions f_1 and f_2 such that $f_1 \leqq f^M \leqq f_2$ on X and $\int_X (f_2(x) - f_1(x))m(dx) = 0$. But, for any $h \in \boldsymbol{C}_0(X)$,

$$(h, G_n(f_2 - f_1)) = (\hat{G}_n h, f_2 - f_1) = 0$$

yielding that $G_n f_1 = G_n f_2$ m-a. e. and hence q. e. owing to (**vi**). Therefore there is a Borel almost polar set B_n such that $G_n f^M(x) = G_n f_1(x)$ for every $x \in X - B_n$. Put $B_M = B_0 \cup \bigcup_{n=1}^{\infty} B_n$, then

$$f^M(x) = \lim_{n\to\infty} nG_n f^M(x) = \lim_{n\to\infty} nG_n f_1(x)\,, \qquad x \in X - B_M\,.$$

Consequently f^M is Borel measurable on $X - B_M$. According to (**vii**), there is a Borel almost polar set $B \supset \bigcup_M B_M$ such that $X - B$ is $\boldsymbol{M}$-invariant. Then $f(x) = \lim_{M\to\infty} f^M(x)$ is Borel measurable on $X - B$, completing the proof.

(**x**) *Let $\{f_n\}$ be a decreasing sequence of α-excessive functions with limit f and suppose that $f = 0$ m-a. e. Then $f = 0$ q. e.*

This proposition corresponds to Blumenthal-Getoor [**2**; VI (3.2)]. The proof is quite the same. We do not know whether in our case every semipolar set is cosemipolar. But by making use of (**x**) and following the same line as in Blumenthal-Getoor [**2**; VI (1.19)], we get

(**xi**) *Each semipolar set is the sum of a cosemipolar set and an almost polar set.*

(**xii**) *For any Borel sets A and B, we have*

$$(g, H_A^\alpha H_B^\alpha G_\alpha h) = (\hat{H}_B^\alpha \hat{H}_A^\alpha \hat{G}_\alpha g, h)$$

for any non-negative Borel functions g and h.

PROOF. This is a consequence of (2.2). Take non-negative g and h in $\boldsymbol{C}_0(X)$. Since $H_B^\alpha G_\alpha h$ (resp. $\hat{H}_A^\alpha \hat{G}_\alpha g$) is an α-excessive (resp. α-coexcessive) function, we have

$$\begin{aligned}(g, H_A^\alpha H_B^\alpha G_\alpha h) &= \lim_{\beta\to+\infty} \beta(g, H_A^\alpha G_\beta H_B^\alpha G_\alpha h)\\ &= \lim_{\beta\to+\infty} \beta(g, H_A^\alpha G_\alpha (I - (\beta-\alpha)G_\beta) H_B^\alpha G_\alpha h)\\ &= \lim_{\beta\to+\infty} \beta(\hat{H}_B^\alpha \hat{G}_\alpha (I - (\beta-\alpha)\hat{G}_\beta)\hat{H}_A^\alpha \hat{G}_\alpha g, h)\\ &= \lim_{\beta\to+\infty} \beta(\hat{H}_B^\alpha \hat{G}_\beta \hat{H}_A^\alpha \hat{G}_\alpha g, h) = (\hat{H}_B^\alpha \hat{H}_A^\alpha \hat{G}_\alpha g, h)\,.\end{aligned}$$

(**xiii**) *Let A be a Borel set. Denote by A^r (resp. ${}^rA$) the totality of regular (resp. coregular) points of A. Then ${}^rA - A^r$ is written as*

$${}^rA-A^r=N_1+N_2$$

with a Borel semi-polar set N_1 and a nearly Borel almost polar set N_2. The same conclusion holds for $A^r-{}^rA$.

PROOF. Since ${}^rA$ is co-nearly Borel, there are Borel sets $\hat{A}'$ and $\hat{A}''$ such that $\hat{A}'\subset{}^rA\subset\hat{A}''$ and $\hat{A}''-\hat{A}'$ is almost copolar. There are also Borel sets A' and A'' such that $A'\subset A^r\subset A''$ and $A''-A'$ is almost polar. Put $F=\hat{A}'-A''$, then F is a Borel set, $F\subset{}^rA-A^r$ and the set $({}^rA-A^r)-F$ is almost polar in view of (**iv**). By the preceding identity, we have

$$(g,\ H_A^\alpha H_F^\alpha G_\alpha h)=(\hat{H}_F^\alpha\hat{H}_A^\alpha\hat{G}_\alpha g,\ h)\,.$$

Since $F\cup{}^rF\subset{}^rA$, we see that $\hat{H}_F^\alpha\hat{H}_A^\alpha\hat{G}_\alpha g=\hat{H}_F^\alpha\hat{G}_\alpha g$. Hence, by (2.2), $(g,\ H_A^\alpha H_F^\alpha G_\alpha h)=(g, H_F^\alpha G_\alpha h)$. Now choose h_n such that $G_\alpha h_n\uparrow 1$. We have $(g, H_A^\alpha e_F^\alpha)=(g, e_F^\alpha)$ for every $g\in C_0(X)$. Using (**vi**) we get $e_F^\alpha=H_A^\alpha e_F^\alpha$ q. e. If $x\in F$, then $x\notin A^r$ and $H_A^\alpha e_F^\alpha(x)\leq H_A^\alpha 1(x)<1$. Thus, there is an almost polar Borel set N' such that $e_F^\alpha(x)<1$ for $x\in N_1=F-N'$. N_1 is then a Borel semipolar set because $e_{N_1}^\alpha(x)\leq e_F^\alpha(x)<1$ for $x\in N_1$. Now ${}^rA-A^r=N_1+N_2$ with $N_2=[({}^rA-A^r)-F]+F\cap N'$ is the desired expression.

(**xiv**) *The following two conditions are equivalent.*

(C_1) *Each semipolar set is almost polar.*

(C_1') *A function is q. e. finely continuous if and only if it is q. e. cofinely continuous.*

PROOF. Assume the condition (C_1). Consider a q. e. finely continuous function f. By (**ix**), there is an almost polar Borel set B such that $X_0=X-B$ is finely open and f is Borel measurable and finely continuous on X_0. For a real number a, put $E_a=\{x\in X_0;\ f(x)<a\}$. Since $X-E_a$ is finely closed,

$$N_a={}^r(X-E_a)-(X-E_a)\subset{}^r(X-E_a)-(X-E_a)^r$$

which is almost polar on account of (**xiii**) and (C_1). Notice that $E_a-N_a=E_a-{}^r(X-E_a)$ is cofinely open. Choose an almost polar Borel set $N_a'\supset N_a$ and set $\hat{B}_0=B\cup(\bigcup_{a:\ \text{rational}}N_a')$. By virtue of (**vii**), there exists an almost polar Borel set $\hat{B}\supset\hat{B}_0$ such that $X-\hat{B}$ is $\hat{\boldsymbol{M}}$-invariant. Now, for any rational a, the set $\{x\in X-\hat{B};\ f(x)<a\}=E_a-\hat{B}$ is cofinely open because $E_a-\hat{B}=(E_a-N_a)\cap(X-\hat{B})$ and both E_a-N_a and $X-\hat{B}$ are cofinely open. This shows that f is q. e. cofinely continuous.

Coming to the converse, fix a compact thin set K. We first note that

$$(2.4)\qquad P_x(\lim_{t\uparrow\sigma_K}e_K^\alpha(X_t)=1,\ \sigma_K<+\infty)=P_x(\sigma_K<+\infty)\qquad\text{for}\quad m\text{-a. e. } x\in X\,.$$

To see this, let $\{G_n\}$ be open sets such that $G_n\supset\bar{G}_{n+1}$ and $\bigcap_n G_n=K$. Using (**xii**), we can easily see that $H_{G_n}^\alpha e_K^\alpha(x)=e_K^\alpha(x)$ m-a. e. Since K is thin, we then have $P_x(\sigma_K<+\infty)=P_x(\sigma_{G_n}<\sigma_K$ for every n, $\lim_n \sigma_{G_n}=\sigma_K$, $\sigma_K<+\infty)$ for m-a. e. $x\in X$. We arrive at (2.4) by virtue of [**2**; II (3.12)].

Now let us assume the condition (C_1') and prove that K is almost polar. Put $B_n=\left\{x\in X;\ e_K^\alpha(x)\geqq 1-\frac{1}{n}\right\}$, then (2.4) implies that $\sigma_{B_n}<\sigma_K$ P_x-almost surely on $\{\sigma_K<+\infty\}$ for m-a. e. $x\in X$. Hence $H_{B_n}^\alpha H_K^\alpha f=H_K f$ m-a. e. for any bounded Borel f. Using (**xii**) again, we finally get

$$(2.5)\qquad (\hat{H}_K^\alpha \hat{H}_{B_n}^\alpha g,\ h)=(\hat{H}_K^\alpha g,\ h)$$

for bounded continuous g and m-integrable bounded Borel h. On the other hand, the function e_K^α is, being α-excessive, q. e. cofinely continuous by (C_1'). On account of (**ix**), there is an m-negligible Borel set N such that $X-N$ is $\hat{\boldsymbol{M}}$-invariant and each set B_n-N is (relatively) cofinely closed in $X-N$. Fix a strictly positive bounded m-integrable function h. Rewriting (2.5) as $(\hat{H}_K^\alpha g,\ h)=(I_{X-N}\hat{H}_{K-N}^\alpha \hat{H}_{B_n-N}^\alpha g,\ h)$, we can observe that the measure $\pi(E)=(\hat{H}_K^\alpha I_E,\ h)$ is concentrated on B_n-N. But $\cap B_n$ is empty and hence π must be a zero measure. In particular, $\hat{e}_K^\alpha(x)=\hat{H}_K^\alpha 1(x)$ vanishes m-a. e., yielding that K is almost polar.

§ 3. An ergodic theorem.

Let us assume the condition (C_1) throughout this section. We put

$$(3.1)\qquad K_t f(x)=\int_0^t T_s f(x)ds\,,\qquad x\in X\,,$$

for a bounded Borel function f and $t>0$.

Our main theorem is this.

THEOREM 3.1. *For any bounded Borel f, $g\in L^1(X;m)$, $g\geqq 0$, the ratio*

$$(3.2)\qquad \frac{K_t f(x)}{K_t g(x)}$$

converges, as $t\to+\infty$, to a finite limit q. e. on the set

$$(3.3)\qquad E_g=\bigcup_{t>0}\{x\in X;\ K_t g(x)>0\}\,.$$

This generalization of the Chacon-Ornstein ergodic theorem is an easy consequence of the following version of Brunel's lemma [**10**; 2.7] and the assertion (**v**) of the preceding section.

LEMMA 3.1. *Let f be a bounded Borel function of $L^1(X;m)$ and A be a Borel subset of*

$$(3.4)\qquad \bigcap_{n=1}^\infty \{x\in X;\ \sup_{t>n} K_t f(x)>0\}\,.$$

Then we have

(3.5) $$(\hat{e}_A, f) \geqq 0,$$

where

(3.6) $$\hat{e}_A(x) = \hat{P}_x(\sigma_A < +\infty), \qquad x \in X.$$

We will have to use our condition (C_1) to prove Lemma 3.1. It is convenient to introduce here the notion of characteristics of the Markov process $\boldsymbol{M}$. A non-negative finite valued function $c_t(x)$, $t>0$, $x \in X$, is called a non-negative characteristic if

(3.7) $c_t(\cdot)$ is universally measurable,

(3.8) $$c_s(x) + T_s c_t(x) = c_{s+t}(x), \qquad s, t>0, \qquad x \in X,$$

(3.9) $$\lim_{t \downarrow 0} c_t(x) = 0.$$

The difference of two non-negative characteristics is called merely a characteristic. $K_t f$ is a simple example of a characteristic. It is easy to see that any characteristic is right continuous in $t>0$. Furthermore any bounded characteristic is nearly Borel measurable and finely continuous in $x \in X$. The proof of this quite useful fact was given in E. B. Dynkin [7; Theorem 6.5].

PROOF OF THE IMPLICATION: LEMMA 3.1. $\Rightarrow$ THEOREM 3.1. We may assume, without loss of generality, that f is also non-negative in the statement of Theorem 3.1. Since $K_t g$ is continuous in $t>0$, $E_g = \bigcup_{\substack{r>0 \\ \text{rational}}} \{x \in X;\ K_r g(x) > 0\}$, which is Borel measurable and finely open because of the corresponding properties of the function $K_r g$. Now let us consider the set

$$N = \left\{x \in E_g;\ \overline{\lim_{t \to +\infty}} \frac{K_t f(x)}{K_t g(x)} = +\infty\right\}.$$

Then, for any $\lambda > 0$,

$$N \subset \bigcap_{n=1}^{\infty} \{\sup_{t \geqq n} K_t(f - \lambda g)(x) > 0\}.$$

By Lemma 3.1, we have

$$+\infty > (\hat{e}_N, f) \geqq \lambda(\hat{e}_N, g) \geqq \frac{\lambda}{r}(\hat{K}_r \hat{e}_N, g) = \frac{\lambda}{r}(\hat{e}_N, K_r g).$$

λ being arbitrary, we must have $\hat{e}_N = 0$ m-a. e. on the set $\{x \in X;\ K_r g(x) > 0\}$ and hence m-a. e. on E_g. In view of § 2 (v), N is almost polar. In the same way, we see that the set

$$N_{ab} = \left\{x \in E_g;\ \underline{\lim_{t \to \infty}} \frac{K_t f(x)}{K_t g(x)} < a,\ b < \overline{\lim_{t \to \infty}} \frac{K_t f(x)}{K_t g(x)}\right\}$$

is also almost polar for any $a < b$. q. e. d.

From now on we will concentrate our attention on the proof of Lemma 3.1. We essentially follow the reasoning of A. M. Garsia [10; Chap. 2] and its refinement by P. A. Meyer [11; Theorem 6]. We have to prepare three lemmas (Lemma 3.2~3.4). The following lemma presents our version of Hopf's maximal ergodic inequality (in Garsia's form).

LEMMA 3.2. *Let $c_t(x)$ be a bounded m-integrable characteristic and h be a positive number. We put $E_h=\{x\in X;\ \sup_n c_{nh}(x)>0\}$. Then we have*

$$(3.10)\qquad (I_{E_h}\cdot v,\ c_h)\geqq 0\,,$$

where v is an arbitrary bounded co-excessive function, namely, v is non-negative bounded and $\hat{T}_t v\uparrow v$ as $t\downarrow 0$.

PROOF. Consider the set

$$E_h^n=\{x;\ \max_{1\leqq\nu\leqq n} c_{\nu h}(x)>0\}=\{x;\ \max_{1\leqq\nu\leqq n} c_{\nu h}^+(x)>0\}\,.$$

For $x\in E_h^n$, we have

$$c_h(x)+\max_{1\leqq\nu\leqq n}(c_{(\nu+1)h}-c_h)^+(x)\geqq \max_{1\leqq\nu\leqq n} c_{\nu h}^+(x)\,.$$

Since c_t is a characteristic,

$$(c_{(\nu+1)h}-c_h)^+=(T_h c_{\nu h})^+\leqq T_h c_{\nu h}^+$$

and hence

$$\max_{1\leqq\nu\leqq n}(c_{(\nu+1)h}-c_h)^+\leqq T_h \max_{1\leqq\nu\leqq n} c_{\nu h}^+\,.$$

Therefore we have

$$\begin{aligned}(I_{E_h^n}\cdot v,\ c_h)&\geqq (I_{E_h^n}\cdot v,\ \max_{1\leqq\nu\leqq n} c_{\nu h}^+-T_h(\max_{1\leqq\nu\leqq n} c_{\nu h}^+))\\ &\geqq (v,\ \max_{1\leqq\nu\leqq n} c_{\nu h}^+-T_h(\max_{1\leqq\nu\leqq n} c_{\nu h}^+))\\ &=(v-\hat{T}_h v,\ \max_{1\leqq\nu\leqq n} c_{\nu h}^+)\geqq 0\,.\end{aligned}$$

Letting n tend to infinity, we get (3.10).

Consider next a Borel set $A\subset X$ and a constant $T>0$. We put

$$^{(0)}T_t g(x)=E_x(g(X_t);\ t<\sigma_A)$$

$$^{(0)}K_t g(x)=\int_0^t {}^{(0)}T_s g(x)ds$$

$$H_{\sigma_A\wedge T}\, g(x)=E_x(g(X_{\sigma_A\wedge T}))\,.$$

LEMMA 3.3. *Let f be a bounded m-integrable Borel function and d_t^f be*

$$(3.11)\qquad d_t^f(x)=K_t f(x)-{}^{(0)}K_T f(x)+T_t\cdot{}^{(0)}K_T f(x)\,,\qquad t>0,\quad x\in X\,.$$

Then

(i) *d_t^f is a bounded m-integrable characteristic.*

(ii) *d_t^f satisfies the equality*

$$(g, d_t^f) = (\hat{H}_{\sigma_A \wedge T} \hat{K}_t g, f) \tag{3.12}$$

for any bounded Borel function g.

PROOF. We write d_t^f simply as d_t.

(i) d_t is bounded and universally measurable. It follows from (1.1) that $\|T_t f\|_{L^1} \leqq \|f\|_{L^1}$ for any Borel $f \in L^1(X; m)$. Hence $d_t \in L^1(X; m)$. The equality (3.8) for d_t is easily verified. Let us verify (3.9). Denote $\sigma_A \wedge T$ by σ. Since ${}^{(0)}K_T f(x) = E_x\Big(\int_0^\sigma f(X_s)\,ds\Big)$, we have $T_t {}^{(0)}K_T f(x) = E_x\Big(\int_t^{t+\sigma(\theta_t\omega)} f(X_s)\,ds\Big)$, θ_t being the shift of the sample path ω: $(\theta_t\omega)_s = \omega_{t+s}$, $s > 0$. Hence we have $d_t(x) = E_x\Big(\int_\sigma^{t+\sigma(\theta_t\omega)} f(X_s)\,ds\Big)$, which tends to zero as $t \downarrow 0$ because $t + \sigma(\theta_t) \to \sigma (t \downarrow 0)$.

(ii) We introduce α-order quantities:

$$\hat{T}_t^\alpha g(x) = e^{-\alpha t} \hat{T}_t g(x), \qquad {}^{(0)}\hat{T}_t^\alpha g(x) = e^{-\alpha t}\, {}^{(0)}\hat{T}_t g(x),$$

$${}^{(0)}\hat{K}_T^\alpha g(x) = \int_0^T {}^{(0)}\hat{T}_s^\alpha g(x)\,ds, \qquad \hat{H}_\sigma^\alpha g(x) = \hat{E}_x(e^{-\alpha\sigma} g(X_\sigma)).$$

Since ${}^{(0)}\hat{K}_T^\alpha g(x) = \hat{E}_x\Big(\int_0^\sigma e^{-\alpha s} g(X_s)\,ds\Big)$, we have by Dynkin's formula

$$\hat{H}_\sigma^\alpha \hat{G}_\alpha g = \hat{G}_\alpha g - {}^{(0)}\hat{K}_T^\alpha g. \tag{3.13}$$

Replacing the function g in (3.13) by $\hat{T}_t^\alpha g$ and subtracting the resulting equality from (3.13), we have $\hat{H}_\sigma^\alpha \hat{K}_t^\alpha g = \hat{K}_t^\alpha g - {}^{(0)}\hat{K}_T^\alpha g + {}^{(0)}\hat{K}_T^\alpha \hat{T}_t^\alpha g$. Letting α tend to zero,

$$\hat{H}_\sigma \hat{K}_t g = \hat{K}_t g - {}^{(0)}\hat{K}_T g + {}^{(0)}\hat{K}_T \hat{T}_t g.$$

Taking the duality relations (1.1) and (2.1) of T_t and ${}^{(0)}T_t$ into consideration, we finally get

$$(\hat{H}_\sigma \hat{K}_t g, f) = (g, K_t f - {}^{(0)}K_T f + T_t {}^{(0)}K_T f) = (g, d_t).$$

Let us write as $A_1 \underset{\text{q.e.}}{\subset} A_2$ if $A_1 - A_2$ is almost polar.

LEMMA 3.4. *Let f be a bounded m-integrable Borel function and A be a Borel subset of*

$$\bigcap_n \{x \in X;\ \sup_{t>n} K_t f(x) > 0\}. \tag{3.14}$$

Fix a constant $T > 0$ and consider the characteristic d_t^f of Lemma 3.3 defined for the present f, A and T.

Then, for any $\varepsilon > 0$,

$$A \underset{\text{q.e.}}{\subset} \{x \in X;\ \sup_{t>0} (\varepsilon K_t |f|(x) + d_t^f(x)) > 0\}. \tag{3.15}$$

PROOF. From (3.9),

$$d_t^f(x)+T_tK_T|f|(x)\geqq K_tf(x)-{}^{(0)}K_Tf(x).$$

Observe that

$$\lim_{t\to\infty}\frac{T_tK_T|f|(x)}{K_t|f|(x)}=0, \qquad x\in X,$$

defining 0/0 to be 0 by convention. Hence, for any $\varepsilon>0$,

$$d_t^f(x)+\varepsilon K_t|f|(x)\geqq K_tf(x)-{}^{(0)}K_Tf(x)$$

for every t greater than some $t_0=t_0(x)>0$. On the other hand, the condition (C_1) implies

(3.16) $$P_x(\sigma_A=0)=1 \qquad \text{for q. e. } x\in A.$$

This is because $A-A^r$ is semipolar (c. f. [**2**; II (3.3)]) and hence almost polar by (C_1). Therefore we have

$${}^{(0)}K_Tf(x)=E_x\Big(\int_0^{\sigma_A\wedge T}f(X_s)ds\Big)=0 \qquad \text{for q. e. } x\in A,$$

arriving at the desired inclusion (3.15).

We are now ready to prove Lemma 3.1.

PROOF OF LEMMA 3.1. Let f be a bounded Borel function of $L^1(X; m)$. Consider any compact subset A of (3.14). Then, by Lemma 3.4, we have the inclusion (3.15) with an arbitrarily fixed $T>0$ and $\varepsilon>0$.

The characteristic d_t^f satisfies the relation (3.12) which can be rewritten as follows:

(3.17) $$(g, d_t^f)=(\hat{H}_A^T\hat{K}_tg, f)+(\hat{Q}_T\hat{K}_tg, f)$$

where

(3.18) $$\begin{cases}\hat{H}_A^Th(x)=\hat{E}_x(h(X_{\sigma_A});\ \sigma_A<T)\\ \hat{Q}_Th(x)=\hat{E}_x(h(X_T);\ T<\sigma_A).\end{cases}$$

Let us put

(3.19) $$c_t(x)=\varepsilon K_t|f|(x)+d_t^f(x),$$

which is a bounded m-integrable characteristic. We then define

(3.20) $$E_{\varepsilon,l}=\{x\in X;\ \max_{\nu} c_{\nu/2^l}(x)>0\}$$

for positive integer l. By virtue of Lemma 3.2, we have $(I_{E_{\varepsilon,l}}\cdot v, c_{1/2^l})\geqq 0$ for any bounded co-excessive function v. Combining this with (3.17) and (3.19), we get

$$\varepsilon(\hat{K}_{1/2^l}v, |f|)+(\hat{Q}_T\hat{K}_{1/2^l}v, |f|)+(\hat{H}_A^T\hat{K}_{1/2^l}(I_{E_{\varepsilon,l}}\cdot v), f)\geqq 0.$$

Multiplying 2^l and letting l tend to infinity,

$$\varepsilon(v, |f|)+(\hat{Q}_T v, |f|)+\overline{\lim_{l\to\infty}}\, 2^l(\hat{H}_A^T\hat{K}_{1/2^l}(I_{E_{\varepsilon,l}}\cdot v), f)\geqq 0\,.$$

Now look at the last term of the left hand side of this inequality. Recalling Lemma 3.4, we see that, there is an almost polar set N_1 such that $A-N_1\subset\bigcup_{l=1}^{\infty}E_{\varepsilon,l}$. Furthermore we can find almost polar sets N_2^l such that each set $E_{\varepsilon,l}-N_2^l$ is cofinely open, because $c_t(x)$ is finely continuous and hence q. e. cofinely continuous by virtue of the condition $(\boldsymbol{C}_1')$ which is equivalent to $(\boldsymbol{C}_1)$ (§ 2, (**xiv**)). Select then an almost polar set $N_2\supset\bigcup_{l=1}^{\infty}N_2^l$ such that $X-N_2$ is $\boldsymbol{M}$-invariant according to § 2 (**vii**). It is easy to see that each $E_{\varepsilon,l}-N_2$ is cofinely open. Thus

(3.21) $$\lim_{l\to\infty} 2^l\hat{K}_{1/2^l}(I_{E_{\varepsilon,l}}\cdot v)(x)=v(x)\,,\qquad x\in A-N_1\cup N_2\,.$$

Since the signed measure $f\cdot\hat{H}_A^T(E)=(\hat{H}_A^T I_E, f)$ is concentrated on the compactum A and does not charge on any almost polar set, we obtain

(3.22) $$\varepsilon(v, |f|)+(\hat{Q}_T v, |f|)+(\hat{H}_A^T v, f)\geqq 0\,.$$

Letting ε tend to zero, $(\hat{Q}_T v, |f|)+(\hat{H}_A^T v, f)\geqq 0$. Finally put $v=\hat{e}_A$ and let T tend to infinity. Observe that $\hat{Q}_T\hat{e}_A(x)=\hat{P}_x(\sigma_A(\theta_T\omega)<+\infty, T<\sigma_A)=\hat{P}_x(T<\sigma_A<+\infty)\to 0$, $T\to\infty$. Using again the property of the measure $f\cdot\hat{H}_A^T$ stated just before (3.22) and recalling (3.16), we are led to $\lim_{T\to\infty}(\hat{H}_A^T\hat{e}_A, f)=\lim_{T\to\infty}(\hat{H}_A^T 1, f)=(\hat{e}_A, f)$, arriving at the desired inequality (3.5).

§ 4. An ergodic decomposition of the state space.

Let Φ be an m-integrable bounded continuous function strictly positive on X and let us put

(4.1) $$\begin{cases} C=\{x\in X;\ G\Phi(x)=+\infty\} \\ D=\{x\in X;\ G\Phi(x)<+\infty\} \end{cases}$$

where $G\Phi(x)=\lim_{t\to\infty}K_t\Phi(x)$. We call C (resp. D) the *conservative* (resp. *dissipative*) part of X. If we assume the condition $(\boldsymbol{C}_1)$, then Theorem 3.1 holds and hence the decomposition $X=C+D$ does not depend on the choice of such a Φ up to an almost polar set.

In order to study some more about this decomposition, let us assume the following conditions throughout this section:

$(\boldsymbol{C}_2)$ A set is semipolar if and only if it is almost polar,

$(\boldsymbol{C}_3)$ Each point $x\in X$ is either polar or stable,

where we say a point $x\in X$ to be stable if almost all sample paths starting at x stay there during some initial time intervals.

The condition (C_2) is the same as (C_1) coupled with the assumption that the measure m is a reference measure for the process $\boldsymbol{M}$ (§ 2, (**viii**)). In particular, (C_2) is met when $\boldsymbol{M}=\hat{\boldsymbol{M}}$ and the resolvent is absolutely continuous with respect to m. Under (C_2), polarity, semipolarity and almost polarity become the equivalent notions. (C_3) corresponds to the condition (L') of J. L. Doob [**6**]. In [**6**] the notion of a (fine) quasinull set was principally used, which is however equivalent, under (C_2) and (C_3), to the polarity of the set (see [**6**; § 8]).

A theorem of Doob [**6**; Theorem 8.1] states that, if $\boldsymbol{M}$ has a reference measure, the fine topology relative to $\boldsymbol{M}$ has the quasi Lindelöf property, that is, every union of finely open sets is equal, up to a quasinull set, to a countable subunion. Therefore, under the present assumptions (C_2) and (C_3), we have

LEMMA 4.1. *Every union of finely open sets is a sum of a countable subunion and a polar set.*

Now following Azema, Kaplan-Duflo and Revuz [**1**], we call a point $x \in X$ *finely transient* if there is a nearly Borel fine neighbourhood V of x such that

$$P_x(\overline{\lim_{t\to\infty}}\, I_V(X_t)=1)=0\,. \tag{4.2}$$

Otherwise x is called *finely recurrent.* A point $x \in X$ is finely transient if and only if there is a non-negative bounded Borel function f such that

$$0<Gf(x)<+\infty \tag{4.3}$$

is valid ([**1**; Proposition.5]). We will soon use this criterion. Denote by T (resp. R) the set of all transient (resp. recurrent) points of X.

THEOREM 4.1. *D is a subset of T and the difference $N=T-D$ is polar. Furthermore $R(=C-N)$ is $\boldsymbol{M}$-invariant.*

PROOF. Since $G\Phi$ is strictly positive everywhere together with Φ, we clearly have $D\subset T$.

Observe that, from the relation (3.8) of the characteristic, we have $K_t f+T_t Gf=Gf$ for any non-negative bounded Borel f, which in turn tells us that Gf is excessive and hence finely continuous. For each point $x\in T$, let us associate a non-negative bounded Borel function f_x such that (4.3) holds. We may further assume that f_x is m-integrable. By the above observation, the set $U(x)=\{y;\ 0<Gf_x(y)<+\infty\}$ $(\subset T)$ is a fine neighbourhood of $x\in T$. By Lemma 4.1, there exist points $x_i\in T$, $i=1,2,\cdots$, and a polar set N such that

$$T=\Big(\bigcup_{i=1}^{\infty}U(x_i)\Big)\cup N\,. \tag{4.4}$$

Applying Theorem 3.1 to Φ and f_{x_i}, we see that each set $C\cap U(x_i)$ is polar.

Therefore $N=C\cap T$ is polar in view of (4.4). Since N is polar, it remains to show

(4.5) $$P_x(\sigma_D<+\infty)=0\,,\qquad x\in R\,.$$

Assume that (4.5) is false, then there are a point $x\in R$ and a set

(4.6) $$\tilde{D}=\{x\in X;\ \delta<\Phi(x),\ G\Phi(x)<M\}\qquad(\subset D)$$

with some $0<\delta<M<+\infty$ such that

(4.7) $$P_x(\sigma_{\tilde{D}}<+\infty)>0\,.$$

Now put $\Psi(y)=\Phi(y)\cdot I_{\tilde{D}}(y)$, $y\in X$, then

(4.8) $$G_\alpha\Psi(x)=E_x(e^{-\alpha\sigma_{\tilde{D}}}G_\alpha\Psi(X_{\sigma_{\tilde{D}}}))\,.$$

Since $G\Phi$ is right continuous along sample paths, (4.8) implies $G_\alpha\Psi(x)\leqq E_x(G\Phi(X_{\sigma_{\tilde{D}}}))\leqq M$ yielding $G\Psi(x)\leqq M$. Furthermore (4.7) and (4.8) imply

$$G\Psi(x)\geqq\delta E_x(e^{-\alpha\sigma_{\tilde{D}}}G_\alpha I_{\tilde{D}}(X_{\sigma_{\tilde{D}}}))>0\,,$$

because, $\tilde{D}$ being finely open, $G_\alpha I_{\tilde{D}}(y)$ is strictly positive for every $y\in\tilde{D}\cup\tilde{D}^r$ (the support of $X_{\sigma_{\tilde{D}}}$). Thus we get $0<G\Psi(x)\leqq M$ contradiction to the fact that x is finely recurrent.

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On an L^p-Estimate of Resolvents of Markov Processes

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§ 1. Introduction

Let $(X, \mathcal{B}, m)$ be a measure space and $L^p = L^p(X; m)$ be the real L^p-space with norm $\| \ \|_p$. L^2-inner product is denoted by $(\ ,\ )$. The notation $\mathcal{E}$ will always means a bilinear form defined on $\mathcal{D}[\mathcal{E}] \times \mathcal{D}[\mathcal{E}]$, $\mathcal{D}[\mathcal{E}]$ being a linear subspace of L^2. We put for $\lambda \geqq 0$

$$\mathcal{E}_\lambda(u, v) = \mathcal{E}(u, v) + \lambda(u, v), \quad u, v \in \mathcal{D}[\mathcal{E}].$$

For $\lambda \geqq 0$ and $f \in L^2$, denote by $R_\lambda f$ a solution u of the next equation:

$$(1) \qquad \begin{cases} u \in \mathcal{D}[\mathcal{E}] \\ \mathcal{E}_\lambda(u, v) = (f, v) \quad v \in \mathcal{D}[\mathcal{E}]. \end{cases}$$

Our aim is to give an a priori estimate of $R_\lambda f$ under the next two conditions on the form $\mathcal{E}$:

($\mathcal{E}$. a) If $u \in \mathcal{D}[\mathcal{E}]$ and $k \geqq 0$, then $v = (u-k)^+ \in \mathcal{D}[\mathcal{E}]$ and $\mathcal{E}(v, v) \leqq \mathcal{E}(u, v)$.

($\mathcal{E}$. b) There exist $\lambda_0 \geqq 0$ and $p_0 > 2$ such that

$$(2) \qquad \|u\|_{p_0}^2 \leqq C\mathcal{E}_{\lambda_0}(u, u) \quad u \in \mathcal{D}[\mathcal{E}],$$

for some constant $C > 0$.

Theorem 1. *Suppose a bilinear form $\mathcal{E}$ satisfies the conditions ($\mathcal{E}$. a) and ($\mathcal{E}$. b), then for* $p > \frac{p_0}{p_0 - 2} \vee 2$

$$(3) \qquad \|R_0 f\|_\infty \leqq C_1 \|f\|_p + C_2 \|R_0 f\|_2, \quad f \in L^p \cap L^2$$

where $C_1 (>0)$ and $C_2 (\geqq 0)$ are constants depending only on λ_0, p_0, C

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and p. *Moreover* C_2 *vanishes when* $\lambda_0=0$ *and* $m(X)<\infty$.[1]

If a bilinear form $\mathcal{E}$ is *non-negative definite* in the sense that $\mathcal{E}(u, u)\geqq 0$, $u\in\mathcal{D}[\mathcal{E}]$, then the solution u of (1) with $\lambda>0$ is unique and satisfies $\|u\|_2\leqq\lambda^{-1}\|f\|_2$. Hence we have

Corollary. *Suppose a non-negative definite bilinear form* $\mathcal{E}$ *satisfies* ($\mathcal{E}$. a) *and* ($\mathcal{E}$. b), *then for* $\lambda>0$ *and* $p>\frac{p_0}{p_0-2}\vee 2$

$$\|R_\lambda f\|_\infty\leqq C_1\|f\|_p+C_2\|f\|_2, \quad f\in L^p\cap L^2 \tag{4}$$

where $C_1(>0)$ *and* $C_2(\geqq 0)$ *depend only on* λ_0, p_0, C, λ *and* p. C_2 *vanishes when* $m(X)<\infty$.

G. Stampacchia [7] obtained an estimate very close to (3) when X is a bounded open set D of R^n, m is the Lebesgue measure and $\mathcal{E}$ is related to the elliptic differential operator $\sum\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right)+\sum b_i\frac{\partial}{\partial x_i}$:

$$\mathcal{E}(u, v)=\sum_{i,j=1}^{n}\int_D\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_j}a_{ij}(x)dx+\sum_{i=1}^{n}\int_D b_i(x)\frac{\partial u(x)}{\partial x_i}v(x)dx \tag{5}$$

with $\mathcal{D}[\mathcal{E}]=H_0^1(D)$. Here a_{ij} is bounded, $\sum a_{ij}(x)\xi_i\xi_j\geqq\nu|\xi|^2$, $\xi\in R^n$, for some constant $\nu>0$ and $b_i\in L^n(D)$. It is known that this form satisfies the condition ($\mathcal{E}$. a) even with the equality $\mathcal{E}(v, v)=\mathcal{E}(u, v)$. Moreover property ($\mathcal{E}$. b) for a sufficiently large λ_0 (which depends only on ν and $\|b_i\|_n$) and for $\frac{1}{2}>\frac{1}{p_0}>\frac{1}{2}-\frac{1}{n}$ follows from the coercivity of $\mathcal{E}_{\lambda_0}$ on $H_0^1(D)$ and the Sobolev inequality.[2]

We show in the next section that Stampacchia's method of the proof still works in our general situation of Theorem 1. In Section 3, Theorem 1 and its Corollary are applied to several kinds of Dirichlet forms includ-

---

[1] More explicitly we may take

$$C_1=2^{1+(p/pp_0-2p-p_0)}\cdot C\cdot[m(X)]^{(p_0-2/p_0)-(1/p)} \quad \text{and} \quad C_2=0$$

when $\lambda_0=0$ and $m(X)<\infty$.

[2] See footnotes 5).

ing the above one of Stampacchia. In particular we are concerned with a form of non-local character

$$(6) \qquad \mathcal{E}(u,v) = \int_{R^n}\int_{R^n} (u(x)-u(y))(v(x)-v(y))k(x,dy)\,dx$$

such that $k(x,dy)$ dominates the Lévy measure of the symmetric stable process with index $0<\alpha<2$:

$$(7) \qquad k(x,E) \geqq \gamma \int_E |x-y|^{-(n+\alpha)}dy$$

for every Borel set E and a.e. $x\in R^n$, γ being a positve constant. By making use of a Sobolev type inequality in N. Aronszajn and K. T. Smith [1], we can see that $(\mathcal{E}.\text{b})$ is valid for $\lambda_0=1$ and $\frac{1}{2}>\frac{1}{p_0}>\frac{1}{2}-\frac{\alpha}{2n}$.

It has been known ([2] and [3]) that, given a regular Dirichlet form $\mathcal{E}$, there exists uniquely up to a certain equivalence a Hunt process $\boldsymbol{M}$ whose resolvent $R_\lambda f$ solves the equation (1). An immediate consequence of our estimate (4) is the absolute continuity of the resolvent kernel R_λ of the process $\boldsymbol{M}$. Further probabilistic consequences of (4) are stated in the last section. We note here that the estimate like (4) also plays important roles in the study of the stochastic differential equations and stochastic controls ([4]).

§ 2. Proof of Theorem 1

1°). Given $f\in L^2\cap L^p$ with $p>2$, consider a solution $u=G_0 f$ of equation (1) with $\lambda=0$. Put $v=(u-k)^+$ for a fixed $k>0$, then $v\in\mathcal{D}[\mathcal{E}]$ and $\mathcal{E}(v,v)\leqq(f,v)$ by $(\mathcal{E}.\text{a})$. $(\mathcal{E}.\text{b})$ and the Hölder inequality then yield

$$(8) \qquad \|v\|_{p_0}^2 \leqq C(f,v) + C\lambda_0[m(A(k))]^{1-2/p_0}\|v\|_{p_0}^2$$

where $A(k)=\{x\in X; u(x)>k\}$.

Take $k_0=(2C\lambda_0)^{p_0/2p_0-4}\|u\|_2$, then the second term of the right hand side of (8) is not greater than $\frac{1}{2}\|v\|_{p_0}^2$ for $k\geqq k_0$, because $m(A(k)) \leqq k_0^{-2}\|u\|_2^2=(2C\lambda_0)^{-p_0/p_0-2}$ in case that $\lambda_0>0$. Hence we get from (8) $\frac{1}{2}\|v\|_{p_0}^2\leqq C(f,v)$, $k\geqq k_0$. Applying the Schwarz inequality to the right hand side and then the Hölder inequality to each factor, we arrive at

$$\|v\|_{p_0}\leqq 2C\|f\|_p[m(A(k))]^{1-1/p-1/p_0}, \quad k\geqq k_0.$$

It is easy to derive from this for $h>k\geqq k_0$

$$m(A(h))\leqq\frac{2^{p_0}C^{p_0}}{(h-k)^{p_0}}\|f\|_p{}^{p_0}[m(A(k))]^{(1-1/p-1/p_0)p_0}. \tag{9}$$

2°). From the inequality (9) and Lemma 4. 1 of G. Stampacchia [7], we are led to the conclusion that, for m-a.e. $x\in X$, $u(x)$ is dominated by the right hand side of (3) with $C_2=(2C\lambda_0)^{p_0/2p_0-4}$. In case that $\lambda_0=0$ and $m(X)=\infty$, we may take $k_0=\|u\|_2$ and hence $C_2=1$ instead of $k_0=C_2=0$. Applying the same argument as above to $-u$ and $-f$, we can see that $-u(x)$ is also dominated by the same bound m-a.e. The proof of Theorem 1 is completed.

§ 3. Application to Dirichlet Forms

Following H. Kunita [5], we consider in this section a bilinear form $\mathcal{E}$ on L^2 such that, for some $\mu_0\geqq 0$, $\mathcal{E}_{\mu_0}$ is non-negative definite, continuous and closed (see Conditions (B. 1)$\sim$(B. 3) of [5]). Then there exists uniquely a semigroup $\{T_t, t>0\}$ of operators T_t on L^2 with $\|T_t\|_2\leqq e^{\mu_0 t}$ whose generator A satisfies $\mathcal{E}(u,v)=(-Au,v)$, $u\in\mathcal{D}(A)$, $v\in\mathcal{D}[\mathcal{E}]$. The Laplace transform $R_\lambda f$ of $T_t f$, $f\in L^2$, is the unique solution of (1) for $\lambda>\mu_0$.

Each T_t is *subMarkov* (namely, $0\leqq T_t f\leqq 1$ whenever $0\leqq f\leqq 1$, $f\in L^2$) if and only if $\mathcal{E}$ fulfills the additional condition ($\mathcal{E}$. a) which is also equivalent to the following [5]:

($\mathcal{E}$. a′) If $u\in\mathcal{D}[\mathcal{E}]$, then $v=(0\vee u)\wedge 1\in\mathcal{D}[\mathcal{E}]$ and $\mathcal{E}(v,v)\leqq\mathcal{E}(v,u)$.
We call $\mathcal{E}$ a *Dirichlet form* if it satisfies the condition ($\mathcal{E}$. a).

An example of a Dirichlet form for which μ_0 is not necessarily zero is the form (5). As was mentioned in Section 1, the form (5) satisfies condition ($\mathcal{E}$. a) as well as ($\mathcal{E}$. b) (with $\lambda_0\geqq\mu_0$). By Corollary to Theorem 1 we have for $\lambda>\mu_0$ and $p>\frac{n}{2}\vee 2$,

$$\|R_\lambda f\|_\infty\leqq C_1\|f\|_p,\quad f\in L^p \tag{10}$$

C_1 depending only on ν, $\|b_i\|_n$, p and λ. The inequality (10) holds even when $0<\lambda\leqq\mu_0$ because R_λ is still bounded on L^2 ([6; Lemma 2. 1]), but it is not clear how the bound is estimated a priori.

We call a Dirichlet form $\mathcal{E}$ on L^2 *symmetric* if $\mu_0=0$ and $\mathcal{E}(u,v)=\mathcal{E}(v,u)$ $u,v\in\mathcal{D}[\mathcal{E}]$. A bilinear form $\mathcal{E}$ on L^2 is a symmetric Dirichlet form if and only if $\mathcal{E}$ is non-negative definite, symmetric, closed and satisfies the condition

($\mathcal{E}$. a$''$) $u\in\mathcal{D}[\mathcal{E}]$ implies $v=(0\vee u)\wedge 1\in\mathcal{D}[\mathcal{E}]$ and $\mathcal{E}(v,v)\leqq\mathcal{E}(u,u)$.

Symmetric Dirichlet forms on L^2 are in one-to-one correspondence with semigroups $\{T_t, t>0\}$ of strongly continuous symmetric subMarkov operators on L^2 by the formula

$$(11)\qquad \begin{cases} \mathcal{D}[\mathcal{E}]=\{u\in L^2;\ \lim\limits_{t\downarrow 0}\dfrac{1}{t}(u-T_tu,u)<\infty\}. \\ \mathcal{E}(u,v)=\lim\limits_{t\downarrow 0}\dfrac{1}{t}(u-T_tu,v). \end{cases}$$

Here we give two examples of symmetric Dirichlet forms.

Let ν_{ij}, $1\leqq i,\ j\leqq n$, be Radon measures on R^n such that

$$(12)\qquad \nu_{ij}=\nu_{ji},\quad \sum_{i,j=1}^{n}\nu_{ij}(E)\xi_i\xi_j\geqq\gamma\cdot|E|\cdot|\xi|^2,\quad \xi\in R^n,\quad E\in\mathcal{B}(R^n),$$

where γ is some positive constant and $|E|$ is the Lebesgue measure of E. For a fixed open set $D\subset R^n$ we put

$$(13)\qquad \mathcal{E}(u,v)=\sum_{i,j=1}^{n}\int_D\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_j}\nu_{ij}(dx),\quad \mathcal{D}[\mathcal{E}]=C_0^\infty(D).$$

We assume that $\mathcal{E}$ is closable on $L^2(D)$. Then its smallest closed extension $\bar{\mathcal{E}}$ is a Dirichlet form on $L^2(D)$ ([3]). Furthermore we see from (12) that $\mathcal{D}[\bar{\mathcal{E}}]\subset H_0^1(D)$ and $\gamma\sum_{i=1}^{n}\int_D\left(\frac{\partial u}{\partial x_i}\right)^2dx\leqq\bar{\mathcal{E}}(u,u)$, $u\in\mathcal{D}[\bar{\mathcal{E}}]$. Hence $\bar{\mathcal{E}}$ satisfies, in view of the Sobolev inequality,[3] the condition ($\mathcal{E}$. b) with $\frac{1}{2}>\frac{1}{p_0}>\frac{1}{2}-\frac{1}{n}$. By Corollary to Theorem 1, the associated resolvent satisfies (4) for any $\lambda>0$ and $p>\frac{n}{2}\vee 2$, with C_1 and C_2 depending only on γ, p and λ. See [3] for examples of ν_{ij} which are not necessarily absolutely continuous with respect to the Lebesgue measure.

Next let $k(x,E)$ be a kernel on $R^n\times\mathcal{B}(R^n)$. We assume the condition (7) as well as the symmetry and finiteness assumptions:

---

[3] See footnotes 5).

$$
(14)\quad \begin{cases} \displaystyle\int_{R^n}\int_{R^n} u(x)v(y)k(x,dy)dx \\ \displaystyle\qquad = \int_{R^n}\int_{R^n} v(x)u(y)k(x,dy)dx\,, \quad u, v\in C_0^+(R^n), \\ C_0^\infty(R^n)\subset \mathcal{D}[\mathcal{E}], \end{cases}
$$

where $\mathcal{E}$ is the form on $L^2(R^n)$ defined by (6) with

$$
\mathcal{D}[\mathcal{E}] = \left\{u\in L^2(R^n);\ \int_{R^n}\int_{R^n}(u(x)-u(y))^2 k(x,dy)dx<\infty\right\}.
$$

It is easy to see that $\mathcal{E}$ is then a symmetric Dirichlet form on $L^2(R^n)$.

Denote by $\mathcal{E}^{(\alpha)}$ the symmetric Dirichlet form associated with the convolution semigroup $T_t u=\nu_t * u$ with $\hat{\nu}_t(\xi)=\exp(-t|\xi|^\alpha)$. By the formula (11), we have[4)]

$$
(15)\quad \mathcal{E}^{(\alpha)}(u, u) = \int_{R^n}|\hat{u}(\xi)|^2|\xi|^\alpha d\xi
$$
$$
= B(\alpha, n)\int_{R^n}\int_{R^n}(u(x)-u(y))^2\frac{1}{|x-y|^{n+\alpha}}dxdy\,,
$$

where $B(\alpha, n)$ is some universal constant ([1]). $\mathcal{D}[\mathcal{E}^{(\alpha)}]$ consists of those $u\in L^2(R^n)$ for which the integral in (15) converges. Therefore condition (7) leads us to

$$
(16)\quad \mathcal{D}[\mathcal{E}]\subset\mathcal{D}[\mathcal{E}^{(\alpha)}],\quad \mathcal{E}^{(\alpha)}(u, u)\leqq\frac{B(\alpha, n)}{\gamma}\mathcal{E}(u, u)\quad u\in\mathcal{D}[\mathcal{E}].
$$

Note that the space $\mathcal{D}[\mathcal{E}^{(\alpha)}]$ and the norm $\sqrt{\mathcal{E}_1^{(\alpha)}(u, u)}$ on it coincide respectively with $\boldsymbol{P}_{\alpha/2}$ and $|u|_{\alpha/2}$ of N. Aronszajn and K. T. Smith [1]. § 10 of [1] contains a neat proof of the inequality

$$
\|u\|_{p_0}\leqq M|||u|||_{\alpha/2}\,,\quad u\in\boldsymbol{P}_{\alpha/2}\,,\quad \frac{1}{2}>\frac{1}{p_0}>\frac{1}{2}-\frac{\alpha}{2n}\,,
$$

where M is a positive constant and $|||u|||_{\alpha/2}$ is a certain norm equivalent to $|u|_{\alpha/2}$.[5)] Combining this with (16), we get the conclusion stated at

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4) $\hat{\nu}_t(\xi)=\int_{R^n}e^{i\xi\cdot x}u(x)dx$ whereas we put $\hat{u}(\xi)=(2\pi)^{-n/2}\int_{R^n}e^{i\xi\cdot x}u(x)dx$. Perseval's formula is then in force and $(T_t u, u)=\int\hat{\nu}_t(\xi)|\hat{u}(\xi)|^2 d\xi$.

5) $|||u|||^2_{\alpha/2}=\int_{R^n}(1+|\xi|^2)^{\alpha/2}|\hat{u}(\xi)|^2 d\xi$. The inequality of Aronszajn-Smith remains valid for $\alpha=2$ in which case the above norm reduces to the norm of the space $H^1(R^n)$. The inequality also holds for $\frac{1}{p_0}=\frac{1}{2}-\frac{\alpha}{2n}$ provided $n>\alpha$.

the end of Section 1. By Corollary to Theorem 1, (4) now holds for $\lambda>0$ and $p>\frac{\alpha}{n}\vee 2$ with C_1 and C_2 depending only on α, γ, p and λ.

§ 4. Probabilistic Consequences

Let X be a locally compact separable Hausdorff space and m be an everywhere dense positive Radon measure on X. If a symmetric Dirichlet form $\mathcal{E}$ on $L^2(X; m)$ is *regular* ([3]), then there exists an m-symmetric standard Markov process $\boldsymbol{M}$ on X whose transition function defines the semigroup on L^2 associated with $\mathcal{E}$ by the formula (8). $\boldsymbol{M}$ is unique up to an equivalence relative to exceptional sets of zero capacity [3]. Here the capacity of a set is evaluated in terms of $\mathcal{E}_1$. We denote by $p(t, x, E)$ and $R_\lambda(x, E)$ the transition function and the resolvent kernel of $\boldsymbol{M}$ respectively.

Theorem 2. *Suppose a symmetric Dirichlet form* $\mathcal{E}$ *on* $L^2(X; m)$ *is regular and satisfies the condition* ($\mathcal{E}$. b), *then the associated standard process* $\boldsymbol{M}$ *possesses the following properties: there exists a Borel set* N *of zero capacity such that* $X-N$ *is* $\boldsymbol{M}$*-invariant and*
(i) $R_\lambda(x, \cdot)$ *is absolutely continuous with respect to* m *for each* $\lambda>0$ *and* $x\in X-N$,
(ii) $p(t, x, \cdot)$ *is absolutely continuous with respect to* m *for each* $t>0$ *and* $x\in X-N$,
(iii) *a set* $B\subset X-N$ *is of zero capacity if and only if* B *is polar, that is, almost all sample paths starting at* $x\in X-N$ *do not hit* B *at positive time.*

In fact the assertion (i) follows from the inequality (4) and the quasi-continuity of $R_\lambda f$, $f\in L^2\cap C_0^+$. Properties (i), (ii) and (iii) are equivalent ([3]). If we replace the notion of capacity by the fine capacity, Theorem 2 remains true for any m-symmetric standard process whose Dirichlet form satisfies ($\mathcal{E}$. b) (without regularity assumption on $\mathcal{E}$) ([3]). Theorem 2 can be extended to the non-symmetric Dirichlet form in view of the work of Carrillo Menendez [2].

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Dirichlet Spaces and Additive Functionals of Finite Energy

M. Fukushima

1. Introduction. In 1959 A. Beurling and J. Deny [1] introduced the notion of the Dirichlet space and revealed all essential substances of the theory, most of which were amplified and proven later in a fine exposition of J. Deny [3].

The link connecting this theory to the Markov process is in the following remarkable fact ([3], [5]): there is a one-to-one correspondence between the family of all Dirichlet forms $\mathcal{E}$ on an L^2-space and the family of all strongly continuous semigroups $\{T_t, t>0\}$ of Markovian symmetric operators on the same L^2-space, the correspondence being specified by

$$(1)\quad \begin{aligned} \mathcal{D}[\mathcal{E}] &= \left\{u\in L^2\colon \lim_{t\downarrow 0}\frac{1}{t}(u, u-T_t u)_{L^2}<\infty\right\},\\ \mathcal{E}(u, v) &= \lim_{t\downarrow 0}\frac{1}{t}(u, v-T_t v)_{L^2}. \end{aligned}$$

If the semigroup $\{T_t, t>0\}$ happens to be transient, then the domain $\mathcal{D}[\mathcal{E}]$ can be extended by completion with respect to the 0-order form $\mathcal{E}$ to a Hilbert space which is continuously embedded into a certain weighted L^1-space. The Dirichlet space in the original sense of [1] can be obtained this way ([3], [15] and Appendices of [5]).

Owing to such connections, the author [4], H. Kunita [11], J. Elliott [10], M. L. Silverstein [15] and Y. Le Jan [13] were able to use the Dirichlet forms on L^2-spaces quite effectively in resolving the so-called "boundary problem of Markov processes", which had been formulated and studied before by W. Feller, A. D. Wentzell et al. mainly in the framework of the semigroup theory on the Banach space $\boldsymbol{C}$ of continuous functions. In the meantime it has been shown that every regular Dirichlet form $\mathscr{E}$ admits a Hunt process M and moreover potential theoretic notions relevant to $\mathscr{E}$ (quasi-continuity, sets of capacity zero, reduced functions and so on) can be interpreted in the language of the Hunt probabilistic potential theory relevant to M (see [5], [6], [15] for symmetric cases and [2], [13] for nonsymmetric cases).

Two different Hunt processes may correspond to the same regular Dirichlet form but their restrictions outside a certain Borel set of capacity zero have the same transition probability [6]. At present we are content with this sort of loose uniqueness of the associated process since the potential theory of the regular Dirichlet form alone can not control inside a set of capacity zero. However it is still important to know whether one can select a nicest version (for instance a Hunt process with a Hölder continuous resolvent). Probably some methods of E. De Giorgi, G. Stampacchia et al. must be brought in before this point is made clearer. See [7] for some related information.

Now we utilize the above mentioned probabilistic potential theory relating the form $\mathscr{E}$ to the process M and study the structure of some important classes of additive functionals of M, namely the class A_c^+ of positive continuous additive functionals (PCAF's) and new classes of additive functionals of finite energy.

We first characterize the class A_c^+ by means of the family S of smooth measures. Since we relax the definition of AF of M slightly by admitting exceptional sets of capacity zero, the family S becomes wider and simpler than the families specified by H. McKean–H. Tanaka, D. Revuz et al. In fact S contains all positive Radon measures charging no set of capacity zero. As an application, H. Nagai [14] has been able to relate the optimal stopping problem of AF's directly to a variational inequality involving the form $\mathscr{E}$ and measures in S.

The *energy* $e(A)$ of a (not necessarily positive) AF A is introduced by

$$e(A)=\lim_{t\downarrow 0}\frac{1}{2t}E_m(A_t^2). \tag{2}$$

The space $\mathring{\mathscr{M}}$ of martingale additive functionals (MAF's) of finite energy is then seen to be complete with metric e. This makes it possible to define the stochastic integrals based on MAF's more simply than M. Motoo–S. Watanabe. Furthermore this leads us to a unique decomposition

$$\tilde{u}(X_t)-\tilde{u}(X_0)=M_t^{[u]}+N_t^{[u]},\ M^{[u]}\in\mathring{\mathscr{M}},\ N^{[u]}\in\mathscr{N}_c, \tag{3}$$

for any function u in the Dirichlet space $\mathscr{F}=\mathscr{D}[\mathscr{E}]$, $\mathscr{N}_c$ being the class of CAF's of zero energy.

The novelty of this decomposition lies in that each AF in $\mathscr{N}_c$ is of quadratic variation zero in a weak sense but not necessarily of bounded variation. Thus we are out of the range of semi-martingales and consequently the generalized Ito formula due to H. Kunita–S. Watanabe does not apply in general. Nevertheless we get the following variant of the Ito formula with the help of a transformation rule of energy measures due to Y. Le Jan [12]:

$$M^{[\Phi(u)]} = \sum_{i=1}^{n} \Phi_{x_i}(u)\cdot M^{[u_i]} \tag{4}$$

for a composite function $\Phi(u)=\Phi(u_1, u_2, \ldots, u_n)$, $u_i \in \mathcal{F}^b_{\mathrm{loc}}$, $1 \leq i \leq n$. For simplicity we assume here that the process M is a diffusion or equivalently that the form $\mathcal{E}$ possesses the local property.

Formulae (1), (2) and (3) tell us that the *resurrected Dirichlet space* $(\mathcal{F}, \mathcal{E}^{\mathrm{res}})$ introduced in § 4 is isometrically embedded into the Hilbert space $(\dot{\mathcal{M}}, e)$. Thus we essentially reduce the study of the Dirichlet space to the study of the space of MAF's of finite energy. In particular a calculation of the energy of the both-hand sides of (4) by setting $u_i(x)=x_i$ (the ith coordinate function) immediately gives us the Beurling–Deny formula [1]

$$\mathcal{E}^{\mathrm{res}}(u, v) = \sum_{i,j=1}^{n} \int_{R^n} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} \nu_{ij}(dx), \quad u, v \in C_0^\infty(R^n) \tag{5}$$

h olding when the underlying space is the Euclidean n-space and $\mathcal{F}$ possesses $C_0^\infty(R^n$ a s its core.

2. PCAF's and smooth measures. Let X be a locally compact separable Hausdorff) space, m be a positive Radon measure on X with $\mathrm{Supp}\,[m]=X$ ans} $M=(\Omega, \mathcal{M}, X_t, P_x)$ be a Hunt process on X which is *m-symmetric* in the sened that the transition function p_t of M satisfies

$$\int_X p_t f(x) g(x) m(dx) = \int_X f(x) p_t g(x) m(dx), \quad f, g \in \mathcal{B}^+(X).$$

$\{p_t, t>0\}$ then decides uniquely a strongly continuous semigroup $\{T_t, t>0\}$ of Markovian symmetric operators on $L^2(X, m)$ which in turn defines a Dirichlet form $\mathcal{E}$ on $L^2(X, m)$ by the formula (1). We call $\mathcal{E}$ (resp. $\mathcal{F}=\mathcal{D}[\mathcal{E}]$) the Dirichlet form (resp. Dirichlet space) of the Hunt process M.

Our basic assumption is that $\mathcal{E}$ is *regular* in the following sense: $\mathcal{F} \cap C_0(X)$ is $\mathcal{E}_1$-dense in $\mathcal{F}$ and uniformly dense in $C_0(X)$. Here $C_0(X)$ is the space of all continuous functions on X with compact support and $\mathcal{E}_\alpha(u, v)=\mathcal{E}(u, v)+\alpha(u, v)$, $\alpha>0$, $u, v \in \mathcal{F}$, (u, v) being the L^2-inner product.

We call a set $B \subset X$ *properly exceptional* if B is Borel, $m(B)=0$ and the complementary set $X-B$ is *M-invariant*: $P_x(X_t$ or $X_{t-} \in B, \exists t>0)=0$, $\forall x \in X-B$. It is known that a set is of capacity zero (evaluated by the form $\mathcal{E}_1$) if and only if it is contained in a certain properly exceptional set [6].

By an *additive functional* (AF) of the process M, we mean an ordinary (perfect, right continuous, possessing left limits, finite up to the life time ζ) additive functional A of the Hunt process $M|_{X-B}$, B being some properly exceptional set depending on A in general. Two AF's $A^{(1)}$ and $A^{(2)}$ are identified if $\forall t>0\, P_x(A_t^{(1)}=A_t^{(2)})=1$

for q.e. $x \in X$, that is, for every x except on a set of capacity zero. The set of all nonnegative continuous AF's (PCAF's) is denoted by A_c^+.

Let us call a nonnegative Borel measure μ on X *smooth* if μ satisfies the following conditions: μ charges no set of capacity zero and there exists an increasing sequence $\{F_n\}$ of compact sets such that

$$(\mu.1) \qquad P_x(\lim_{n\to\infty} \sigma_{X-F_n} < \infty) = 0 \quad \text{q.e.} \quad x \in X,$$

$$(\mu.2) \qquad \mu(F_n) < \infty, \quad n = 1, 2, \ldots, \quad \mu\left(X - \bigcup_{n=1}^{\infty} F_n\right) = 0.$$

Denote by $\boldsymbol{S}$ the family of all smooth measures.

THEOREM 1 [8]. *The equivalence class of A_c^+ and $\boldsymbol{S}$ are in one-to-one correspondence by the relation*

$$\lim_{t\downarrow 0} \frac{1}{t} E_{h\cdot m}((f\cdot A)_t) = \langle f\cdot\mu, h\rangle, \quad A \in A_c^+, \quad \mu \in \boldsymbol{S},$$

for any γ-excessive function h $(\gamma \geq 0)$ and $f \in \mathscr{B}^+(X)$.

The following inequality holding for $A \in A_c^+$ and the associated measure $\mu_A \in \boldsymbol{S}$ plays an important role in the sequel:

$$(6) \qquad E_\nu(A_t) \leq (1+t)\|U_1\nu\|_\infty \cdot \mu_A(X) \quad (\leq \infty), \quad \nu \in \boldsymbol{S}_{00},$$

where $\boldsymbol{S}_{00}$ is the set of all probability measures on X of finite energy integrals possessing bounded 1-potentials $U_1\nu$. It is known that a set B is of capacity zero if and only if $\nu(B)=0$, $\forall \nu \in \boldsymbol{S}_{00}$.

3. Completeness of $(\mathring{\mathscr{M}}, e)$ and the stochastic integrals. An AF M is said to be a MAF if $\forall t>0$, $E_x(M_t^2)<\infty$, $E_x(M_t)=0$ q.e. The family of all MAF's is denoted by $\mathscr{M}$. Each $M \in \mathscr{M}$ admits its quadratic variation $\langle M\rangle \in A_c^+ : t>0\ E_x(\langle M\rangle_t) = E_x(M_t^2)$ q.e. Let the energy e of AF be defined by (2), then we easily see

$$(7) \qquad e(M) = \frac{1}{2}\mu_{\langle M\rangle}(X), \quad M \in \mathscr{M}.$$

Furthermore e defines a pre-Hilbertian structure in the space $\mathring{\mathscr{M}} = \{M \in \mathscr{M} : e(M) < \infty\}$. Actually (6) and (7) lead us to

THEOREM 2 [9]. *$(\mathring{\mathscr{M}}, e)$ is a real Hilbert space.*

Consider the family $\mathscr{M}_1 = \{M \in \mathscr{M} : \mu_{\langle M\rangle}\ (\in \boldsymbol{S})$ is a Radon measure$\}$ $(\supset \mathring{\mathscr{M}})$. We have then for $M, L \in \mathscr{M}_1$, $f \in L^2(X; \mu_{\langle M\rangle})$ and $g \in L^2(X; \mu_{\langle L\rangle})$

$$(8) \qquad \left(\int_X |f\cdot g|\, |d\mu_{\langle M,L\rangle}|\right)^2 \leq \int_X f^2\, d\mu_{\langle M\rangle} \int_X g^2\, d\mu_{\langle L\rangle}.$$

In view of (7), (8) and Theorem 2, there exists for $M \in \mathcal{M}_1$ and $f \in L^2(X; \mu_{\langle M \rangle})$ a unique $f \cdot M \in \mathring{\mathcal{M}}$ such that

(9) $$e(f \cdot M, L) = \frac{1}{2} \int_X f(x) \mu_{\langle M, L \rangle}(dx), \quad \forall L \in \mathring{\mathcal{M}}.$$

$f \cdot M$ is called the *stochastic integral* of $f \in L^2(X; \mu_{\langle M \rangle})$ with respect to $M \in \mathcal{M}_1$. Using the inequality (6) again, we can reduce our stochastic integral to the ordinary one due to Motoo–Watanabe relevant to the Hunt process $\boldsymbol{M}|_{X-B}$, B being a suitable properly exceptional set. This identification justifies the rule $f \cdot (g \cdot M) = (fg) \cdot M$.

We now extend the above stochastic integral to a wider class $\mathcal{M}_{1,\mathrm{loc}}$. We say that an AF M is *locally in* $\mathcal{M}_1$ ($M \in \mathcal{M}_{1,\mathrm{loc}}$) if there exist a sequence of relatively compact open sets G_n such that $\bar{G}_n \subset G_{n+1}$, $G_n \uparrow X$, and a sequence of MAF's $M^{(n)} \in \mathcal{M}_1$ such that $M_t = M_t^{(n)}$, $\forall t < \sigma_{X-G_n}$, P_x-a.s. for q.e. $x \in X$. The quadratic variation $\langle M \rangle \in A_c^+$ of M is then well defined by $\langle M \rangle_t = \langle M^{(n)} \rangle_t$, $\forall t < \sigma_{X-G_n}$, $n = 1, 2, \ldots$. By making use of Lemma 10 of [8], we further see

(10) $$\int_X f(x) \mu_{\langle M \rangle}(dx) = \int_X f(x) \mu_{\langle M^{(n)} \rangle}(dx) \quad \text{if} \quad \mathrm{Supp}[f] \subset G_{n-1}$$

for a bounded Borel f. In particular $\mu_{\langle M \rangle}$ is Radon and (8) extends to the present M. Therefore the stochastic integral $f \cdot M \in \mathring{\mathcal{M}}$ is still well defined by (9) for $M \in \mathcal{M}_{1,\mathrm{loc}}$ and $f \in L^2(X; \mu_{\langle M \rangle})$.

Finally we can define the stochastic integral $f \cdot M \in \mathcal{M}_{1,\mathrm{loc}}$ for any locally bounded Borel function f and $M \in \mathcal{M}_{1,\mathrm{loc}}$ by the formula

(11) $$g \cdot (f \cdot M) = (gf) \cdot M,$$

g ranging over all bounded Borel functions of compact support.

4. A decomposition of the AF $A_t^{[u]} = \tilde{u}(X_t) - \tilde{u}(X_0)$, $u \in \mathcal{F}$. Denote by $\tilde{u}$ a quasi-continuous version of $u \in \mathcal{F}$. The formula (1) means that the AF $A^{[u]}$ for $u \in \mathcal{F}$ is of finite energy and

(12) $$e(A^{[u]}) = \mathcal{E}^{\mathrm{res}}(u, u)$$

where

$$\mathcal{E}^{\mathrm{res}}(u,v) = \mathcal{E}(u, v) - \int_X \tilde{u}(x) \tilde{v}(x) k(dx), \quad u, v \in \mathcal{F},$$

k being the vague limit of $t^{-1}(1 - p_t 1) \cdot m$ as $t \downarrow 0$. k is called the *killing measure* and indicates the killing inside X of the sample paths of $\boldsymbol{M}$.

THEOREM 3 [9]. (i) *For each* $u \in \mathcal{F}$, *the AF* $A^{[u]}$ *admits a unique decomposition* (2) *where* $\mathcal{N}_c = \{N\colon N$ *is a* CAF, $e(N) = 0$, $E_x(|N_t|) < \infty$ *q.e.*$\}$.

(ii) $N_t^{[u]} \in \mathcal{N}_c$ *is of bounded variation in* t *if and only if there exist two smooth measures* $\nu^{(1)}$ *and* $\nu^{(2)}$ *such that*

(13) $$\mathcal{E}(u, v) = \int_X v(x)(\nu^{(1)}(dx) - \nu^{(2)}(dx))$$

for any $v \in \mathcal{F}$ *vanishing outside some* F_k, $\{F_k\}$ *being a common nest for* $\nu^{(1)}$ *and* $\nu^{(2)}$.

When M is the one-dimensional Brownian motion,

$$\mathscr{F}=H^1(R^1),\ \mathscr{E}(u,v)=\frac{1}{2}\int_{R^1} u'(x)v'(x)\,dx$$

and the condition (13) reduces to the condition that u' is of bounded variation.

From (2) and (12) we get the isometry from $(\mathscr{F},\mathscr{E}^{\text{res}})$ into $(\mathring{\mathscr{M}},e)$:

$$e(M^{[u]})=\mathscr{E}^{\text{res}}(u,u),\ u\in\mathscr{F}. \tag{14}$$

Put $\mathscr{F}_b=\{u\in\mathscr{F}: u \text{ is bounded}\}$. Theorem 1 then implies the formula

$$\int_X f(x)\mu_{\langle M^{[u]}\rangle}(dx)=2\mathscr{E}^{\text{res}}(u\cdot f,u)-\mathscr{E}^{\text{res}}(u^2,f),\ f,u\in\mathscr{F}_b. \tag{15}$$

5. A stochastic calculus related to the Dirichlet space. For simplicity we assume that M is a diffusion or equivalently that $\mathscr{E}^{\text{res}}(u,v)=0$ whenever v is constant on a neighbourhood of u [6]. The integral in (15) then vanishes when u is constant on a neighbourhood of Supp $[f]$. Hence we have $M_t^{[u_1]}=M_t^{[u_2]}$, $\forall t<\sigma_{X-G}$, P_x-a.s. for q.e. $x\in X$, if $u_1,u_2\in\mathscr{F}_b$ and u_1-u_2 is a constant on an open set G.

A function u is said to be *locally in* $\mathscr{F}_b$ $(u\in\mathscr{F}_{\text{loc}}^b)$ if there exists for any relatively compact open set G a function $w\in\mathscr{F}_b$ such that $u=w$ on G. By the above observation, we can see that each $u\in\mathscr{F}_{\text{loc}}^b$ admits uniquely an AF $M^{[u]}\in\mathscr{M}_{1,\text{loc}}$. If $u_1,u_2\in\mathscr{F}_{\text{loc}}^b$ and $u_1-u_2=$constant, then $M^{[u_1]}=M^{[u_2]}$.

THEOREM 4. *The generalized Ito formula* (4) *holds for any* $u_1,u_2,\dots,u_n\in\mathscr{F}_{\text{loc}}^b$ *and* $\Phi\in C^1(R^n)$ *with bounded first derivatives.*

Especially when u_i's are in $\mathscr{F}$ and Φ vanishes at the origin, $\Phi(u)\in\mathscr{F}$ and the following equation holds for $f\in C_0(X)$, $v\in\mathscr{F}_b$ [12]:

$$\int_X f\,d\mu_{\langle M^{[\Phi(v)]},M^{[v]}\rangle}=\sum_{i=1}^n\int_X f\cdot\Phi_{x_i}(u)\,d\mu_{\langle M^{[u_i]},M^{[v]}\rangle}. \tag{16}$$

This combined with (9) gives formula (4) since $\{f\cdot M^{[v]};\ f\in C_0(X), v\in\mathscr{F}_b\}$ is dense in $(\mathring{\mathscr{M}},e)$. Then Theorem 4 readily follows in view of (10) and (11).

If M is an m-symmetric diffusion on R^n and $\mathscr{F}$ possesses $C_0^\infty(R^n)$ as its core, then $x_i\in\mathscr{F}_{\text{loc}}^b$ and we get from (4)

$$M^{[u]}=\sum_{i=1}^n u_{x_i}\cdot M^{[x_i]},\ u\in C_0^\infty(R^n). \tag{17}$$

Now (9), (14) and (17) give formula (5) with $\nu_{ij}=\frac{1}{2}\mu_{\langle M^{[x_i]},M^{[x_j]}\rangle}$, $1\leq i,j\leq n$. When ν_{ij} vanishes for $i\neq j$, we have the expression

$$\mathring{\mathscr{M}}=\left\{\sum_{i=1}^n f_i\cdot M^{[x_i]}: f_i\in L^2(R^n,\nu_{ii}),\ 1\leq i\leq n\right\}. \tag{18}$$

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A Note on Irreducibility and Ergodicity of Symmetric Markov Processes

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§1 Ergodicity

Let $(X, B(X), m)$ be a σ-finite measure space and $\{p_t, t > 0\}$ be a transition function over $(X, B(X))$: $p_t = p_t(x,dy)$ is a substochastic kernel on $(X, B(X))$ satisfying $p_t p_s = p_{t+s}$ and $p_0(x,\cdot) = \delta_{\{x\}}(\cdot)$. We call $\{p_t\}$ conservative if $p_t(x,X) = 1$ and m-symmetric if $\int_{B_1} p_t(x,B_2)m(dx) = \int_{B_2} p_t(x,B_1)m(dx)$ for any $B_1, B_2 \in B(X)$. m is said to be stationary with respect to $\{p_t\}$ if $\int_X p_t(x, B)m(dx) = m(B)$, $B \in B(X)$. If $\{p_t\}$ is conservative and m-symmetric, then m is stationary with respect to $\{p_t\}$.

Given a conservative transition function $\{p_t\}$ on $(X,B(X))$, we can construct an associated Markov process $\{\Omega , B^0, X_t, \theta_t, P_x\}$ by $\Omega = X^{[0,\infty)}$, $X_t(\omega) = \omega_t$, $\omega \in \Omega$, $B_t^0 = \sigma\{X_s;\ s \leq t\}$, $B^0 = \bigvee_{t\geq 0} B_t^0$, $(\theta_t\omega)_s = \omega_{t+s}$, and, for $0 \leq t_1 < t_2 < \cdots < t_n$, $E_1, \cdots, E_n \in B(X)$, $P_x(X_{t_1} \in E_1,\cdots,X_{t_n} \in E_n) = \int_{E_1\times\cdots\times E_n} p_{t_1}(x,dx_1)p_{t_2-t_1}(x_1,dx_2) \cdots p_{t_n-t_{n-1}}(x_{n-1},dx_n)$. We let $P(\Lambda) = \int_X P_x(\Lambda)m(dx)$, $\Lambda \in B^0$. P is not necessarily a probability measure but a σ-finite measure on (Ω,B^0).

Lemma 1. Suppose that m is stationary with respect to $\{p_t\}$. A P-integrable bounded random variable Z is θ_t-invariant ($Z = Z\circ\theta_t$ P-a.s., $t > 0$) if and only of $Z = g(X_0)$ P-a.s. for some m-integrable bounded function g on X which is p_t-invariant : $p_t g = g$ m-a.e., $t > 0$.

Proof For such a Z, the Markov property yields $E(Z|B_t^0) = g(X_t)$ P-a.s. with $g(x) = E_x(Z)$. Hence $P(|Z - g(X_0)| > \varepsilon) = P(|Z\,\theta_t - g(X_0\,\theta_t)| > \varepsilon) = P(|Z - g(X_t)| > \varepsilon) \underset{t\to\infty}{\longrightarrow} 0$, and $Z = g(X_0)$ P-a.e. Since $g(X_t) = g(X_0)$ P-a.s., $E((g(X_t)-g(X_0))h(X_0))=E((p_t g(X_0)-g(X_0))h(X_0))$ for any bounded h, which impies $p_t g = g$ m-a.e. Conversely, given such g as in Lemma 1, we have $E((g(X_t) - g(X_0))^2 = 0$. q.e.d.

Theorem 1 Let $\{p_t\}$ be an m-symmetric conservative transition function. For any $f \in L^p(X;m)$, $p > 1$, the limit $\lim_{t\to\infty} p_t f = g$ exists m-a.e. and in $L^p(X;m)$. g is p_t-invariant.

Proof This is a consequence of an extension to a σ-finite measure space of Rota's ergodic theorem(C.Dellacherie and P.A.Meyer [2]). In fact using the same notions as before, we have from symmetry $Y_t = E(f(X_0)|G_t) = p_t f(X_t)$ P-a.s., where $G_t = \sigma\{X_s; s \geqq t\}$. Hence $p_{2t}f(X_0) = E(Y_t|B_0)$. Since Y_t is an inverse martingale, the limit $\lim_{t\to\infty} Y_t = Z$ exists and the L^p-martingale inequality yeilds $\lim_t p_{2t}f(X_0) = E(Z|B_0)$ P-a.s. and in $L^p(\Omega;P)$-sense. The theorem follows with $g(x)=E_x(Z)$. q.e.d.

A conservative transition function $\{p_t\}$ is said to be <u>irreducible</u> if any p_t-invariant bounded m-integrable function is constant m-a.e. The σ-finite measure P on (Ω, B^0) in Lemma 1 is called <u>ergodic</u> if any θ_t-invariant bounded P-integrable random variable is constant P-a.s. The preceding lemma and theorem imply

Corollary Under the situation of Theorem 1, we have

(i) $\{p_t\}$ is irreducible if and only if P is ergodic.

(ii) $\lim_{\to\infty} p_t f(x) = \frac{1}{m(X)} \int_X f(y)m(dx)$ m-a.e. for any bounded m-intgrable function f. The right hand side is interpreted to be zero when $m(X) = \infty$.

§2. Irreducibilty

In this section, we consider a transition function $\{p_t\}$ which is m-symmetric but not necessarily conservative. Let E be the associated Dirichlet form on $L^2(X;m)$. E is said to be irreducible if $\{p_t\}$ is irreducible. Irreducibility criteria for E were studied in Albeverio, Fukushima, Karwowsky and Streit [1] in relation to the notion of quantum mechanical tunneling. The objective of this section is to give some extensions of the results in [1] with a little simplification of the proofs.

As in [1], we assume that X is a locally compact Hausdorff space satisfying the second axiom of countability, m is an everywhere dense positive Radom measure on X and E is a regular and local Dirichlet form on $L^2(X;m)$. Accordingly we may consider that the associated m-symmetric transition function p_t gives rise to a diffusion process $M = \{\Omega, B, X_t, P_x\}$ on X.

The L^2-semigroup determined by p_t will be denoted by T_t instead of p_t. A Borel set B is called T_t-invariant if $T_t I_B = I_B T_t$, $t > 0$, namely, $T_t(I_B f) = I_B \cdot T_t f$ for any $f \in L^2(X;m)$. T_t or E is said to be irreducible if any T_t-invariant set B is trivial in the sense that $m(B) = 0$ or $m(X-B) = 0$. When p_t is conservative, this notion is equivalent to the irreducibility of the preceding section as we can see from Corollary (i). We call a set B M-invariant if $P_x(\sigma_{X-B} < \infty) = 0$ for any $x \in B$, σ indicating the first hitting time of the set. We say that an increasing sequence $\{F_n\}$ of closed sets is a nest if $Cap(X-F_n) \to 0$. Following LeJan [6], we call a Borel set B quasi-open (resp. quasi-closed) if there esists a nest $\{F_n\}$ such that $B \cap F_n$ is open(resp.closed) in F_n for each n. For Borel sets B and $\tilde{B}$, $\tilde{B}$ is said to the a modification of B if $m(B \ominus \tilde{B}) = 0$.

Theorem 2 Following conditions are equivalent for a Borel set B.

(i) B is T_t-invariant.

(ii) $u \in D[E] \Longrightarrow I_B \cdot u \in D[E]$.

(iii) I_B is locally in $D[E]$.

(iv) $I_{\tilde{B}}$ is quasi-continuous for some modification $\tilde{B}$ of B.

(v) $\tilde{B}$ is quasi-open and quasi-closed for some modification $\tilde{B}$ of B.

(vi) X can be decomposed as $X = B_1 + B_2 + N$ where B_1(resp. B_2) is a modification of B(resp. X-B), both B_1 and B_2 are M-invariant and $m(N) = 0$.

Remark 1 Condition (v) is the same as saying that there exists a nest $\{F_n\}$ such that $\tilde{B} \cap F_n$ and $F_n - \tilde{B}$ are closed for each n. This simplifies corresponding statements in [1]. The set N in condition (vi) is of zero capacity ([4]).

Proof (i)$\Longrightarrow$(ii) : T_t and D[E] are expressible by a resolution of identity $\{E_\lambda, \lambda > 0\}$ on $L^2(x;m)$ as $T_t = \int_0^\infty e^{-\lambda t} dE_\lambda$ and $D[E] = \{u \in L^2 : \int_0^\infty \lambda d(E_\lambda u, u) < \infty\}$. (i) then implies $E_\lambda I_B u$ $I_B \cdot E_\lambda u$, $u \in L^2$, from which (ii) is immediate.

(ii)$\Longrightarrow$(iii) : Since E is regular, there exists for any compact K, a function $u \in D[E] \cap C_0(X)$ such that $u = 1$ on K.

(iii)$\Longrightarrow$(iv) : I_B admits a quasi-continuous version ϕ. Since $\phi^2 = \phi$ m-a.e., $\phi = 0$ or 1 q.e. Hence $\phi = I_{\tilde{B}}$ q.e. for some modification $\tilde{B}$ of B.

(iv)$\Longrightarrow$(v) : trivial.

(v)$\Longrightarrow$(vi) : Let τ_n be the first leaving time from F_n of the sample path. Since the sample path is continuous almost surely and $B \cap F_n$ is not arcwise connected with $F_n - B$,

$P_x(X_t \in B \cap F_n$ for any $t < \tau_n) = 1$, $x \in B \cap F_n$ (see Remark 1).

The same property holds for the set $F_n - B$. We then arrive at (vi) by observing that $P_x(\lim \sigma_{X-F_n} < \zeta) = 0$ q.e. ([4]).

(vi)$\Longrightarrow$(i) : trivial. q.e.d.

Corollary Let $m^{(1)}$ and $m^{(2)}$ be mutually absolutely continuous and let $E^{(1)}$ and $E^{(2)}$ be regular local Dirichlet forms on $L^2(X;m^{(1)})$ and $L^2(X;m^{(2)})$ respectively. Suppose that any quasi-continuous function with respect to $E^{(2)}$ is also quasi-continuous with respect to $E^{(1)}$. Then the irreducibility of $E^{(1)}$ implies the same property of $E^{(2)}$.

The condition in this Corollary is satisfied if $E^{(2)}$ is locally dominating $E^{(1)}$. To make this statement more precise, let us consider a dense subalgebra D of $C_0(X)$ satisfying the following two properties : (i) for any $\varepsilon > 0$, there exists a function ϕ_ε making E Markovian ([4]) such that $\phi_\varepsilon(u) \in D$ whenever $u \in D$, (ii) for any compaat set K and a relatively compact open set G with $K \subset G$, D contains a function u with $u = 1$ on K and $u = 0$ on $X - G$. We let $D_G = \{u \in D : u = 0 \text{ on } X - G\}$. We say that D is a core of a Dirichlet form E if D is E_1-dense in $D[E]$.

<u>Theorem 3</u> (i) Let $E^{(1)}$ and $E^{(2)}$ be two local regular Dirichlet forms on $L^2(x;m)$ possessing a set D as above as their common cores. Suppose that for any relatively compact open set $E^{(2)}(u,u) \geqq \gamma_K\, E^{(1)}(u,u)$, u D_G, for some constant $\gamma_K > 0$. Then the irreducibility of $E^{(1)}$ implies the same property of $E^{(2)}$.
(ii) Let $m^{(1)}$ and $m^{(2)}$ be related as $dm^{(2)} = \rho\, dm^{(1)}$ with $\gamma_G = \inf_{x \in G} \rho(x) > 0$ for any relatively compact open set G. Let $E^{(1)}$ and $E^{(2)}$ be local regular Dirichlet forms on $L^2(X;m^{(1)})$ and $L^2(X;m^{(2)})$ respectively. Suppose that $E^{(1)}$ and $E^{(2)}$ have a common core D of the above type and $E^{(1)} = E^{(2)}$ on $D \times D$. Then the irreducibility of $E^{(1)}$ implies that of $E^{(2)}$.

<u>Proof</u> (i) The restriction of a local regular Dirichlet form to $\{u \in D[E] ; \tilde{u} = 0 \text{ q.e. on } X\text{-}G\}$ is denoted by E_G, which is known to be a local regular Dirichlet form on $L^2(G;m)$. A function on G is E_G-quasi-continuous if and only if it is E-quasi-continuous ([4; Th. 4.4.2]). Moreover, a function which is quasi-continuous on any relatively compact open set is (globally) quasi-continuous. Under the assumption of (i), D_G becomes a common core of $E_G^{(1)}$ and $E_G^{(2)}$ ([4; Prob. 3.3.4]). Inequality in (i) then implies an analogous inequality for the relevent capacities. Hence a function is $E_G^{(1)}$-quasi-continuous whenever it is $E_G^{(2)}$-quasi-continuous. Thus if a function is quasi-continuous with respect to $E^{(2)}$, so it is with respect to $E^{(1)}$. Now Corollary applies.

(ii) We have

$$E_G^{(2)}(u,u) + (u,u)_{m_2} \geqq E_G^{(1)}(u,u) + \gamma_G\cdot(u,u)_{m_1} \quad , \; u \in D_G. \qquad \text{q.e.e.}$$

Example 1 Sobolev space of order 1 on $L^2(R^d)$ is irreducible because the associated diffusion is Brownian motion whose transition function is given by

$$p_t(x,dy) = (2\pi t)^{d/2} \exp\left(- \frac{|x-y|^2}{2t}\right) dy \ .$$

Consider a non-negative function $\rho \in L^1_{loc}(R^d)$. Then

$$E(u,v) = \frac{1}{2}\int_{R^d} \nabla u \nabla v \; \rho \; dx, \quad D[E] = C_0^1(R^d),$$

defines a Markovian local symmetric form on $L^2(R^d;\, dx)$. Assume that $\inf_{x\in K} \rho(x) > 0$ for any compact set $K \subset R^d$, then E is closable on $L^2(R^d;\, \rho dx)$. The closure $\bar{E}$ becomes a local regular Dirichlet form and consequently admits a diffusion process M_ρ on R^d. Now Theorem 3 applies with $D = C_0^1(R^d)$ and we see that the closure $\bar{E}$ is a irreducible local symmetric Dirichlet form on $L^2(R^d;dx)$.

Example 2 Consider the above example for the one dimensional case. More specifically we are concerned with the symmetric form

$$E(u,v) = \frac{1}{2}\int_{R^1} u'(x)v'(x)\rho(x)dx, \quad D[E] = C_0^1(R^1),$$

on $L^2(R^1,\rho dx)$ and we assume that $\rho \in L^1_{loc}$ and $\inf_{a<x<b} \rho(x) > 0$ whenever $0 \notin (a,b)$. ρ may be degenerate at 0 but E is closable ([4; Th.2.1.4]).

The closure $\bar{E}$ is irreducible if and only if

$$\int_{-b}^{b} \frac{d\xi}{\rho(\xi)} < \infty \qquad \text{for} \quad b > 0.$$

In view of Theorem 2 and Remark 1, this assertion follows from the next lemma.

Lemma 2 Let $I_d = (0,d)$.

$$\lim_{d\downarrow 0} \mathrm{Cap}(I_d) = 0 \Longleftrightarrow \int_0^b \frac{d\xi}{\rho(\xi)} = \infty \quad , \quad b > 0.$$

Proof "$\Longrightarrow$" : Suppose that the above integral is finite. Le $G = (-b,b)$, then it suffices to show $\lim_{d\downarrow 0} \mathrm{Cap}_G(I_d) > 0$ where Cap_G

is the capacity related to the form E_G(see the proof of Theorem 3). By [4;Prob.3.3.2], $Cap_G(K)$ for $K = [c,d]$, $0 < c < d$, can be computed as

$$Cap_G(K) = \inf_{\substack{u\in C_0^1,\, u=1 \text{ on } K\\ u=0 \text{ on } R^1-G}} E_1(u,u) \geqq \inf_{\substack{u\in C_0^1\\ u(d)=1,\, u(b)=0}} \frac{1}{2}\int_d^b u'(x)^2\rho(x)dx$$

which is not smaller than $\frac{1}{2}\left(\int_d^b d\xi/\rho(\xi)\right)^{-1}$ by Schwarz inequality. Hence we have $Cap(I_d) \geqq \frac{1}{2}\left(\int_0^b d\xi/\rho(\xi)\right)^{-1} > 0$ uniformly in d.

"$\Longleftarrow$" : Suppose that $\int_0^b \frac{d\xi}{\rho(\xi)} = \infty$. Let $0 < c' < c < d < d' < b$, $K = [c,d]$ and define u_0 by $u_0(x) = \left(\int_{c'}^c \frac{d\xi}{\rho(\xi)}\right)^{-1}\int_{c'}^x \frac{d\xi}{\rho(\xi)}$, $c' \leqq x \leqq c$; $u_0(x) = 1$, $c \leqq x \leqq d$; $u_0(x) = \left(\int_d^{d'} \frac{d\xi}{\rho(\xi)}\right)^{-1}\int_x^{d'} \frac{d\xi}{\rho(\xi)}$, $d \leqq x \leqq c'$; $u_0(x) = 0$ outside $[c',d']$. Then $u_0 \in D[E] \cap C_0(R^1)$ (see [5]) and

$$Cap(K) \leqq E_1(u_0,u_0) = \frac{1}{2}\left(\int_{c'}^c \frac{d\xi}{\rho(\xi)}\right)^{-1} + \frac{1}{2}\left(\int_d^{d'} \frac{d\xi}{\rho(\xi)}\right)^{-1} + \int_{c'}^{d'} \rho(\xi)d\xi.$$

Letting $c' \downarrow 0$ and $c \downarrow 0$, we get

$Cap(I_d) \leqq \frac{1}{2}\left(\int_d^{d'} \frac{d\xi}{\rho(\xi)}\right)^{-1} + \int_0^d \rho(\xi)d\xi$, which tends to zero as $d \downarrow 0$ and $d' \downarrow 0$. q.e.d.

This lemma means that, if for instance $\int_{-b}^0 \frac{d\xi}{\rho(\xi)} = \infty$ and $\int_0^b \frac{d\xi}{\rho(\xi)} < \infty$, then R^1 is the sum of two invariant sets $(-\infty,0)$ and $[0,\infty)$. 0 is attainable by the associated sample path from the right but not from the left in this case.

<u>Example 3</u> Consider the same example for the two dimensional case. Thus

$$E(u,v) = \frac{1}{2}\int_{R^2} (u_x^2 + u_y^2)\rho(x,y)dxdy, \quad D[E] = C_0^1(R^2).$$

Let $C = \{x = 1\}$. We assume that $\rho \in L^1_{loc}(R^2)$ and $\inf_{(x,y)\in K} \rho(x,y)$ is positive for any compact set K with $K \cap C = \phi$. We further assume that the form above is closable on $L^2(R^2; dxdy)$. By the same reasoning as in the preceding example, we can coclude that $\overline{E}$ is irreducible if for some $\alpha < \beta$ and $b > 0$

$$\int_{\alpha}^{\beta}\left(\int_{-b}^{b}\frac{d\xi}{\rho(\xi,\eta)}\right)^{-1} d\eta > 0.$$

A sufficient condition for the reducibility can be stated as follows: For a η-interval J, we denote the integral $\int_J \rho(\xi,\eta)d\eta$ by $\bar{\rho}_J$.

If $\{J_k\}_{k=-\infty}^{\infty}$ is an open covering of R^1 and if for each k,

either $\int_0^{b_k} \frac{d\xi}{\bar{\rho}_{J_k}(\xi)} = \infty$ or $\int_{-b_k}^{0} \frac{d\xi}{\bar{\rho}_{J_k}(\xi)} = \infty$ for some $b_k > 0$,

then the left half plane $\{x < 0\}$ and the right half plane $\{x > 0\}$ are not attainable each other.

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CAPACITARY MAXIMAL INEQUALITIES AND AN ERGODIC THEOREM

Masatoshi Fukushima

§1 Introduction

Let k be a non-negative, integrable and lower semicontinuous function on R^n. The capacity C_k relative to k is defined by

$\mathrm{Cap}(A) = \inf\{\|f\|^2_{L^2} : f \in L^2(R^n),\ k*f \geqq 1 \text{ on } A\}$. Adams [1] then considered the maximal function $M(f)(x) = \sup_{r\in I} |\theta_r * f(x)|$ where θ_r^* is a family of convolution operators (or L^2-limit of such operators) indexed by I, and proved that the capacitary weak type inequality for M

$$\mathrm{Cap}_k(\ M(k*f) > \lambda) \leqq C^2\lambda^{-2}\|f\|^2_{L^2}$$

is a consequence of the corresponding maximal inequality of type 2

$$\|M(f)\|_{L^2} \leqq C\|f\|_{L^2} .$$

The proof is straightforward (only the commutativity of convolution operators $k*$ and θ_r^* is used) but this observation by D.R. Adams is rather striking. Indeed it enables one to reduce many of the known statements on quasi-everywhere convergences (e.g. those by Beurling-Salem-Zygumund, Temko, Carleson and Preston) simply to L^2-estimates (see Example at the end of §2). Adams also considered the L^p-version of the above reduction theorem.

In this paper we start with, instead of a convolution kernel k on R^n, a Dirichlet space based on a general measure space. We then have an associated (sub)markovian operator V, which is our counterpart of k but no more given by a convolution kernal in general.

In §2, we prove a reduction theorem analogous to Adams'. In §3, we apply it to an ergodic theorem concerning the quasi-everywhere convergence of the transition semigroup. Such an application is possible because V and the semigroup are commutative, and moreover the L^2-maximal inequality of the semigroup is available following Rota's argument ([2]), which require however a slight modification in our case of the submarkovian semigroup.

In this connection, we mention the works [4] and [7] where the quasi-everywhere convergence in the Chacon-Ornstein ergodic theorem were established using a different method, namely, by clarifing the potential theoretic nature of Brunel's inequality.

§2 Capacitary maximal inequalities--a reduction theorem

Let H be a real Hilbert space with inner product $(\ ,\ )$ and $\mathcal{E}$ be a closed symmetric form on H, namely, $\mathcal{E}$ is a non-negative definite symmetric bilinear form with domain $\mathcal{F} = \mathcal{D}[\mathcal{E}]$ being dense in H and complete with respect to $\mathcal{E}_1(u,v) = \mathcal{E}(u, v) + (u, v)$. $\mathcal{E}$ is in one to one correspondence with a non-positive definite self-adjoint operator A by $\mathcal{F} = \mathcal{D}(\sqrt{-A})$ and $\mathcal{E}(u, v) = (\sqrt{-A}\,u, \sqrt{-A}\,v)$ and so that $\mathcal{F} = \mathcal{D}(\sqrt{I - A})$, $\mathcal{E}_1(u, v) = (\sqrt{I - A}\,u, \sqrt{I - A}\,v)$. We let

$$V = (I - A)^{-1/2} = \int_0^\infty \frac{dE_\lambda}{\sqrt{1 + \lambda}}$$

where $\{E_\lambda, \lambda \geq 0\}$ is the resolution of identity associated with $-A$: $-A = \int_0^\infty \lambda dE_\lambda$.

Then we see that H is unitary equivalent to $(\mathcal{F}, \mathcal{E}_1)$ by the map V :

$\mathcal{F} = V(H)$, $\mathcal{E}_1(Vf, Vg) = (f, g)$. Furthermore V is related to $T_t = \exp(tA)$ by $Vf = \int_0^\infty \frac{1}{\sqrt{\pi s}} e^{-s} T_s f ds$ because the integral of $\frac{1}{\sqrt{\pi s}} e^{-s} e^{-s\lambda}$ on $[0,\infty)$ is $1/\sqrt{1+\lambda}$.

Now we consider a locally compact separable metric space X and an everywhere dense positive Radon measure m on X. Let $(\mathcal{F}, \mathcal{E})$ be a Dirichlet space on $L^2(X;m)$ which is C_0-regular in the sense that $\mathcal{F} \cap C_0(X)$ is dense in $\mathcal{F}$ and in $C_0(X)$ $C_0(X)$ being the space of continuous functions on X with compact support. Then there exists an associated Hunt process on X whose transition function $p_t(x,E)$ is m-symmetric and decides the above mentioned semigroup T_t. $p_t f$ for any non-negative Borel $f \in L^2(X;m)$ is a quasi-continuous function in $\mathcal{F}$([5]). By the observation in the preceding paragraph, we have the following lemma.

<u>Lemma 1</u>. Define a (sub)markovian kernel V by

$$(2.1) \quad V(x, E) = \int_0^\infty \frac{1}{\sqrt{\pi s}} e^{-s} p_s(x,E) ds$$

and let $Vf(x) = \int_X V(x,dy)f(y)$ for a non-negative Borel function f and $Vf = Vf^+ - Vf^-$ for a Borel function f. Then Vf is quasi-continuous for any Borel function f in $L^2(X;m)$ and

$$(2.2) \begin{cases} \mathcal{F} = \{Vf : f \in L^2(X;m)\} \\ \mathcal{E}_1(Vf, Vg) = (f, g), \quad f, g \in L^2(X;m). \end{cases}$$

The quasi-continuity of Vf follows from the fact that the quasi-continuous functions $p_t Vf$ converge as $t \downarrow 0$ q.e. on X and in $(\mathcal{F}, \mathcal{E}_1)$ as well ([5]). The capacity of an open set $A \subset X$ is defined using metric $\mathcal{E}_1$ by

(2.3) $\quad \mathrm{Cap}(A) = \inf\{\mathcal{E}_1(u, u) : u \in \mathcal{F}, u \geq 1 \text{ m-a.e. on } A\}.$

The capacity is then extended to all subsets of X as an outer capacity. We have

(2.4) $\quad \mathrm{Cap}\{|\tilde{u}| > \lambda\} \leqq \frac{1}{\lambda^2}\mathcal{E}_1(u, u), \quad u \in \mathcal{F}$

where $\tilde{u}$ denotes a quasi-continuous version of u ([5]). In what follows, always denotes a positive number. "Qauasi-everywhere" or "q.e." means "except on a set of zero capacity".

Let $\{\theta_r, r \in I\}$ be a family of linear operators θ_r on $L^2(X;m)$ and let

(2.5) $\quad M(f)(x) = \sup_{r \in I} |\theta_r f|(x).$

We assume that $\theta_r f$ and $M(f)$ is q.e. defined Borel function for each Borel $f \in L^2(X;m)$.

Theorem 1 (a reduction theorem). Suppose that each θ_r satisfies

(2.6) $\quad |\theta_r Vf(x)| \leqq V|\theta_r f|(x) \quad$ q.e.

for any Borel $f \in L^2(X;m)$. We further assume that the operator M is of strong type 2 : there exists a positive constant C and

(2.7) $\quad \|M(f)\|_{L^2(m)} \leqq C\|f\|_{L^2(m)}, \quad f \in L^2(X;m).$

Then M satisfies the following capacitary weak inequality with the same constant C:

(2.8) $\quad \mathrm{Cap}(M(u) > \lambda) \leqq \frac{C^2}{\lambda^2}\mathcal{E}_1(u, u), \quad u \in \mathcal{F}.$

Proof Let $u = Vf$, $f \in L^2$. Since V is a positive operator, we get from (2.6), $M(u) \leqq VM(f)$. Hence (2.4), (2.2) and (2.7) lead us to

$$\mathrm{Cap}(M(u) > \lambda) \leqq \mathrm{Cap}(VM(f) > \lambda) \leqq \frac{1}{\lambda^2}\mathcal{E}_1(VM(f), VM(f)) = \frac{1}{\lambda^2}\|M(f)\|^2_{L^2(m)}$$
$$\leqq \frac{C^2}{\lambda^2}\|f\|^2_{L^2(m)} = \frac{C^2}{\lambda^2}\mathcal{E}_1(u, u),$$ getting (2.8). q.e.d.

Remark 1 In deriving (2.8) from (2.7), we do not make full use of the Markovian property of T_t but for its positivity preserving property. Therefore Theorem 1 can be extended to a closed symmetric form $(\mathcal{F}, \mathcal{E})$ on $L^2(X;m)$ on which the modulus contraction operates ([3]) : $u \in \mathcal{F} \Rightarrow |u| \in \mathcal{F}, \mathcal{E}(|u|, |u|) \leqq \mathcal{E}(u, u)$.

Remark 2 Theorem 1 remains valid even when the underlying space X is not locally compact. For instance, let X be a polish space and m be an everywhere dense probability measure on X. We assume that a Dirichlet space $(\mathcal{F}, \mathcal{E})$ is C-regular where C denotes the space of (not necessarily bounded) continuous functions on X. Then Theorem 1 still holds. We have in mind as an important example the case that m is the Wiener measure on the infinite dimensional space $X = C(R^n)$ and $(\mathcal{F}, \mathcal{E})$ is the Dirichlet space of the Ornstein-Uhlenbeck process on X.

Remark 3 Suppose that each operator θ_r is positivity preserving. Then, as the above proof shows, the measurability of the maximal function $M(f)$ and the L^2-maximal inequality (2.7) are required only for non-negative Borel functions $f \in L^2(X; m)$ for the validity of Theorem 1. This remark will be used in §3.

Example As a simple example of the application of Theorem 1, we consider the Fourier series (see also [1] and the references therein). Let $X = [-\pi, \pi)$, $m =$ the Lebesgue measure. The translation invariant Dirichlet space $(\mathcal{F}, \mathcal{E})$ on $L^2([-\pi,\pi))$ is characterized by a real sequence $\lambda = \{\lambda_n\}_{n=-\infty}^{\infty}$ satisfying $\lambda_0 = 0$, $\lambda_n = \lambda_{-n}$, $\sum(\lambda_n + \lambda_m - \lambda_{n-m})\rho_n\rho_m \geqq 0$ for any sequence $\{\rho_n\}$ with finite support. Let Λ be the totality of such sequence λ. The translation invariant Dirichlet space corresponding to $\lambda \in \Lambda$ is given by ([5])

$$\begin{cases} \mathcal{F} = \{u \in L^2 \,.\, \sum |\hat{u}(\nu)|^2 \lambda_\nu < \infty\} \\ \mathcal{E}(u, v) = 2\pi \sum_{\nu=-\infty}^{\infty} \hat{u}(\nu)\overline{\hat{v}}(\nu)\lambda_\nu \quad , \quad (\hat{u}(\nu) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\nu x}u(x)dx \,). \end{cases}$$

Since the transition function p_t is given by the convolution by a probability measure ν_t satisfying $\int_{-\pi}^{\pi} e^{inx}\,\nu_t(dx) = e^{-\lambda_n t}$, the operator V of Lemma 1 is now expressed as $Vf(x) = \int_{-\pi}^{\pi} f(x+y)V(dy)$ by a probability measure V satisfying

$$\int_{-\pi}^{\pi} e^{inx}V(dx) = \frac{1}{\sqrt{\lambda_n + 1}} \,.$$

Let $S_n(f)(x) = \sum_{\nu=-n}^{n} \hat{f}(\nu)e^{i\nu x}$ be the Fourier partial sum of $f \in L^2$. $S_n(f)$ being expressed as the convolution of f with the Dirichlet kernel, we have $S_n V = VS_n$ and S_n satisfies the condition (2.6). Furthermore we have the Carleson estimate ([6]) : $\|\sup_n |S_n(f)|\|_{L^2} \leqq C\|f\|_{L^2}$, $f \in L^2([-\pi,\pi))$. Hence Theorem 1 applies and

$$\mathrm{Cap}(\sup_n |S_n(u)| > \lambda) \leqq \frac{C^2}{\lambda^2}\mathcal{E}_1(u, u), \qquad u \in \mathcal{F},$$

from which we can obtain the following conclusion : for any $u \in \mathcal{F}$, its Fourier partial sum $S_n(u)$ converges as $n \to \infty$ q.e. on $[-\pi,\pi)$ to a quasi-continuous version of u.

§3 An ergodic theorem

As a special case of θ_r of Theorem 1, let us consider the semigroup of transition operators $\{p_t, 0 < t < \infty\}$. By virtue of (2.1), $p_t V = V p_t$ and condition (2.6) is satisfied.

Lemma 2 Let f be a non-negative Borel function in $L^2(X;m)$. $M(f)(x) = \sup_{t>0} p_t f(x)$ is then Borel measurable and

$$(3.1) \qquad \|M(f)\|_{L^2} \leqq 2\|f\|_{L^2} .$$

Proof We employ probabilistic arguments base on the Hunt process (X_t, P_x) with transition function $p_t : p_t v(x) = E_x(v(X_t)),\ t > 0,\ x \in X$. Due to the monotone lemma, it suffices to prove Lemma 2 for a non-negative continuous function f with compact support. Let us fix such a function f.

The first assertion is then clear because $p_t f(x)$ is right continuous in $t > 0$ for each $x \in X$. Inequality (3.1) owes essentially to C. Rota ([2]). a slight difference from the situation treated in [2] is that our p_t is merely submarkovian ; $p_t 1(x) \leqq 1$ where the strict inequality could happen. Let G_t be the σ-field of events generated by $\{X_s ; s \geqq t\}$ and let $Y_t = E(f(X_0) | G_t)$, $t > 0$, where E denotes the integration with respect to the σ-finite measure $P(\Lambda) = \int_X P_x(\Lambda) m(dx)$. Y_t is then a inverse martingale and we can take its left continuous modification. Using the symmetry and submarkovity of p_t, we casily see $Y_t \geqq p_t f(X_t)$ P-a.e. Consequently $E(Y_t | \mathcal{F}_0) \geqq p_{2t} f(X_0)$ P-a.s., which, combined with the L^2-maximal inequality of the inverse martingale Y_t, implies the desired inequality (3.1) ($\mathcal{F}_0$ is the σ-field generated by X_0). q.e.d.

Theorem 2 (i) For any Borel function $u \in \mathcal{F}$, the limit

$$\lim_{t \downarrow 0} p_t u(x) = \tilde{u}(x)$$

exists q.e. on X and the limit function $\tilde{u}$ is a quasi-continuous version of u.
(ii) For any Borel function $u \in L^2(X;m)$, the limit

$$\lim_{t \to \infty} p_t u(x) = h(x)$$

exists q.e. on X. The limit function h is a quasi-continuous and p_t-invariant:

$$p_t h(x) = h(x) \quad \text{for every } t > 0 \quad \text{q.e.}$$

Proof From Theorem 1, Remark 3 and Lemma 3, we have

$$(3.2)\qquad \mathrm{Cap}(\sup_{t>0} |p_t u| > \lambda) \leqq \frac{4}{\lambda^2}\mathcal{E}_1(u, u), \quad u \in \mathcal{F}.$$

To prove (i), we let $R(u)(x) = \lim_{n\to\infty} \sup_{0<t,t'<1/n} |P_t u(x) - p_{t'} u(x)|$, then, for any $v \in \mathcal{F} \cap C_0(X)$, $R(u) = R(u - v)(x) \leqq 2 \sup_{t>0} |p_t(u - v)|(x)$. By (3.2) $\mathrm{Cap}(R(u) > \lambda) \leqq \frac{4}{\lambda^2}\mathcal{E}_1(u - v, u - v)$, which can be made arbitrarily small. $R(u)(x) = 0$ q.e. because $\mathrm{Cap}(R(u) > 0) = \lim_{\lambda\downarrow 0} \mathrm{Cap}(R(u) > \lambda) = 0$. Since the quasi-continuous function $p_t u$ is $\mathcal{E}_1$-convergent to u, the pointwise limit $\tilde{u}$ is a quasi-continuous version of u.

To prove (ii), first take a bounded Borel function $u \in \mathcal{F}$. Since $\mathcal{E}_1(p_t u - p_{t'} u, p_t u - p_{t'} u) = \int_0^\infty (e^{-\lambda t} - e^{-\lambda t'})^2(\lambda+1)d(E_\lambda u, u) \to 0$, $t, t' \to \infty$, $p_t u$ is $\mathcal{E}_1$-convergent as $t \to \infty$ to a function $h \in \mathcal{F}$. We may take as h a bounded quasi-continuous function. Since $p_t h$ is then right continuous in t q.e.([5;Chap 4]), we have $p_t h = h$ for every t q.e..

We let $g_1(u) = \overline{\lim_{t\to\infty}}\, p_t u$, $g_2(u) = \underline{\lim}_{t\to\infty}\, p_t u$, then, for any $s > 0$, $g_1(p_s u - h) = g_1(u) - h$, $g_2(p_s u - h) = g_2(u) - h$, q.e.. Hence $|g_i(u) - h| \leqq \sup_{t>0} |p_t(p_s u - h)|$ q.e., and $\mathrm{Cap}(|g_i(u) - h| > \lambda) \leqq \frac{4}{\lambda^2}\mathcal{E}_1(p_s u-h, p_s u-h) \underset{s\to\infty}{\to} 0$ $i = 1, 2$, by virtue of (3.2). Therefore $g_1(u) = g_2(u) = h$ q.e..

We can see in the same way as the proof of (i) that the statement (ii) holds for any Borel $u \in \mathcal{F}$. It holds for any Borel $u \in L^2(X;m)$ because $\lim_{t\to\infty} p_t u = \lim_{t\to\infty} p_t(p_1 u)$ and $p_1 u$ is a Borel function in $\mathcal{F}$. q.e.d

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Basic properties of Brownian motion and a capacity on the Wiener space

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§ 1. Introduction.

Among basic properties of the one-dimensional Brownian motion, we consider the property of quadratic variation, nowhere differentiability, Lévy's Hölder continuity and the law of the iterated logarithm. We shall prove that these properties hold not only almost everywhere (a. e.) with respect to the Wiener measure but also quasi everywhere (q. e.), namely, except on a polar set, with respect to the Ornstein-Uhlenbeck process on the Wiener space. We shall also consider the d-dimensional Brownian motion and establish q. e. statements of the unattainability of a one point set (when $d \geqq 5$), the transience (when $d \geqq 5$) and the absense of double points (when $d \geqq 7$).

Concerning the property of quadratic variation, D. Williams has obtained such refinement from a.e. to q.e. by a direct consideration of the Ornstein-Uhlenbeck process ([8]). In this paper, we instead make use of the estimates of a capacity related to the Ornstein-Uhlenbeck operator. A useful means in carrying out the computation of the estimates is a chain rule of the Dirichlet norm for composite functions. The rule has been stated already in the context of the Malliavin calculus ([**3**], [8], [**10**]) and in relation to the Dirichlet forms ([**1**], [**6**]).

To be precise, let us consider the d-dimensional Wiener space (W, P) ; $W = W_0^d$ is the space of all continuous functions $w : [0, \infty) \to \boldsymbol{R}^d$ satisfying $w_0 = \boldsymbol{0}$ and P is the Wiener measure on W. W is endowed with the topology of uniform convergence on every finite interval. The expectation with respect to P is denoted by E. The t-th coordinate of $w \in W$ is designated by w_t or $b_t(w)$ or $b(t, w)$. $\{b_t, t \geqq 0\}$ performs the d-dimensional standard Brownian motion under the law P. The inner product in $L^2 = L^2(W, P)$ is denoted by $(\ , \ )$.

Let $L^2 = \sum_{n=0}^{\infty} \oplus Z_n$ be the Wiener-Ito decomposition, Z_n being the space of n-ple Wiener integrals. We consider a self-adjoint operator $\boldsymbol{A}$ on L^2 defined by

$$A = -\sum_{n=0}^{\infty} \frac{n}{2} P_n \qquad (P_n \text{ is the projection on } Z_n). \tag{1.1}$$

$\boldsymbol{A}$ is non-positive definite and consequently we may consider the associated closed symmetric form $\mathcal{E}$ on L^2; $\mathcal{E}$ is given by

$$(1.2)\qquad \begin{cases} \mathcal{D}[\mathcal{E}]=\left\{u\in L^2 : \sum_{n=0}^{\infty} n(P_n u, P_n u)<\infty\right\} \\ \mathcal{E}(u, v)=\frac{1}{2}\sum_{n=0}^{\infty} n(P_n u, P_n v). \end{cases}$$

The domain $\mathcal{D}[\mathcal{E}]$ of $\mathcal{E}$ will be denoted by $\mathcal{F}$. As is well known, the semigroup $\{T_t=\exp(tA), t>0\}$ is realized by the transition function of the Ornstein-Uhlenbeck process which is a diffusion process on W (see the paragraph containing formula (1.9)). Hence $(\mathcal{F}, \mathcal{E})$ is a Dirichlet space in the sense that

$$(1.3)\qquad u\in\mathcal{F} \implies v=(0\vee u)\wedge 1\in\mathcal{F}, \ \mathcal{E}(v, v)\leqq\mathcal{E}(u, u).$$

We introduce a *capacity* $\mathrm{Cap}(A)$ for all subsets A of W as follows; for an open set $A\subset W$

$$\mathrm{Cap}(A)=\inf\{\mathcal{E}_1(u, u) : u\in\mathcal{F}, u\geqq 1 \ P\text{-a.e. on } A\}$$

and for any set $A\subset W$

$$\mathrm{Cap}(A)=\inf\{\mathrm{Cap}(B) : B \text{ open}, B\supset A\},$$

where $\mathcal{E}_1(u, v)=\mathcal{E}(u, v)+(u, v)$, $u, v\in\mathcal{F}$. Cap is a non-negative increasing set function on W such that

$$(1.4)\qquad P(A)\leqq\mathrm{Cap}(A) \quad \text{for Borel } A, \ \mathrm{Cap}(W)=1$$

$$(1.5)\qquad A_n\uparrow \implies \mathrm{Cap}\left(\bigcup_{n=1}^{\infty} A_n\right)=\sup_n \mathrm{Cap}(A_n)$$

$$(1.6)\qquad \mathrm{Cap}\left(\bigcup_{n=1}^{\infty} A_n\right)\leqq\sum_{n=1}^{\infty}\mathrm{Cap}(A_n).$$

Properties (1.5) and (1.6) follow from property (1.5) for open A_n's and the strong subadditivity of Cap for open sets, which are in turn easy consequences of the feature (1.3) of the Dirichlet form ([**1**; §3.1]). A trivial but important observation we want to mention here is that the countable subadditivity (1.6) implies the capacitary version of the *first Borel Cantelli lemma*

$$(1.7)\qquad \sum_{n=1}^{\infty}\mathrm{Cap}(A_n)<\infty \implies \mathrm{Cap}(\overline{\lim_{n\to\infty}} A_n)=0$$

which will be repeatedly used in concluding our quasi-everywhere statements.

We use the term "*quasi-everywhere*" or "*q.e.*" to mean "except on a subset of W of capacity zero". We can now state our theorems. The first five theorems concern the one dimensional Brownian motion.

THEOREM 1 (*$d=1$, quadratic variation*). *Fix $t>0$ and consider a sequence Δ_n of partitions of $[0, t]$: $0=t_0^{(n)}<t_1^{(n)}<\cdots<t_m^{(n)}=t$, $|\Delta_n|=\max_k(t_k^{(n)}-t_{k-1}^{(n)})$. We let*

$$S_n(w)=\sum_k (b(t_k^{(n)}, w)-b(t_{k-1}^{(n)}, w))^2, \qquad w\in W.$$

If $|\Delta_n|\to 0$, $n\to\infty$, *then* S_n *converges to* t *in capacity* and a subsequence of* S_n *converges to* t *q.e. If* $\sum_{n=1}^{\infty}|\Delta_n|<\infty$, *then* S_n *converges to* t *q.e.*

THEOREM 2 ($d=1$, *non-differentiability*). $b(t)$ *is nowhere differentiable in* t *q.e.*

THEOREM 3 ($d=1$, *Lévy's Hölder continuity*).

$$\lim_{\delta\downarrow 0}\sup_{\substack{0\le t_1<t_2\le 1\\ t=t_2-t_1<\delta}}\frac{|b(t_2)-b(t_1)|}{(2t\log 1/t)^{1/2}}=1 \qquad q.e.$$

THEOREM 4 ($d=1$, *law of the iterated logarithm at* 0).

$$\overline{\lim_{t\downarrow 0}}\frac{b(t)}{(2t\log_2 1/t)^{1/2}}=1 \qquad q.e.$$

THEOREM 5 ($d=1$, *law of the iterated logarithm at* ∞).

$$\overline{\lim_{t\to\infty}}\frac{b(t)}{(2t\log_2 t)^{1/2}}=1 \qquad q.e.$$

The next three theorems concern higher dimensional Brownian motion.

THEOREM 6 (*unattainability of a one point set*). *Let* $d\geqq 5$. *Fix any point* $\boldsymbol{a}=(a_1,\cdots,a_d)\in\boldsymbol{R}^d$ *and let* $\sigma_{\{\boldsymbol{a}\}}(w)=\inf\{t>0 : b(t,w)=\boldsymbol{a}\}$ ($\inf\emptyset=\infty$ *by convention*). *Then* $\sigma_{\{\boldsymbol{a}\}}=\infty$ *q.e.*

THEOREM 7 (*Transience*). *Let* $d\geqq 5$. *Then* $\lim_{t\to\infty}|b(t)|=\infty$ *q.e.*

THEOREM 8 (*Absence of double points*). *Let* $d\geqq 7$. *Then* $b(t)$ *has no double point q.e.*

In §2, we prove Theorem 1 and then present three propositions concerning some basic capacitary estimates. Other theorems will be proved in §3 and §4. We will refer to McKean [7], Ito-McKean [4] and Kakutani [5] for the proof of the classical a.e. statements. The idea is that our propositions of §2 together with our version (1.7) of the first Borel Cantelli lemma enable us to proceed along the same lines as in [7], [4] and [5] to our q.e. statements.

In the remainder of the introduction, we give some remarks on the probabilistic and analytic significance of the W-set of zero capacity. Let us consider the Ornstein-Uhlenbeck process $(\boldsymbol{Y}_t, \boldsymbol{P})$ on W with initial (and stationary) distribution being the Wiener measure P. We have then the relation

$$\text{(1.8)} \qquad \mathrm{Cap}(A)=0 \iff \boldsymbol{P}(\boldsymbol{Y}_t\in A \text{ for some } t>0)=0.$$

In fact, a simple application of the optional sampling theorem to a supermar-

* $\lim_{n\to\infty}\mathrm{Cap}(|S_n-t|>\varepsilon)=0$ for any $\varepsilon>0$.

tingale enables one to identify the capacitary potential of an open set A with the 1-order hitting probability of A with respect to the Ornstein-Uhlenbeck process ([**1**; Lemma 4.3.1]). The implication "$\Rightarrow$" in (1.8) is immediate from this ([**1**; Theorem 4.3.1]). Since our space W is not locally compact, the argument of [**1**] does not work directly in obtaining the converse implication "$\Leftarrow$". However, we can show that the converse is also true by embedding W continuously onto a dense Borel subset of a compact space and thus reducing the situation to the standard setting of [**1**] ([**11**]). This method of embedding has been used by Kusuoka [**6**] in constructing a diffusion on a Banach space.

In view of (1.8), we see that Theorem 1 includes William's result [**8**]. Moreover (1.8) allows us to translate each of our q.e. statements into an a.e. statement concerning the "Brownian sheet". Let $W(t, \tau)$, $t \geqq 0$, $\tau \geqq 0$, be a two dimensional parameter Wiener process taking values in $\boldsymbol{R}^d$ defined on a probability space $(\Omega, \boldsymbol{P})$: $W(t, \tau)=(W^1(t, \tau), \cdots, W^d(t, \tau))$ possesses independent components and each component $W^i(t, \tau)$ is continuous, vanishing on axes and centered Gaussian with $\boldsymbol{E}(W^i(t, \tau)W^i(t', \tau'))=t\wedge t' \cdot \tau \wedge \tau'$. $W(t, \tau)$ is the so called *Brownian sheet*. For each $t \geqq 0$ and $\omega \in \Omega$, $\boldsymbol{X}_t(\omega)=W(t, \cdot)(\omega)$ takes value in W and $(\Omega, \boldsymbol{P}, \boldsymbol{X}_t)$ is a version of the Brownian motion on the space W. Following Meyer [**8**], we let

$$Y_t^w = e^{-t/2}(w + X_{e^t-1}), \qquad w \in W, \tag{1.9}$$

then $\boldsymbol{Y}_t^w$ becomes a realization of the Ornstein-Uhlenbeck process on W starting at w and indeed $p_t f(w)=\boldsymbol{E}(f(\boldsymbol{Y}_t^w))$ is a version of $\exp(t\boldsymbol{A})f(w)$ for the operator $\boldsymbol{A}$ of (1.1) ([**8**]).

Let us use a convenient fact that, as a realization of the Ornstein-Uhlenbeck process on W with initial distribution being the Wiener measure P, we may take

$$Y_t = e^{-t/2} W(e^t, \cdot), \qquad t \geqq 0. \tag{1.10}$$

Consider a statement S concerning each element $w \in W$ and let $A=\{w \in W : S(w)$ *is true*$\}$. We know from (1.8) and (1.10) that

$$\mathrm{Cap}(A) > 0$$

if and only if

(1.11) $\boldsymbol{P}(e^{-t/2}W(e^t, \tau)$ satisfies S as a function of $\tau \geqq 0$ for some $t>0)>0$.

Therefore the stronger assertion

"*S is true q.e.*", namely, "$\mathrm{Cap}(A^c)=0$",

holds if and only if

(1.12) $\boldsymbol{P}(e^{-t/2}W(e^t, \tau)$ satisfies S as a function of $\tau \geqq 0$ for every $t>0)=1$.

If S holds P-a.e., then

(1.13) $\quad \boldsymbol{P}(e^{-t/2}W(e^t, \tau)$ satisfies S as a function of $\tau \geqq 0)=1$ for each $t>0$,

because $\{e^{-t/2}W(e^t, \tau), \tau \geqq 0\}$ is under $\boldsymbol{P}$ a Brownian motion for each $t>0$. However (1.13) does not always imply (1.12). For instance, let $d=1$ and take as S the statement "$w_1 \neq 0$". Then (1.13) is fulfilled but (1.12) is not because $\{e^{-t/2}W(e^t, 1), t \geqq 0\}$ is under $\boldsymbol{P}$ a one-dimensional Ornstein-Uhlenbeck process and consequently hits the origin almost surely. In other words, the W-set

$$A_1 = \{w_1 = 0\} \tag{1.14}$$

is P-negligible but of positive capacity. In the same way we can see that any finite dimensional set which is of zero Lebesgue measure and yet of positive Newtonian capacity always gives rise to a P-negligible W-set of positive capacity (see the final remark in this section).

We exhibit a more interesting P-negligible W-set of positive capacity which is not given by a finite dimensional projection as above. An intensive study of the sample function behaviours of the Brownian sheet $W(t, \tau)$ has been given by Orey-Pruitt [**9**]. In particular, they proved that the $\boldsymbol{P}$-measure of the Ω-set

$$W(t, \tau)=\boldsymbol{a} \qquad \text{for some} \quad (t, \tau) \in (0, \infty) \times (0, \infty)$$

is either 1 or 0 according as $d<4$ or $d \geqq 4$, where $\boldsymbol{a}$ is an arbitrarily fixed point of $\boldsymbol{R}^d$. Hence we see by (1.11) that the capacity of W-set

$$A_2 = \{w_\tau = \boldsymbol{0} \text{ for some } \tau > 0\} \tag{1.15}$$

is either positive or zero according as $d<4$ or $d \geqq 4$, where $\boldsymbol{0}$ is the origin of $\boldsymbol{R}^d$. When $d=2$ or 3, $\mathrm{Cap}(A_2)>0$ but, as is well known, $P(A_2)=0$.

Note that the statement $\mathrm{Cap}(A_2)=0$ for $d \geqq 5$ constitutes a part of our Theorem 6. We also note that the interpretation by (1.12) allows us to derive Theorem 5 (law of the iterated logarithm at ∞) and Theorem 7 (transience for $d \geqq 5$) from the corresponding statements for the Brownian sheet by Zimmerman [**12**; (2.15)] and by Orey-Pruitt [**9**; Theorem 3.1] respectively.

Finally we add a remark that our Dirichlet form $\mathcal{E}$ of (1.2) is reduced by a finite dimensional projection to a familiar expression

$$\mathcal{E}^{(n)}(u, v) = \frac{1}{2}\int_{\boldsymbol{R}^n} \nabla u \cdot \nabla v \, d\mu \qquad u, v \in C_0^\infty(\boldsymbol{R}^n), \tag{1.16}$$

where $d\mu = (2\pi)^{-n/2} e^{-|x|^2/2} dx_1 \cdots dx_n$. In fact, let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be elements of W' constituting an orthonormal system in $L^2([0, \infty))$ and let

$$P_{\alpha_1, \cdots, \alpha_n}(w) = (\{\alpha_1, w\}, \cdots, \{\alpha_n, w\}) \in \boldsymbol{R}^n, \qquad w \in W$$

(see §2 for the definitions of W' and $\{\ ,\ \}$). Then we can see by Lemma 1 of

§ 2 that

$$(1.17)\qquad \mathcal{E}(u\circ P_{\alpha_1,\cdots,\alpha_n}, v\circ P_{\alpha_1,\cdots,\alpha_n})=\mathcal{E}^{(n)}(u, v), \qquad u, v\in C_0^\infty(\boldsymbol{R}^n).$$

$\mathcal{E}^{(n)}$ is a Dirichlet form on $L^2(\boldsymbol{R}^n;\mu)$ and indeed associated with the Ornstein-Uhlenbeck process on $\boldsymbol{R}^n$. We can also see that the Ornstein-Uhlenbeck process on $\boldsymbol{R}^n$ with initial distribution μ is realized by the projection $P_{\alpha_1,\cdots,\alpha_n}(\boldsymbol{Y}_t)$.

The expression (1.16) tells us that a set $E\subset\boldsymbol{R}^n$ is of zero Newtonian capacity if and only if E has zero capacity with respect to $\mathcal{E}^{(n)}$, which is in turn equivalent to the probabilistic condition that the process $P_{\alpha_1,\cdots,\alpha_n}(\boldsymbol{Y}_t)$ does not hit E almost surely. Therefore, in view of (1.8), we can conclude that $E\subset\boldsymbol{R}^n$ is of zero Newtonian capacity if and only if the cylindrical W-set $A=P^{-1}_{\alpha_1,\cdots,\alpha_n}(E)$ satisfies $\mathrm{Cap}(A)=0$. In this sense, we may well claim that the present notion of capacity is a natural extension of the classical Newtonian capacity on the n-space toward the infinite dimensional space W.

§ 2. Basic estimates of capacity.

Denote by $C(W)$ the totality of (not necessarily bounded) continuous functions on W. Since the polynomials of the coordinate functions $b(t)=(b^1(t), b^2(t), \cdots, b^d(t))$ belongs to $\mathcal{F}\cap C(W)$ and $\mathcal{E}_1$-dense in $\mathcal{F}$, we see in the same way as in [**1**; § 3.1] that each $u\in\mathcal{F}$ admits a quasi-continuous version $\tilde{u}$ and

$$(2.1)\qquad \mathrm{Cap}(|\tilde{u}|>\lambda)\leqq\frac{1}{\lambda^2}\mathcal{E}_1(u, u), \qquad \lambda>0, \quad u\in\mathcal{F}.$$

Theorem 1 is a straightforward application of this estimate:

PROOF OF THEOREM 1. Suppose $d=1$. Since

$$S_n-t=\sum_k[(b(t_k^{(n)})-b(t_{k-1}^{(n)}))^2-(t_k^{(n)}-t_{k-1}^{(n)})]\in Z_2,$$

we have from the expression (1.2)

$$\mathcal{E}_1(S_n-t, S_n-t)=2\|S_n-t\|_{L^2}^2=2E((S_n-t)^2)\leqq C|\Delta_n|\cdot t.$$

Now (2.1) and our capacitary version of the first Borel Cantelli lemma (1.7) lead us to Theorem 1. q.e.d.

The proof of other theorems are not so simple and we must go a little beyond the algebraic expression (1.2) and use a chain rule in computing the Dirichlet norm. In what follows, we denote $\sqrt{\mathcal{E}(u, u)}$ and $\sqrt{\mathcal{E}_1(u, u)}$ by $\|u\|_{\mathcal{E}}$ and $\|u\|_{\mathcal{E}_1}$ respectively, $u\in\mathcal{F}$.

Following Meyer [8], we denote by W' the set of all functions

$$\alpha:[0, \infty)\longrightarrow\boldsymbol{R}^d$$

with finite variation and compact support. We define the pairing of W' and W by

$$\{\alpha, w\} = -\int_0^\infty w_s \cdot d\alpha_s, \qquad \alpha \in W', \quad w \in W,$$

which is a version of the stochastic integral $\int_0^\infty \alpha_s \cdot db_s(w)$ and belongs as a function of w to the space $Z_1 \subset \mathcal{D}(\boldsymbol{A})$. We view W' as a subspace of $L^2([0, \infty)) = L^2([0, \infty) \to \boldsymbol{R}^d)$ with inner product

$$q(\alpha, \beta) = \int_0^\infty \alpha_s \cdot \beta_s ds.$$

LEMMA 1. *Take* $\alpha_1, \alpha_2, \cdots, \alpha_n \in W'$ *and* $f \in C_0^\infty(\boldsymbol{R}^n \to \boldsymbol{R}^1)$. *Then*

$$f(\{\alpha_1, \cdot\}, \{\alpha_2, \cdot\}, \cdots, \{\alpha_n, \cdot\}) \in \mathcal{F}$$

and

$$\|f(\{\alpha_1, \cdot\}, \{\alpha_2, \cdot\}, \cdots, \{\alpha_n, \cdot\})\|_{\mathcal{E}}^2$$
$$= \frac{1}{2} \sum_{i,j=1}^n E[f_{x_i}(\{\alpha_1, \cdot\}, \cdots, \{\alpha_n, \cdot\}) f_{x_j}(\{\alpha_1, \cdot\}, \cdots, \{\alpha_n, \cdot\})] q(\alpha_i, \alpha_j).$$

PROOF. Consider functions $F \in \mathcal{D}(\boldsymbol{A})$ $(\subset \mathcal{F})$ satisfying $F^2 \in \mathcal{D}(\boldsymbol{A})$, F, $\boldsymbol{A}F \in L^4(W, P)$. For such F and G, let

$$\langle F, G\rangle = \boldsymbol{A}(FG) - (\boldsymbol{A}F) \cdot G - F \cdot (\boldsymbol{A}G) \qquad (\in L^2(W; P)),$$

then

$$\mathcal{E}(F, G) = \frac{1}{2} E(\langle F, G\rangle)$$

because the both hand sides equal $-E((\boldsymbol{A}F) \cdot G)$. Let $F_1, F_2, \cdots, F_n$ be of the above type and f be as in Lemma 1. Take quasi-continuous versions $\tilde{F}_i$ of F_i $(\in \mathcal{F})$. Then $\tilde{F}_i(\boldsymbol{Y}_t)$ are continuous semimartingales and the Ito formula yields the chain rule ([8], [**10**]):

$$\langle f(F_1, F_2, \cdots, F_n), G\rangle = \sum_{i=1}^n f_{x_i}(F_1, F_2, \cdots, F_n) \langle F_i, G\rangle.$$

On the other hand, in view of the trivial actions of $\boldsymbol{A}$ on Z_1 and Z_2, we readily have

$$\langle \{\alpha_i, \cdot\}, \{\alpha_j, \cdot\}\rangle = q(\alpha_i, \alpha_j).$$

Since F is closed under the composition with a uniformly Lipschitz function, the proof is complete. q.e.d.

For an interval $I = (s, t) \subset \boldsymbol{R}$, we put

$$X_I(w) = \frac{b(t, w) - b(s, w)}{\sqrt{t-s}}, \qquad X_I(w) = (X_I^1(w), \cdots, X_I^d(w)).$$

PROPOSITION 1. *For disjoint intervals* $I_1, I_2, \cdots, I_N \subset \boldsymbol{R}$ *and* $a_i^j < b_i^j$, $c_i^j > 0$, $i = 1, 2, \cdots, N$, $j = 1, 2, \cdots, d$, *we have*

$$\operatorname{Cap}\Big(\bigcap_{i=1}^{N}\bigcap_{j=1}^{d}\{a_i^j<X_{I_i}^j<b_i^j\}\Big)$$
$$\leqq\Big(\frac{Nd}{2c^2}+1\Big)P\Big(\bigcap_{i=1}^{N}\bigcap_{j=1}^{d}\{a_i^j-c_i^j<X_{I_i}^j<b_i^j+c_i^j\}\Big)$$

where $c=\min_{i,j} c_i^j$.

PROOF. From Lemma 1

(2.2) $$\|f(X_{I_1}^1,\cdots,X_{I_1}^d,X_{I_2}^1,\cdots,X_{I_2}^d,\cdots,X_{I_N}^1,\cdots,X_{I_N}^d)\|_{\mathcal{E}}^2$$
$$=\frac{1}{2}\sum_{k=1}^{Nd}E[f_{x_k}(X_{I_1}^1,\cdots,X_{I_N}^d)^2],\qquad f\in C_0^\infty(\boldsymbol{R}^{Nd}\to\boldsymbol{R}).$$

Fix $\delta>0$. For each $1\leqq i\leqq N$ and $1\leqq j\leqq d$, choose a C^∞-function $f_i^j(x)$ on $\boldsymbol{R}$ such that $f_i^j(x)=1$ on (a_i^j, b_i^j), $f_i^j(x)=0$ outside $(a_i^j-c_i^j, b_i^j+c_i^j)$, $0\leqq f_i^j(x)\leqq 1$ and $|(f_i^j)'(x)|\leqq\frac{1}{c-\delta}$. Then put $f(x_1,\cdots,x_{Nd})=f_1^1(x_1)\cdots f_1^d(x_d)f_2^1(x_{d+1})\cdots f_N^d(x_{Nd})$. Since $F(w)=f(X_{I_1}^1,\cdots,X_{I_1}^d,X_{I_2}^1,\cdots,X_{I_N}^d)\in\mathcal{F}$ is equal to 1 on the open set $B=\bigcap_{i=1}^{N}\bigcap_{j=1}^{d}\{a_i^j<X_{I_i}^j<b_i^j\}$, we have, by the definition of Cap(B) and (2.2),

$$\operatorname{Cap}(B)\leqq\mathcal{E}_1(F,F)=\|F\|_{\mathcal{E}}^2+E(F^2)$$
$$\leqq\Big(\frac{Nd}{2(c-\delta)^2}+1\Big)P\Big(\bigcap_{i=1}^{N}\bigcap_{j=1}^{d}\{X_{I_i}^j\in(a_i^j-c_i^j,\ b_i^j+c_i^j)\}\Big).\qquad \text{q.e.d.}$$

LEMMA 2. *Let* $0\leqq s<t_1<t_2<\cdots<t_n$ *and put* $b_{s;t_1\cdots t_n}=(b(t_1)-b(s), b(t_2)-b(s),\cdots,b(t_n)-b(s))$ $(\in\boldsymbol{R}^{nd})$. *Suppose* $f\in H_{loc}^1(\boldsymbol{R}^{nd})$ *satisfies* $f(b_{s;t_1\cdots t_n})\in L^2(W;P)$ *and* $f_{x_k}(b_{s;t_1\cdots t_n})\in L^2(W;P)$, $1\leqq k\leqq nd$, *then* $f(b_{s;t_1\cdots t_n})\in\mathcal{F}$ *and*

(2.3) $$\|f(b_{s;t_1\cdots t_n})\|_{\mathcal{E}}^2$$
$$=\frac{1}{2}\sum_{l,m=1}^{n}\sum_{k=1}^{d}E[f_{x_{(l-1)d+k}}(b_{s;t_1\cdots t_n})f_{x_{(m-1)d+k}}(b_{s;t_1\cdots t_n})](t_l-s)\wedge(t_m-s).$$

PROOF. When $f\in C_0^\infty(\boldsymbol{R}^{nd})$, (2.3) follows from Lemma 1 by setting

$$\alpha_{(l-1)d+k}(s)=(\overbrace{0,\cdots,I_{[s,t_l]}(s)}^{k},\cdots,0).$$

Suppose next $f\in H^1(\boldsymbol{R}^{nd})$ is of compact support. Then the sum of the right hand side of (2.3) and $E(f(b_{s;t_1\cdots t_n})^2)$ is dominated by

$$\Big(\frac{1}{2}n(t_n-s)+1\Big)[(2\pi)^{n/2}(t_1-s)(t_2-t_1)\cdots(t_n-t_{n-1})]^{-d/2}\|f\|_{H^1}^2$$

where $\|f\|_{H^1}^2=\int_{\boldsymbol{R}^{nd}}\{|\nabla f|^2+f^2\}dx$. Let ρ_ε be a mollifier. Then $\rho_\varepsilon*f\in C_0^\infty(\boldsymbol{R}^{nd})$ satisfies Lemma 2.2 and so does f because $\rho_\varepsilon*f$ converges as $\varepsilon\downarrow 0$ to f in the space $H^1(\boldsymbol{R}^{nd})$.

Now take any function f satisfying the condition of the lemma and let $f^{(p)}=f\cdot g^{(p)}$ where $g^{(p)}$ is a C^1-function such that $g^{(p)}=1$ for $|x|\leqq p$, $g^{(p)}=0$ for $|x|\geqq p+1$ and $0\leqq g^{(p)}\leqq 1$ and $|g^{(p)}_{x_k}|\leqq M$ for some M independent of x, p and k. Then $f^{(p)}$ is a function of the preceding type and

$$E[\{(f^{(p)}-f^{(q)})_{x_k}(b_{s;t_1\cdots t_n})\}^2]\leqq 8\int_{|x|\geqq p}\{M^2 f(x)^2+f_{x_k}(x)^2\}\,d\mu(x)\,,\quad p\leqq q\,,$$

where μ is the distribution of $b_{s;t_1\cdots t_n}$. Hence we have Lemma 2. q.e.d.

For $b(t)=(b^1(t),\cdots,b^d(t))$, we put

$$M^i_{s,t}=\max_{s\leqq v\leqq t}(b^i(v)-b^i(s))\,,\qquad 1\leqq i\leqq d\,,\quad 0\leqq s<t\,.$$

LEMMA 3. *Let $1\leqq i\leqq d$ and $0\leqq s<t$. If $u(x)$ is a non-negative non-decreasing convex and absolutely continuous function on $\boldsymbol{R}$ such that $u(M^i_{s,t})$ and $u'(M^i_{s,t})$ are in $L^2(W;P)$, then $u(M^i_{s,t})\in\mathcal{F}$ and*

$$\|u(M^i_{s,t})\|^2_{\mathcal{E}}\leqq\frac{t-s}{2}E(u'(M^i_{s,t})^2)\,.$$

PROOF. Take $s<t_1<t_2<\cdots<t_n\leqq t$ and let $f(x_1,\cdots,x_{nd})=u(x_i\vee x_{d+i}\vee\cdots\vee x_{(n-1)d+i})$. Then $f(b_{s;t_1\cdots t_n})=u(M^i_{s;t_1\cdots t_n})$ where $M^i_{s;t_1\cdots t_n}=\max_{1\leqq l\leqq n}(b^i(t_l)-b^i(s))$. It is easy to see that f satisfies the condition of Lemma 2 which implies

$$\|u(M^i_{s;t_1\cdots t_n})\|^2_{\mathcal{E}}=\frac{1}{2}\sum_{l=1}^n E(u'(M^i_{s;t_1\cdots t_n})^2\,;\,M^i_{s;t_1\cdots t_n}=b^i(t_l)-b^i(s))(t_l-s)$$

$$\leqq\frac{t-s}{2}E(u'(M^i_{s;t_1\cdots t_n})^2)\leqq\frac{t-s}{2}E(u'(M^i_{s\,t})^2)\,.$$

In the present case, the cross terms of the right hand side of (2.3) vanish because

$$P(M^i_{s;t_1\cdots t_n}=b^i(t_l)-b^i(s)=b^i(t_m)-b^i(s))\leqq P(b^i(t_l)-b^i(t_m)=0)=0\,,\quad l\neq m\,.$$

Let $\{t_1,t_2,\cdots,t_n\}$ increase to a countable dense subset of $[s,t]$. Then $u(M^i_{s;t_1\cdots t_n})$ increases to $u(M^i_{s,t})$. Since $\mathcal{E}_1$-norm of $u(M^i_{s;t_1\cdots t_n})$ is uniformly bounded by the above estimate, the Cesaro means of a subsequence are $\mathcal{E}_1$-convergent to a function of $\mathcal{F}$ by virtue of the Banach-Saks theorem. The limit function must coincide with $u(M^i_{s,t})$. q.e.d.

PROPOSITION 2. *Let $1\leqq i\leqq d$ and $0\leqq s<t$. Then*

$$\text{Cap}\Big(M^i_{s,t}-\frac{\alpha}{2}(t-s)>\beta\Big)\leqq\Big(\frac{\alpha^2(t-s)}{4}+2\Big)e^{-\alpha\beta}\,,\qquad \alpha,\ \beta>0\,.$$

PROOF. The right hand side equals

$$\text{Cap}\,(e^{(\alpha/2)M^i_{s,t}}>e^{(\alpha^2(t-s)/4)+(\alpha\beta/2)})\leqq e^{-(\alpha^2/2)(t-s)-\alpha\beta}\|e^{(\alpha/2)M^i_{s,t}}\|^2_{\mathcal{E}_1}\qquad\text{by (2.1).}$$

By virtue of Lemma 3, we see

$$\|e^{(\alpha/2)M_{s,t}^i}\|_{\mathcal{E}_1}^2 \leqq \left(\frac{\alpha^2(t-s)}{8}+1\right)E(e^{\alpha M_{s,t}^i}) \leqq \left(\frac{\alpha^2(t-s)}{4}+2\right)e^{(\alpha^2/2)(t-s)}. \quad \text{q.e.d.}$$

PROPOSITION 3. *For* $1 \leqq i \leqq d$, $0 \leqq s < t$ *and* $\eta > 0$

$$\text{Cap}\,(\max_{s \leqq v \leqq t} |b^i(v)-b^i(s)| > \eta) \leqq \left(\frac{\eta^2}{2(t-s)}+4\right)e^{-\eta^2/(2(t-s))}$$

$$\text{Cap}\,(\max_{s \leqq v \leqq t} |b^i(t)-b^i(v)| > \eta) \leqq \left(\frac{\eta^2}{2(t-s)}+4\right)e^{-\eta^2/(2(t-s))}.$$

PROOF. By letting $\alpha = \frac{\eta}{t-s}$ and $\beta = \frac{\eta}{2}$ in Proposition 2, $\text{Cap}\,(M_{s,t}^i > \eta) \leqq \left(\frac{\eta^2}{4(t-s)}+2\right)e^{-\eta^2/(2(t-s))}$. Since Proposition 2 still holds by replacing $b(t) = (b^1(t), \cdots, b^i(t), \cdots, b^d(t))$ with $-b(t)$, we get the first inequality. The second inequality also follows from an obvious modification of Proposition 2. q.e.d.

§3. q.e. properties of the one dimensional Brownian motion.

We assume $d=1$ and prove Theorems 2, 3, 4 and 5 stated in §1.

[3.1] **Nowhere differentiability** (proof of Theorem 2).

Fix a positive integer l and let

$$B_n = \bigcup_{1 \leqq i \leqq n-5} \bigcap_{i<j \leqq i+5} \left\{ \left| b\left(\frac{j}{n}\right) - b\left(\frac{j-1}{n}\right) \right| < \frac{11l}{n} \right\}.$$

It suffices to prove $\text{Cap}\,(\varlimsup_{n\to\infty} B_n) = 0$ ([7; p.9]). But, by Proposition 1, we have, for $c > 0$

$$\begin{aligned}
\text{Cap}\,(B_n) &\leqq \sum_{i=1}^{n-5} \text{Cap}\left(\bigcap_{i<j\leqq i+5}\left\{\sqrt{n}\left|b\left(\frac{j}{n}\right)-b\left(\frac{j-1}{n}\right)\right| < \frac{11l}{\sqrt{n}}\right\}\right) \\
&\leqq \sum_{i=1}^{n-5}\left(\frac{5n}{2c^2}+1\right)P\left(\bigcap_{i<j\leqq i+5}\left\{\sqrt{n}\left|b\left(\frac{j}{n}\right)-b\left(\frac{j-1}{n}\right)\right| < \frac{11l+c}{\sqrt{n}}\right\}\right) \\
&\leqq \frac{n(5n+2c^2)}{2c^2}[P(\sqrt{n}\,|b(1)| < 11l+c)]^5 \\
&= \frac{n(5n+2c^2)}{2c^2}\left[\frac{1}{\sqrt{2\pi n}}\int_{-11l-c}^{11l+c} e^{-(x^2/2n)}dx\right]^5 \to 0, \qquad n \to \infty.
\end{aligned}$$

q.e.d.

[3.2] **Lévy's Hölder continuity** (proof of Theorem 3).

Let us put

$$F(w) = \lim_{\gamma \downarrow 0} \sup_{\substack{0 \leqq t_1 < t_2 \leqq 1 \\ t = t_2 - t_1 < \gamma}} \frac{|b(t_2)-b(t_1)|}{(2t \log 1/t)^{1/2}}.$$

PROOF OF "$F(w) \geqq 1$ q.e.". Let $h(t) = (2t \log 1/t)^{1/2}$, $0 < \delta < 1$ and $0 < c < \delta$, then by Proposition 1,

$$\mathrm{Cap}\,(\max_{k\leqq 2^n}|b(k2^{-n})-b((k-1)2^{-n})|\leqq(1-\delta)h(2^{-n}))$$

$$=\mathrm{Cap}\left[\max_{k\leqq 2^n}\left|\frac{b(k2^{-n})-b((k-1)2^{-n})}{\sqrt{2^{-n}}}\right|\leqq(1-\delta)\sqrt{2\log 2^n}\right]$$

$$\leqq\left(\frac{2^n}{4c^2\log 2^n}+1\right)P\left[\max_{k\leqq 2^n}\left|\frac{b(k2^{-n})-b((k-1)2^{-n})}{\sqrt{2^{-n}}}\right|\leqq(1-\delta+c)\sqrt{2\log 2^n}\right]$$

$$\leqq\left(\frac{2^n}{4c^2 n\log 2}+1\right)\exp\left(-2^{n(\delta-c)}\right),$$

the last inequality holding for sufficiently large n according to [7; p. 15]. Since the sum in n of the last expression is finite, we have from (1.7)

$$\max_{k\leqq 2^n}\frac{|b(k2^{-n})-b((k-1)2^{-n})|}{h(2^{-n})}>1-\delta \qquad \text{from some } n \text{ on,} \quad \text{q.e.,}$$

which implies the desired inequality because $\delta>0$ is arbitrary. q.e.d.

PROOF OF "$F(w)\leqq 1$ q.e.". By Proposition 3, we have for $\varepsilon>0$ and $0\leqq s<t$

$$\mathrm{Cap}\,[|b(t)-b(s)|>(1+\varepsilon)h(t-s)]\leqq\left((1+\varepsilon)^2\log\frac{1}{t-s}+4\right)(t-s)^{(1+\varepsilon)^2}.$$

Hence, using the subadditivity of the capacity, we find

$$\mathrm{Cap}\left[\max_{\substack{0<k=j-i\leqq 2^{n\delta}\\ 0<i<j\leqq 2^n}}\frac{|b(j2^{-n})-b(i2^{-n})|}{h(k2^{-n})}\geqq 1+\varepsilon\right]$$

$$\leqq\sum_{\substack{0<k=j-i\leqq 2^{n\delta}\\ 0<i<j\leqq 2^n}}\mathrm{Cap}\left(\frac{|b(j2^{-n})-b(i2^{-n})|}{h(k2^{-n})}\geqq 1+\varepsilon\right)$$

$$\leqq 2^n\sum_{1\leqq k\leqq 2^{n\delta}}\left\{(1+\varepsilon)^2\log\frac{2^n}{k}+4\right\}(k2^{-n})^{(1+\varepsilon)^2}$$

$$\leqq 2^{n(1+\delta)}\{(1+\varepsilon)^2 n\log 2+4\}\,2^{-n(1-\delta)(1+\varepsilon)^2}.$$

Choose small $\delta>0$ satisfying $(1+\delta)<(1-\delta)(1+\varepsilon)^2$, then by (1.7) again, we see that, for q.e. $w\in W$, there exists $m=m(w)$ such that for any $n\geqq m(w)$

$$|b(j2^{-n})-b(i2^{-n})|<(1+\varepsilon)h(k2^{-n}),\ 0<i<j\leqq 2^n,\ k=j-i\leqq 2^{n\delta}.$$

But we have for such w $F(w)\leqq 1+3\varepsilon+2\varepsilon^2$ ([7; p. 16]). q.e.d.

[3.3] **Law of the iterated logarithm at** 0 (proof of Theorem 4).

Let us put

$$F(w)=\overline{\lim_{t\downarrow 0}}\,\frac{b(t)}{(2t\log_2 1/t)^{1/2}}.$$

PROOF OF "$F(w)\leqq 1$ q.e.". Let $h(t)=(2t\log_2 1/t)^{1/2}$ and take just as in

[7; p.13] $0<\theta<1$, $0<\delta<1$, $\alpha=(1+\delta)\theta^{-n}h(\theta^n)$, $\beta=h(\theta^n)/2$, $t=\theta^{n-1}$ and $s=0$ so that

$$\left(\frac{\alpha^2(t-s)}{4}+2\right)e^{-\alpha\beta}=C_1(\log n+C_2)n^{-(1+\delta)}$$

is a general term of a convergent sum. By Proposition 2 and (1.7),

$$\max_{0<v<\theta^{n-1}} b(v)\leqq\left[\frac{1+\delta}{2\theta}+\frac{1}{2}\right]h(\theta^n) \qquad \text{from some } n \text{ on, q.e.}$$

The rest of the proof is the same as in [7].

PROOF OF "$F(w)\geqq 1$ q.e.". We choose $0<\theta<1$ and put $\xi=\log 1/\theta$, $h(t)=(2t\log_2 1/t)^{1/2}$ and $W_k=\{b(\theta^k)-b(\theta^{k+1})\leqq(1-\sqrt{\theta})h(\theta^k)\}$. By Proposition 1

$$\begin{aligned}\mathrm{Cap}\left(\bigcap_{k=l}^{n}W_k\right)&=\mathrm{Cap}\left[\bigcap_{k=l}^{n}\frac{b(\theta^k)-b(\theta^{k+1})}{\sqrt{\theta^k(1-\theta)}}\leqq\frac{1-\sqrt{\theta}}{\sqrt{1-\theta}}(2\log\xi k)^{1/2}\right]\\ &\leqq\left(\frac{(1-\theta)n}{4c^2\log\xi l}+1\right)P\left[\bigcap_{k=l}^{n}\left\{\frac{b(\theta^k)-b(\theta^{k+1})}{\sqrt{\theta^k(1-\theta)}}<\frac{1-\sqrt{\theta}+c}{\sqrt{1-\theta}}(2\log\xi k)^{1/2}\right\}\right]\\ &=\left(\frac{(1-\theta)n}{4c^2\log\xi l}+1\right)\prod_{k=l}^{n}(1-I_k)\leqq\left(\frac{(1-\theta)n}{4c^2\log\xi l}+1\right)\exp\left(-\sum_{k=l}^{n}I_k\right).\end{aligned}$$

Here I_k denotes the integral of $\dfrac{\exp(-c^2/2)}{\sqrt{2\pi}}dc$ on $\left[\dfrac{1-\sqrt{\theta}+c}{\sqrt{1-\theta}}\sqrt{2\log\xi k},\ \infty\right)$.

Choose $c>0$ so small that $\dfrac{1-\sqrt{\theta}+c}{\sqrt{1-\theta}}<1$, then

$$\sum_{k=l}^{n}I_k\geqq C\sum_{k=l}^{n}\frac{k^{-(1-\sqrt{\theta}+c)^2/(1-\theta)}}{\sqrt{\log k}}\geqq C'(n^{1-\kappa}-l^{1-\kappa}),$$

where κ is such that $\dfrac{(1-\sqrt{\theta}+c)^2}{\sqrt{1-\theta}}<\kappa<1$. Hence

$$\mathrm{Cap}\left(\bigcap_{k=l}^{\infty}W_k\right)\leqq\mathrm{Cap}\left(\bigcap_{k=l}^{n}W_k\right)\leqq C'n\,e^{-C'(n^{1-\kappa}-l^{1-\kappa})}\to 0, \qquad n\to\infty.$$

Therefore $\mathrm{Cap}(\varliminf_{k\to\infty}W_k)=0$, namely, $b(\theta^n)-b(\theta^{n+1})>(1-\sqrt{\theta})h(\theta^n)$ infinitely often q.e. The rest of the proof is again the same as in [7; p.14]. q.e.d.

[3.4] **Law of the iterated logarithm at ∞** (proof of Theorem 5).

Trivial modifications of the proof of Theorem 4 suffice. In fact, we let $h(t)=(2t\log_2 t)^{1/2}$, $\theta>1$, $0<\delta<1$, $\alpha=(1+\delta)\theta^{-n}h(\theta^n)$, $\beta=h(\theta^n)/2$, $t=\theta^{n+1}$ and $s=0$ in Proposition 2 to get inequality "$\leqq$". To get the converse, we let $\theta>1$, $\xi=\log\theta$ and $W_k=\{b(\theta^{k+1})-b(\theta^k)\leqq(1-1/\sqrt{\theta})h(\theta^{k+1})\}$. q.e.d.

REMARK. Here is another way to prove Theorem 5. Note that we could start with, instead of $W=W_0^d$, the space $\widetilde{W}=C((0,\infty)\to\boldsymbol{R}^d)$ and the Wiener measure P on $\tilde{W}$. Then $Tw(t)=tw(1/t)$, $t>0$, defines a P-measure preserving transform T on $\tilde{W}$ which also preserves the associated capacity $\widetilde{\mathrm{Cap}}$ on $\tilde{W}$, because the induced transform on $L^2(\tilde{W}, P)$ makes spaces Z_n and consequently the Dirichlet form (1.2) invariant. Hence Theorem 5 (for $\tilde{W}$) follows from Theorem 4 (for $\tilde{W}$) directly. Now our space W is continuously embedded into $\tilde{W}$ and $P(\tilde{W}-W)=0$. This means that $\mathrm{Cap}(B)\leqq\widetilde{\mathrm{Cap}}(B)$ for $B\subset W$ (actually the equality holds by virtue of the compactification argument ([**11**]) and we can recover q.e. statements on W from those on $\tilde{W}$. It may be also possible to derive the second inequality of Proposition 3 from the first one in a similar way.

§4. q.e. properties of higher dimensional Brownian motions.

As in §2, we write as $b(t)=(b^1(t), \cdots, b^d(t))$. We first note an obvious remark that each component $b^i(t)$, $1\leqq i\leqq d$, satisfies the q.e. statements of Theorems 1 through 5. We shall now prove Theorems 6, 7 and 8 stated in §1.

[**4.1**] **Unattainability of a one point set** (proof of Theorem 6).

Assume $d\geqq 5$. Fix $\boldsymbol{a}=(a_1, \cdots, a_d)\in\boldsymbol{R}^d$ and define as in [**4**; p. 62]

$$B_n=\bigcup_{n\leq k<2n}\bigcap_{1\leq i\leq d}\left\{\left|b^i\left(\frac{k}{n}\right)-a_i\right|<\sqrt{\frac{3\log n}{n}}\right\}.$$

It suffices to show $\lim_{n\to\infty}\mathrm{Cap}(B_n)=0$, because, by virtue of Theorem 3 holding for each component $b^i(t)$, we see that the set

$$\{(b^1(t), \cdots, b^d(t))=(a_1, \cdots, a_d)\ \textit{for some}\ t\in[1, 2)\}$$

is contained q.e. in the set $\overline{\lim_{n\to\infty}}B_n$.

By Proposition 1, we have for each k,

$$\begin{aligned}
&\mathrm{Cap}\left(\bigcap_{i\leq d}\left\{\left|b^i\left(\frac{k}{n}\right)-a_i\right|<\sqrt{\frac{3\log n}{n}}\right\}\right)\\
&=\mathrm{Cap}\left[\bigcap_{i\leq d}\left\{\sqrt{\frac{n}{k}}b^i\left(\frac{k}{n}\right)\in\left(\sqrt{\frac{n}{k}}a_i-\sqrt{\frac{3\log n}{k}},\ \sqrt{\frac{n}{k}}a_i+\sqrt{\frac{3\log n}{k}}\right)\right\}\right]\\
&\leqq\left(\frac{kd}{2c^2\log n}+1\right)P\left[\bigcap_{i\leq d}\left\{\sqrt{\frac{n}{k}}b^i\left(\frac{k}{n}\right)\in\left(\sqrt{\frac{n}{k}}a_i-(\sqrt{3}+c)\sqrt{\frac{\log n}{k}},\right.\right.\right.\\
&\qquad\qquad\left.\left.\left.\sqrt{\frac{n}{k}}a_i+(\sqrt{3}+c)\sqrt{\frac{\log n}{k}}\right)\right\}\right]\\
&\leqq 2\left(\frac{kd}{2c^2\log n}+1\right)(\sqrt{3}+c)\left(\frac{\log n}{2\pi k}\right)^{d/2}.
\end{aligned}$$

Hence $\mathrm{Cap}(B_n)\leqq Cn\left(\frac{\log n}{n}\right)^{(d/2)-1}\to 0,\quad n\to\infty.$ q.e.d.

[4.2] **Transience** (proof of Theorem 7).

Assume $d\geqq 5$. If we let, just as in [5], $t_k=k^{3/4}$, $k=1,2,\cdots$, and $\gamma_k=\mathrm{Cap}(|b^i(t)|<M,\ 1\leqq i\leqq d,\ \textit{for some}\ t\in[t_k,t_{k+1}])$ for $M>0$, then

$$\gamma_k\leqq\mathrm{Cap}(|b^i(t_k)|<2M,\ 1\leqq i\leqq d)+\sum_{i=1}^{d}\mathrm{Cap}(\max_{t_k\leqq t\leqq t_{k+1}}|b^i(t)-b^i(t_k)|>M)\,. \tag{4.1}$$

The first term of the right hand side of (4.1) equals

$$\begin{aligned}&\mathrm{Cap}\left(\frac{1}{\sqrt{t_k}}|b^i(t_k)|<\frac{2M}{\sqrt{t_k}},\ 1\leqq i\leqq d\right)\\&\leqq\left(\frac{dt_k}{2M^2}+1\right)P\left(\frac{1}{\sqrt{t_k}}|b^i(t_k)|<\frac{3M}{\sqrt{t_k}},\ 1\leqq i\leqq d\right)\\&\leqq\left(\frac{dt_k}{2M^2}+1\right)\left(\frac{6M}{\sqrt{2\pi t_k}}\right)^d\leqq C_1k^{-3(d-2)/8}\end{aligned}$$

by virtue of Proposition 1. By Proposition 3, the second term of (4.1) is dominated by

$$d\left(\frac{M^2}{2(t_{k+1}-t_k)}+4\right)\exp\left(-\frac{M^2}{2(t_{k+1}-t_k)}\right)\leqq C_2k^{1/4}\exp(-C_3k^{1/4})\,.$$

Hence $\sum_{k=1}^{\infty}\gamma_k<\infty$ and we see from (1.7) that, for q.e. $w\in W$, there exists $k_0=k_0(w)$ and $\sqrt{\sum_{i=1}^{d}b^i(t)^2}>M$ for any $t>t_{k_0}$. q.e.d.

[4.3] **Absence of double points** (proof of Theorem 8).

Assume $d\geqq 7$. We still follow [5] to consider disjoint intervals $I=(s_0,s_1)$ and $J=(t_0,t_1)$ with $s_1<t_0$ and let $\gamma=\mathrm{Cap}(b^i(s)=b^i(t),\ 1\leqq i\leqq d,$ for some $s\in I$ and $t\in J)$. Then, for $\eta>0$,

$$\begin{aligned}\gamma&\leqq\mathrm{Cap}(|b^i(s_1)-b^i(t_0)|<2\eta,\ i=1,2,\cdots,d)\\&\quad+\sum_{i=1}^{d}\mathrm{Cap}(\max_{s\in I}|b^i(s_1)-b^i(s)|>\eta)+\sum_{i=1}^{d}\mathrm{Cap}(\max_{t\in J}|b^i(t)-b^i(t_0)|>\eta)\\&\leqq\frac{d(t_0-s_1)+2\eta^2}{2\eta^2}\left(\frac{6\eta}{\sqrt{2\pi(t_0-s_1)}}\right)^d\\&\quad+d\left(\frac{\eta^2}{2(s_1-s_0)}+4\right)\exp\left(-\frac{\eta^2}{2(s_1-s_0)}\right)\\&\quad+d\left(\frac{\eta^2}{2(t_1-t_0)}+4\right)\exp\left(-\frac{\eta^2}{2(t_1-t_0)}\right),\end{aligned}\tag{4.2}$$

where we have used Proposition 1, the second inequality of Proposition 3 and the first one of Proposition 3 successively.

Divide I and J into p-subintervals of equal length:

$$I=\sum_{k=1}^{p} I_k, \quad J=\sum_{k=1}^{p} J_k, \quad |I_k|=\frac{|I|}{p}, \quad |J_k|=\frac{|J|}{p}, \quad k=1, 2, \cdots, p.$$

Applying the above inequality to I_k and J_l, we get

$$\gamma \leqq \sum_{k=1}^{p} \sum_{l=1}^{p} \mathrm{Cap}\,(b^i(s)=b^i(t),\ 1\leqq i\leqq d, \text{ for some } s\in I_k \text{ and } t\in J_l)$$

$$\leqq p^2\Big\{\frac{6^d(d(t_1-s_0)+2\eta^2)}{2(2\pi(t_0-s_1))^{d/2}}\eta^{d-2}+d\Big(\frac{p\eta^2}{2(s_1-s_0)}+4\Big)$$

$$\times\exp\Big(-\frac{p\eta^2}{2(s_1-s_0)}\Big)+d\Big(\frac{p\eta^2}{2(t_1-t_0)}+4\Big)\exp\Big(-\frac{p\eta^2}{2(t_1-t_0)}\Big)\Big\}$$

which tends to zero if we set $\eta=p^{-\sigma}$ with $\frac{2}{d-2}<\sigma<\frac{1}{2}$ and let $p\to\infty$.

q. e. d.

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A TRANSFORMATION OF A SYMMETRIC MARKOV PROCESS AND THE DONSKER-VARADHAN THEORY

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1. Introduction

In obtaining the lower bound in the celebrated law of large deviation of the occupation distribution for the one dimensional Brownian motion, Donsker and Varadhan [3] performed a transformation of the absorbing Brownian motion on an interval (a, b) by a drift $b=(\log \rho)'$, or equivalently, by a multiplicative functional

$$\frac{\rho(B_t)}{\rho(B_0)}\exp\left(-\int_0^t \frac{\rho''}{2\rho}(B_s)ds\right)I_{\{t<\zeta\}}$$

into a conservative process on (a, b) with invariant probability measure $\rho^2 dx$, to which the ergodic theorem was well applied. Here ρ^2 is assumed to be a probability density C^2-function on R^1, positive inside (a, b) and vanishing outside.

We show in this paper that their method works for any symmetric Hunt process corresponding to a regular and irreducible Dirichlet form. In the present general case, we take function ρ from the range of the resolvent. In order to prove the conservativeness of transformed process, we make full use of an explicit expression of the transformed Dirichlet form, while the Feller test of non-explosion was available in the special case of [3].

We consider a locally compact separable metric space X and a positive Radon measure m on X such that $\mathrm{Supp}[m]=X$. The inner product in real L^2 space $L^2(X; m)$ is denoted by $(\ , \ )$ and $C_0(X)$ stands for the space of continuous functions on X with compact support. Let $M=(X_t, P_x, \zeta)$ be a Hunt process on X which is *m-symmetric* in the sense that the transition function P_t satisfies $(P_t f, g)=(f, P_t g)$, $f, g \in C_0(X)$. Then the *Dirichlet form* E of M can be defined by $F=D[E]=D(\sqrt{-A})$, $E(u, v)=(\sqrt{-A}u, \sqrt{-A}v)$ where A is the infinitesmal generator of the semigroup on $L^2(X; m)$ determined by P_t. We always assume that E is regular: $F \cap C_0(X)$ is dense both in F and in $C_0(X)$,

In § 2, we derive the Beurling-Deny formula

$$E(u, v) = \frac{1}{2}\int_X d\mu^c_{<u,v>} + \int_{X\times X-d} (\tilde{u}(x)-\tilde{u}(y))(\tilde{v}(x)-\tilde{v}(y))J(dx, dy) + \int_X \tilde{u}\tilde{v}k\,, \quad u, v\in F\,,$$

together with the derivation property of the local energy measure $\mu^c_{<u,v>}$ due to Le Jan ([11]). Our approach is more comprehensive than [6] and [11], and indeed we first decompose the martingale part $M^{[u]}_t$ of the additive functional $\tilde{u}(X_t)-\tilde{u}(X_0)$ and then compute the energy of each term.

In § 3, we perform the above mentioned transformation of $\boldsymbol{M}$ to get a $\rho^2 m$-symmetric process $\tilde{\boldsymbol{M}}$. Using the fact that the multiplicative functional involved is a solution of a Doléans-Dade equation related to the martingale $M^{[\rho]}$, the Dirichlet form $\tilde{E}$ of $\tilde{\boldsymbol{M}}$ is shown to have the expression

$$\tilde{E}(u, v) = \frac{1}{2}\int_X \rho^2 d\mu^c_{<u,v>} + \int_{X\times X-d} \tilde{u}((x)-\tilde{u}(y))(\tilde{v}(x)-\tilde{v}(y))\rho(x)\rho(y)J(dx, dy)\,.$$

We can then conclude that $1\in D[\tilde{E}]$ and $\tilde{E}(1, 1)=0$, which simply means the conservativeness of $\tilde{\boldsymbol{M}}$.

In § 4, we assume that there exist relatively compact open sets G_n increasing to X and the part of the Dirichlet form E to each set G_n is irreducible. Let

$$L(t, A) = \frac{1}{t}\int_0^t I_A(X_s)ds\,, \qquad t<\zeta\,,$$

then the set function $L(t, \cdot)$ called the *occupation distribution* is an element of the space $\mathcal{M}$ of probability measures on X. We then have just as in [3] the estimate

$$\varliminf_{t\to\infty} \frac{1}{t} \log \boldsymbol{E}_x(e^{-t\Phi(L(t,\cdot))};\, t<\zeta) \geqq -\inf_{\rho\in V}[\Phi(\rho^2)+E(\rho, \rho)]\,,$$

where $V=\{\rho\in F;\ \rho^2 m\in\mathcal{M},\ \mathrm{Supp}[\rho^2 m]$ is compact$\}$ and Φ is any functional on $\mathcal{M}$ such that $\Phi(\mu_n)\to\Phi(\mu)$ whenever $\mu_n\in\mathcal{M}$ converge weakly to μ and the support of μ_n is contained in a common compact set. The above inequality holds for every $x\in X$ except possibly on a set of zero capacity which is independent of Φ.

Since any regular Dirichlet form admits an associated symmetric Hunt process, we may say that the present lower estimate is an intrinsic property of Dirichlet form which is regular and irreducible. These two conditions on Dirichlet form are very mild and directly verifiable. We do not assume the Feller property nor the absolute continuity of the associated transition function, although the transition function is always symmetric in our setting.

For instance, consider locally integrable functions $a_{i,j}(x)$, $1\leqq i, j\leqq d$, on the d-space R^d satisfying $a_{i,j}=a_{j,i}$ and $\inf_{x\in K, |\xi|=1} \sum_{i,j=1}^{d} a_{i,j}(x)\xi_i\xi_j>0$ for any compact set K, then the form

$$E(u, v)=\sum_{i,j=1}\int_{R^d}\frac{\partial u}{\partial x_i}(x)\frac{\partial v}{\partial x_j}(x)a_{i,j}(x)dx\,, \qquad u, v\in C_0^1(R^d)\,,$$

is closable on $L^2(R^d)$ and the closure $\bar{E}$ becomes a regular and irreducible Dirichlet form on $L^2(R^d)$. Here $C_0^1(R^d)$ is the space of C^1-functions in $C_0(R^d)$. More generally, we may replace $a_{i,j}(x)dx$ by a Radon measure $\nu_{i,j}$ such that $\nu_{i,j}=\nu_{j,i}$ and $\inf_{|\xi|=1} \sum_{i,j=1}^{d} \nu_{i,j}(K)\xi_i\xi_j \geqq \delta_K |K|$ for any compact K where δ_K is positive constant and $|K|$ is the Lebesgue measure of K. Under the closability assumption of the associated form E on $C_0^1(R^d)$, we have a same kind of form $\bar{E}$. We can instead start with a jumping measure of the type $J(dx, dy)=N(x, dy)dx$ which makes the associated symmetric form

$$E(u, v)=\int_{R^d x R^d-d}(u(x)-u(y))\,(v(x)-v(y))N(x, dy)dx\,, \qquad u, v\in C_0^1(R^d)\,,$$

closable on $L^2(R^d)$. In this case, it suffices to assume $\mathrm{Supp}[N(x, \cdot)]=R^d$ for almost all x in order to obtain the Dirichlet form $\bar{E}$ possessing the required properties. In each of the above three examples, the stated lower estimate holds for the Hunt process on R^d associated with $\bar{E}$. See [6] for closability criteria for the second and third examples. We know from the works in theory of partial differential equations due to Nash et al. that the transition function in the first example is Feller and absolutely continuous. We do not know about this for the second and third examples in general. But see Tomisaki [12] for relevant information.

Donsker and Varadhan have extended their result in [3] to wider classes of Markov processes by finding sufficient conditions on the transition functions ([4], [5]). As for the lower estimate in the case of the complete separable metrizable state space, their conditions on the transition function (Hypothesis $H_1 \sim H_4$ in [5]) include the Feller property and an absolutely continuity in addition to a transitivity assumption. The intrinsic quantity appearing in their upper and lower bounds is the I-functional rather than the Dirichlet form, but those two characteristics have been identified when the transition function is symmetric and absolutely continuous ([4]). As for the upper estimate, we only note that the relevant statements in [3] remain true for any Markov process with Feller transition function.

2. Decomposition of martingale additive functionals and the Beurling-Deny formula

Let (E, F) be a C_0-regular Dirichlet space and $\boldsymbol{M}=(X_t, \boldsymbol{P}_x, \zeta)_{x\in X}$ be the associated m-symmetric Hunt process, and we use the relevant notions and notations in Fukushima [6], Dellacherie-Meyer [1]. By Theorem 5.2.2 in [6], the additive functional (AF in abbriviation) $A_t^{[u]}=\tilde{u}(X_t)-\tilde{u}(X_0)$, $u\in F$, has a unique decomposition

$$A^{[u]}=M^{[u]}+N^{[u]} \tag{2.1}$$

where $M^{[u]}$ is a square integrable martingale AF of finite energy and $N^{[u]}$ is a continuous AF of zero energy. We further decompose $M^{[u]}$ as

$$M^{[u]}=\overset{c}{M}{}^{[u]}+\overset{d}{M}{}^{[u]}=\overset{c}{M}{}^{[u]}+\overset{j}{M}{}^{[u]}+\overset{k}{M}{}^{[u]} \tag{2.2}$$

where $\overset{c}{M}{}^{[u]}$ and $\overset{d}{M}{}^{[u]}$ are continuous and purely discontinuous part of $M^{[u]}$ respectively, and $\overset{j}{M}{}^{[u]}$ and $\overset{k}{M}{}^{[u]}$ are defined by

$$\overset{k}{M}{}^{[u]}=-\overset{c}{\widehat{\tilde{u}(X_{\zeta-})I_{\{\zeta\leq t\}}}}\,,\quad \overset{j}{M}{}^{[u]}=\overset{d}{M}{}^{[u]}-\overset{k}{M}{}^{[u]}\,. \tag{2.3}$$

Here, for an additive functional A, it's compensator $\overset{c}{\widehat{A}}$ is defined by $A-A^p$ with A^p being the dual predictable projection of A (see [1; Definition 73]). Using the co-energy e of AF's, we can define three symmetric forms by

$$\begin{aligned} &E^{(c)}(u,v)=e(\overset{c}{M}{}^{[u]},\overset{c}{M}{}^{[v]})\quad E^{(j)}(u,v)=e(\overset{j}{M}{}^{[u]},\overset{j}{M}{}^{[v]})\\ &E^{(k)}(u,v)=2e(\overset{k}{M}{}^{[u]},\overset{k}{M}{}^{[v]})\,. \end{aligned} \tag{2.4}$$

Let $(N(x,dy),H)$ be a Lévy system of Hunt process $\boldsymbol{M}$ and ν be a smooth measure corresponding to H. We put

$$J(dx,dy)=\frac{1}{2}N(x,dy)\nu(dx)\,,\quad k(dx)=N(x,\Delta)\nu(dx) \tag{2.5}$$

and call them the *jumping measure* and the *killing measure* respectively. Since $\langle\overset{j}{M}{}^{[u]},\overset{j}{M}{}^{[v]}\rangle_t=[M^{[u]},M^{[v]}]_t^p=\int_0^t\int_X(\tilde{u}(X_s)-\tilde{u}(y))(\tilde{v}(X_s)-\tilde{v}(y))N(x,dy)dH_s$ we have

$$E^{(j)}(u,v)=\int_{X\times X-d}(\tilde{u}(x)-\tilde{u}(y))(\tilde{v}(x)-\tilde{v}(y))J(dx,dy)\,. \tag{2.6}$$

Further, since $\langle\overset{k}{M}{}^{[u]},\overset{k}{M}{}^{[v]}\rangle_t=\int_0^t\tilde{u}(X_s)\tilde{v}(X_s)N(X_s,\Delta)dH_s$, we have

$$E^{(k)}(u,v)=\int_X\tilde{u}(x)\tilde{v}(x)k(dx)\,. \tag{2.7}$$

Lemma 2.1. 1) *For any* $u\in F^+$ ($u\in F$ *and* $u\geqq 0$, *m-a.e.*) *and* $\alpha>0$, $\tilde{u}\cdot k$ *is of finite energy integral and* $\boldsymbol{E}_x(e^{-\alpha\zeta}u(X_{\zeta-}))$ *is a quasi-continuous version of* $U_\alpha(\tilde{u}\cdot k)$.

2) *For* $u \in F$

$$\lim_{t \downarrow 0} \frac{1}{t} \boldsymbol{E}_{u^2 m}(I_{\{\zeta \leqq t\}}) = \langle k, \tilde{u}^2 \rangle . \tag{2.8}$$

Proof. 1) Since (2.2) is an orthogonal decomposition with respect to e, $\frac{1}{2} E^{(k)}(u, u)$ is dominated by $e(M^{[u]}, M^{[u]}) \leqq E(u, u)$. Therefore, by Schwarz inequality and (2.7),

$$\int_X |f| \tilde{u} dk \leqq C \sqrt{E(f, f)}\,, \qquad f \in F \cap C_0$$

which means $\tilde{u} \cdot k$ is of finite energy integral. By noticing that $\boldsymbol{E}_x(e^{-\alpha\zeta} \tilde{u}(X_{\zeta-}))$ equals $\boldsymbol{E}_x(\int_0^\infty e^{-\alpha s} \tilde{u}(X_s) d(I_{\{X_{\zeta-} \neq \Delta, \zeta \leqq s\}^p}))$, we get the second assertion of 1).
2) For any $h \in C_0^+$ and $f \in F_b^+$,

$$\begin{aligned} \boldsymbol{E}_{R_\alpha h \cdot fm}(I_{\{\zeta \leqq t\}}) &= (h, R_\alpha(f \cdot (1 - P_t 1))) \\ &= \boldsymbol{E}_{hm}(\int_{\zeta - t}^{\zeta} e^{-\alpha s} \tilde{f}(X_s) ds;\ t < \zeta) + \boldsymbol{E}_{hm}(\int_0^\zeta e^{-\alpha s} \tilde{f}(X_s) ds;\ t \geqq \zeta)\,. \end{aligned}$$

But, since $\frac{1}{t^2} \boldsymbol{E}_{hm}(\int_0^\zeta e^{-\alpha s} \tilde{f}(X_s) ds;\ t \geqq \zeta)^2 \leqq \frac{1}{t} \boldsymbol{E}_{hm}((\int_0^t e^{-\alpha s} \tilde{f}(X_s) ds)^2) \cdot \frac{1}{t} \boldsymbol{P}_{hm}(t \geqq \zeta)$ $\to 0$ $(t \to 0)$, we have

$$\lim_{t \downarrow 0} \frac{1}{t} \boldsymbol{E}_{R_\alpha h \cdot fm}(I_{\{\zeta \leqq t\}}) = \boldsymbol{E}_{hm}(e^{-\alpha\zeta} \tilde{f}(X_{\zeta-})) = (h, U_\alpha(\tilde{f} \cdot k)) = \langle k, R_\alpha h \cdot \tilde{f} \rangle.$$

In particular, $\lim_{t \downarrow 0} \frac{1}{t} \boldsymbol{E}_{(R_\alpha h)^2 m}(I_{\{\zeta \leqq t\}}) = \langle k, (R_\alpha h)^2 \rangle$. We can prove the relation (2.8) for general $u \in F$ in the same way as in [6; Lemma 4.5.2]. q.e.d.

Denote by $\mu_{\langle u \rangle}$, $\mu^c_{\langle u \rangle}$, and $\mu^d_{\langle u \rangle}$ the smooth measure of $\langle M^{[u]} \rangle$, $\langle \overset{c}{M}{}^{[u]} \rangle$ and $\langle \overset{d}{M}{}^{[u]} \rangle$ respectively, $u \in F$.

Lemma 2.2. *If* $u \in F$ *is constant on an open set* G, *then* $\mu^c_{\langle u \rangle} = 0$ *on* G.

Proof. Define $B_t^{(n)} = \sum_{k=1}^n (\tilde{u}(X_{\frac{k}{n}t}) - \tilde{u}(X_{\frac{k-1}{n}t}))^2$, then $B_t^{(n)}$ equals zero on $t < \tau_G = \inf\{t;\ X_t \notin G\}$. On the other hand, $\lim_{n \to \infty} B_t^{(n)} = [M^{[u]}]_t$, $t < \tau_G$, $\boldsymbol{P}_m$-a.e., because of the property of $N^{[u]}$ that $\sum_{k=1}^n (N^{[u]}_{\frac{k}{n}t} - N^{[u]}_{\frac{k-1}{n}t})^2$ tends to zero in $L^1(\boldsymbol{P}_m)$. Since $[M^{[u]}]_t = \langle \overset{c}{M}{}^{[u]} \rangle_t + \sum_{0 < s \leqq t} (\tilde{u}(X_s) - \tilde{u}(X_{s-}))^2$,

$$\langle \overset{c}{M}{}^{[u]} \rangle_t = 0\,, \quad t \leqq \tau_G\,, \quad \boldsymbol{P}_m\text{-a.e.} \tag{2.9}$$

which implies the lemma by virtue of [6; Lemma 5.1.5]. q.e.d.

Lemma 2.3. *For any* $u, v \in F \cap C_0^+$ *such that* $\mathrm{Supp}[u] \cap \mathrm{Supp}[v] = \phi$,

$$\int_{X \times X - d} u(x) v(x) J(dx, dy) = -\frac{1}{2} E(u, v). \tag{2.10}$$

Proof. Consider a relatively compact open set satisfying $\mathrm{Supp}[u] \subset G \subset \bar{G} \subset (\mathrm{Supp}[v])^c$. We put $Nf(x) = \int_X f(y) N(x, dy)$. Since

$$e(\overset{j}{M}{}^{[|f|]}, \overset{j}{M}{}^{[v]}) = -\frac{1}{2}\left(\int_X v \cdot N|f| \cdot \nu + \int_X |f| \cdot Nv \cdot \nu\right) \qquad \text{for } f \in F_G \cap C_0,$$

we have

$$\int_X |f| \cdot Nv \cdot \nu \leqq 2e(\overset{j}{M}{}^{[|f|]})^{1/2} \cdot e(\overset{j}{M}{}^{[v]})^{1/2} \leqq C\sqrt{E(f, f)},$$

and hence $I_G \cdot Nv \cdot \nu$ is of finite energy integral with respect to E_G. We then have $H_\alpha^{X-G} v(x) = E_x\left(\int_0^{\tau_G} e^{-\alpha s} Nv(X_s) dH_s\right) = U_\alpha^G(Nv \cdot \nu)$ on G and $H_\alpha^{X-G} v = U^G(Nv \cdot \nu) + v$. It follows from $E_\alpha(H_\alpha^{X-G} v, u) = 0$ that

$$E(U_\alpha^G(Nv \cdot \nu), u) = 2\int_{X \times X - d} u(x) v(y) J(dx, dy) = -E(u, v). \qquad \text{q.e.d.}$$

Theorem 2.1 (Representation of the Dirichlet form E). *It holds that*

$$E(u, v) = E^{(c)}(u, v) + E^{(j)}(u, v) + E^{(k)}(u, v), \qquad u, v \in F. \tag{2.11}$$

$E^{(c)}$ *has the local property*; $E^{(c)}(u, v) = 0$ *whenever* u *is constant on the support of* v. $E^{(j)}$ *and* $E^{(k)}$ *are expressed by* (2.6) *and* (2.7) *respectively. Moreover the measure* J *is symmetric.*

Proof. By virtue of Lemma 2.1,

$$\begin{aligned} e(M^{[u]}) &= \lim_{t \downarrow 0} \frac{1}{2t} E_m((\tilde{u}(X_t) - \tilde{u}(X_0))^2) \\ &= \lim_{t \downarrow 0} \frac{1}{t}\left\{(u, u - P_t u) - \frac{1}{2}(u^2, 1 - P_t 1)\right\} \\ &= E(u, u) - \frac{1}{2}\int_X \tilde{u}^2 dk, \end{aligned}$$

and we have (2.11) in view of (2.2) and (2.4). Other assertions follow from Lemma 2.2 and Lemma 2.3. q.e.d.

This theorem says that formula (2.11) is nothing but the Beurling-Deny formula ([6]), and moreover the symmetric measure J admits a specific expression (2.5).

We put $\mu_{\langle u,v\rangle}=\frac{1}{4}(\mu_{\langle u+v\rangle}-\mu_{\langle u-v\rangle})$, $u, v\in F$. $\overset{c}{\mu}_{\langle u,v\rangle}$ and $\overset{d}{\mu}_{\langle u,v\rangle}$ are defined in the same way.

Lemma 2.4. *It holds that*

$$d\overset{c}{\mu}_{\langle u^2,v\rangle}=2ud\overset{c}{\mu}_{\langle u,v\rangle} \qquad \text{for } u, v\in F\cap C_0 \tag{2.12}$$

Proof. By the same method as in [6; Lemma 5.4.1] we have

$$\begin{aligned}\int_X fd\mu_{\langle u^2,v\rangle}-2\int_X fud\mu_{\langle u,v\rangle} \\ =2\int_{X\times X-d}(u(x)-u(y))^2(v(x)-v(y))f(x)J(dx, dy)-\int_X fu^2vk\,.\end{aligned} \tag{2.13}$$

According to the representation theorem, $\int_X fd\overset{d}{\mu}_{\langle u^2,v\rangle}-2\int_X fud\overset{d}{\mu}_{\langle u,v\rangle}$ equals the right hand side of (2.13), and consequently $\int_X fd\overset{c}{\mu}_{\langle u^2,v\rangle}=2\int_X fud\overset{c}{\mu}_{\langle u,v\rangle}$. q.e.d.

Theorem 2.2 (derivation property of $\overset{c}{\mu}_{\langle u\rangle}$).

$$d\overset{c}{\mu}_{\langle uv,w\rangle}=\tilde{u}d\overset{c}{\mu}_{\langle v,w\rangle}+\tilde{v}d\overset{c}{\mu}_{\langle u,w\rangle} \qquad \text{for } u, v, w\in F_b\,. \tag{2.14}$$

Proof. This follows from Lemma 2.4 in the same way as in [6; Lemma 5.4.2]. q.e.d.

3. A transformation by a multiplicative functional into a conservative process

In § 2, we have proved that C_0-regular Dirichlet form E on $L^2(X; m)$ can be represented as

$$\begin{aligned}E(u, v)=\frac{1}{2}\int_X d\overset{c}{\mu}_{\langle u,v\rangle}+\int_{X\times X-d}(\tilde{u}(x)-\tilde{u}(y))(\tilde{v}(x)-\tilde{v}(y))J(dx, dy) \\ +\int_X \tilde{u}(x)\tilde{v}(x)k(dx)\,.\end{aligned} \tag{3.1}$$

We introduce the space

$$\mathcal{X}=\left\{\rho\,;\ \begin{matrix}\rho=R_\alpha g,\ \alpha>0,\ g \text{ is a strictly positive function in} \\ C_b\cap L^1(m),\ \int_X \rho^2dm=1\end{matrix}\right\}$$

Take $\rho=R_\alpha g\in\mathcal{X}$. Then, $\rho(X_t)-\rho(X_0)$ can be expressed as $M^{[\rho]}+N^{[\rho]}=\overset{c}{M}{}^{[\rho]}+\overset{d}{M}{}^{[\rho]}+N^{[\rho]}$, because $\rho\in F$. We consider the transformation of $\boldsymbol{M}$ by the multiplicative functional

$$(3.2)\qquad L_t=\exp\Big(\int_0^t\frac{1}{\rho(X_{s-})}dM_s^{[\rho]}-\frac{1}{2}\int_0^t\frac{1}{\rho^2(X_s)}d\langle M^{c[\rho]}\rangle_s\Big)$$
$$\times\prod_{\substack{0<s\leq t\\ X_{s-}\neq X_s}}\frac{\rho(X_s)}{\rho(X_{s-})}\exp\Big(-\Big(\frac{\rho(X_s)}{\rho(X_{s-})}\Big)-1\Big)I_{\{t<\zeta\}}\,.$$

Denote by $\tilde{M}$ the transformed Markov process. The transition function $\tilde{P}_t$ of $\tilde{M}$ has the expression

$$\tilde{P}_t g(x)=E_x(L_t g(X_t))\,,\qquad x\in X\,.$$

REMARK 3.1. L_t is a solution of Doléans-Dade equation ([2])

$$(3.3)\qquad Z_t-1=\int_0^t Z_{s-}\frac{1}{\rho(X_{s-})}dM_s^{[\rho]}\,,\qquad t<\zeta\,,$$

and L_t admits a simpler expression

$$(3.4)\qquad L_t=\frac{\rho(X_t)}{\rho(X_0)}\exp\Big(-\int_0^t\frac{A\rho}{\rho}(X_s)ds\Big)I_{\{t<\zeta\}}$$

by virtue of Ito formula applied to the semi-martingale $\rho(X_t)$. Here, $A\rho=\alpha\rho-g$.

Theorem 3.1. 1) *$\tilde{M}$ is ρ^2m-symmetric.*
2) *Let $(\tilde{E},\tilde{F})$ be the Dirichlet space generated by $\tilde{M}$, then $F\subset\tilde{F}$ and for $u, v\in F$*

$$(3.5)\qquad \tilde{E}(u,v)=\frac{1}{2}\int_X\rho^2 d\mu^c_{\langle u,v\rangle}+\int_{X\times X-d}(\tilde{u}(x)-\tilde{u}(y))(\tilde{v}(x)-\tilde{v}(y))$$
$$\times\rho(x)\rho(y)J(dx,dy)\,.$$

Proof. 1) Let $\{K_n\}_{n=1}^\infty$ be a compact nest satisfying $\inf_{x\in K_n}\rho(x)\geq r_n>0$ and $\overset{0}{K}_n$ be a fine interior of K_n, and denote by $M^{(n)}$ (resp. $\tilde{M}^{(n)}$) the part of M (resp. $\tilde{M}$) on $\overset{0}{K}_n$.

For any $f, g\in\beta_b^+$

$$((\tilde{P})_t^{(n)}g,f)_{\rho^2m}=(E_\cdot^{(n)}(L_t g(X_t)),f)_{\rho^2m}$$
$$=\Big(E_\cdot^{(n)}\Big(\frac{\rho(X_t)}{\rho(X_0)}\exp\Big(-\int_0^t\frac{A\rho}{\rho}(X_s)ds\Big)g(X_t)\Big),f\rho^2\Big)_m$$
$$=E_m^{(n)}\Big(f(X_0)\rho(X_0)\exp\Big(-\int_0^t\frac{A\rho}{\rho}(X_s)ds\Big)g(X_t)\rho(X_t)\Big)$$
$$=\lim_{k\to\infty}E_m^{(n)}\Big(f(X_0)\rho(X_0)\prod_{p=0}^{k-1}\exp\Big(-\frac{A\rho}{\rho}(X_{\frac{p}{k}t})\frac{t}{k}\Big)g(X_t)\rho(X_t)\Big)\,.$$

On account of symmetry of $M^{(n)}$, the last expression equals

$\lim_{k\to\infty} \boldsymbol{E}_m^{(n)}(g(X_0)\rho(X_0)\prod_{p=0}^{k-1}\exp\Big(-\frac{A\rho}{\rho}(X_{\frac{p-1}{k}})\frac{t}{k}\Big)f(X_t)\rho(X_t)\Big)$ by [6; Lemma 4.2.2]. Hence, we have

$$((\tilde{P})_t^{(n)}g, f)_{\rho^2 m} = (g, (\tilde{P})_t^{(n)}f)_{\rho^2 m}$$

and by letting $n\to\infty$, we get equality

$$((\tilde{P})_t g, f)_{\rho^2 m} = (g, (\tilde{P})_t f)_{\rho^2 m}\,.$$

2) First of all, we prove this for $u=R_1^{(n)}g$ $(g\in C_0)$. We let

$$\begin{aligned}(3.6)\qquad &(u-\boldsymbol{E}_\cdot^{(n)}(L_t u(X_t)),\, u)_{\rho^2 m}\\ &= (u-P_t^{(n)}u,\, u)_{\rho^2 m}-(\boldsymbol{E}_\cdot^{(n)}((L_t-1)u(X_t)),\, u)_{\rho^2 m}\\ &= (I)_t-(II)_t\,.\end{aligned}$$

Since $L_t-1=\frac{1}{\rho(X_0)}\int_0^t \exp\Big(-\int_0^s\frac{A\rho}{\rho}(X_u)du\Big)dM_s^{[\rho]}$, $t<\zeta$, by Remark 3.1, $(II)_t$ equals

$$\boldsymbol{E}_{u\rho m}\Big(u(X_t)\int_0^t \exp\Big(-\int_0^s\frac{A\rho}{\rho}(X_u)du\Big)dM_s^{[\rho]};\ t<\tau^{(n)}\Big)\,.$$

If we set

$$(III)_t = \boldsymbol{E}_{u\rho m}\Big(u(X_t)\int_0^{t\wedge\tau^{(n)}} \exp\Big(-\int_0^s\frac{A\rho}{\rho}(X_u)du\Big)dM_s^{[\rho]};\ t\geqq\tau^{(n)}\Big)$$

then, by Schwarz inequality,

$$\begin{aligned}\frac{1}{t^2}(III)_t^2 \leqq &\frac{1}{t}\boldsymbol{E}_{|u|\rho m}\Big(\int_0^{t\wedge\tau^{(n)}} \exp\Big(-2\int_0^s\frac{A\rho}{\rho}(X_u)du\Big)d\langle M^{[\rho]}\rangle_t\Big)\\ &\times\frac{1}{t}\boldsymbol{E}_{|u|\rho m}(u(X_t)^2;\ t\geqq\tau^{(n)})\,.\end{aligned}$$

But the first factor of the right hand side is not greater than

$$\frac{1}{t}e^{ct}\boldsymbol{E}_{|u|\rho m}(\langle M^{[\rho]}\rangle_{t\wedge\tau^{(n)}})\leqq C'\frac{1}{t}\boldsymbol{E}_m(\langle M^{[\rho]}\rangle_t)\leqq 2C'E(\rho,\rho)\,,$$

and the second factor equals

$$\frac{1}{t}(|u|\rho,\, P_t u^2-u^2)_m-\frac{1}{t}(|u|\rho,\, P_t^{(n)}u^2-u^2)_m \xrightarrow[t\to 0]{} E(|u|\rho,\, u^2)-E(|u|\rho,\, u^2)=0\,.$$

Hence we get

$$(3.7)\qquad \frac{1}{t}(III)_t\to 0\qquad (t\to 0)\,.$$

This implies

$$\lim_{t\downarrow 0}\frac{1}{t}(II)_t=\lim_{t\downarrow 0}\frac{1}{t}\boldsymbol{E}_{u\rho m}\Big(u(X_t)\int_0^{t\wedge\tau^{(n)}}\exp\Big(-\int_0^s\frac{A\rho}{\rho}(X_u)du\Big)dM_s^{[\rho]}\Big),$$

which in turn equals

$$\lim_{t\downarrow 0}\frac{1}{t}\boldsymbol{E}_{u\rho m}\Big((u(X_t)-u(X_0))\int_0^{t\wedge\tau^{(n)}}\exp\Big(-\int_0^s\frac{A\rho}{\rho}(X_u)du\Big)dM_s^{[\rho]}\Big)$$
$$=\lim_{t\downarrow 0}\frac{1}{t}\boldsymbol{E}_{u\rho m}\Big(\int_0^{t\wedge\tau^{(n)}}\exp\Big(-\int_0^s\frac{A\rho}{\rho}(X_u)du\Big)d\langle M^{[u]},M^{[\rho]}\rangle_s\Big).$$

Noticing that there is a constant N such that

$$\left|\frac{\exp\Big(-\int_0^s\frac{A\rho}{\rho}(X_u)du\Big)-1}{t}\right|\leqq N\qquad\text{for } s\leqq t\wedge\tau^{(n)},$$

we have

$$\text{(3.8)}\qquad \lim_{t\downarrow 0}\frac{1}{t}(II)_t=\lim_{t\downarrow 0}\frac{1}{t}\boldsymbol{E}_{u\rho m}(\langle M^{[u]},M^{[\rho]}\rangle_{t\wedge\tau^{(n)}})=\int_X u\rho\, d\mu_{\langle\rho,u\rangle}.$$

Last equality holds by [6; Lemma 5.1.5].

On the other hand, we have

$$\text{(3.9)}\qquad \lim_{t\downarrow 0}\frac{1}{t}(I)_t=E(u,u\rho^2).$$

From (3.6), (3.8) and (3.9), we have finally that $u\in\tilde{F}$ and

$$\text{(3.10)}\qquad \tilde{E}(u,u)=E(u,u\rho^2)-\int_X u\rho\, d\mu_{\langle\rho,u\rangle}.$$

We can see that the right hand side of (3.10) equals the right hand side of (3.5). To see this, rewrite the right hand side of (3.10) as a sum of two terms I and II where

$$I=\frac{1}{2}\int_X d\mu^c_{\langle u,u\rho^2\rangle}-\int_X u\rho\, d\mu^c_{\langle\rho,u\rangle}$$
$$II=\int_{X\times X-d} w(x,y)J(dx,dy)$$

with
$$w(x,y)=(u(x)-u(y))(u(x)\rho^2(x)-u(y)\rho^2(y))$$
$$-2u(x)\rho(x)(\rho(x)-\rho(y))(u(x)-u(y)).$$

Note that the contribution from the killing parts cancels out at this stage. By the derivation property, we easily see that

$$I = \frac{1}{2}\int_X \rho^2 d\tilde{\mu}^c_{\langle u\rangle}.$$

On the other hand, since $w(x, y)-(u(x)-u(y))^2\rho(x)\rho(y)=(u(x)-u(y))\times(\rho(x)-\rho(y))u(y)\rho(y)-(u(x)-u(y))(\rho(x)-\rho(y))u(x)\rho(x)$ and the measure J is symmetric,

$$II = \int_{X\times X-d}(u(x)-u(y))^2\rho(x)\rho(y)J(dx, dy)\,.$$

We have proved (3.5) for $u=R_1^{(n)}g$, $g\in C_0$, and consequently $\tilde{E}_1(u, u)\leqq \|\rho\|_\infty^2 E_1(u, u)$ for $u=R_1^{(u)}g$, $g\in C_0$. Now we can get the conclusion of Theorem 3.1 because $\lim_{n\to\infty} R_1^{(n)}g=R_1g$ in E_1-norm for $g\in C_0$ and $\{R_1g, g\in C_0\}$ is a dense subset of F. q.e.d.

Lemma 3.1. *If B is a nearly Borel set and $v\in F_b$ satisfies $\inf_{x\in B}|v(x)|=\delta>0$, then $\frac{u}{v}\in F_B$ for $u\in F_B^b$.*

Proof. Since

$$\left|\frac{u(x)}{v(x)}\right|\leqq\frac{1}{\delta}|u(x)|,\ \left|\frac{u(x)}{v(x)}-\frac{u(y)}{v(y)}\right|\leqq\frac{2}{\delta}|u(x)-u(y)|+\frac{1}{\delta}|u(x)v(x)-u(y)v(y)|,$$

and $uv\in F_B$, we can conclude $\frac{u}{v}\in F_B$. q.e.d.

We may assume that $\tilde{\boldsymbol{M}}$ is a standard process on X and has the elements Ω, X_t, ζ in common with $\boldsymbol{M}$. We then have ([10])

$$\tilde{\boldsymbol{P}}_x(\Lambda; t<\zeta) = \boldsymbol{E}_x(L_t; \Lambda)\,,\qquad \Lambda\in\beta_t,\ x\in X\,, \tag{3.11}$$

where β_t is the σ-field of Ω generated by X_s, $s\leqq t$. It follows from (3.11) that the notion of the exceptionality of sets and consequently the q.e. statements are the same for $\boldsymbol{M}$ and $\tilde{\boldsymbol{M}}$.

Theorem 3.2. *$\tilde{\boldsymbol{M}}$ is conservative: $\tilde{\boldsymbol{P}}_x(\tilde{\zeta}<+\infty)=0$, q.e. $x\in X$.*

Proof. It suffices to prove that $1\in\tilde{F}$ and $\tilde{E}(1, 1)=0$ ([7]). Let us put $h_n=\frac{R_\alpha^{(n)}g}{R_\alpha g}$. By Theorem 3.1 and Lemma 3.1,

$$h_n\in F_{K_n}^0\cap\tilde{F}\,. \tag{3.12}$$

Since

$$\begin{aligned} d\tilde{\mu}^c_{\langle R_\alpha^{(n)}g-R_\alpha^{(m)}g\rangle} &= d\tilde{\mu}^c_{\langle R_\alpha g(h_n-h_m)\rangle}\\ &= 2(h_n-h_m)d\tilde{\mu}^c_{\langle R_\alpha g(h_n-h_m), R_\alpha g\rangle}+(R_\alpha g)^2 d\tilde{\mu}^c_{\langle h_n-h_m\rangle}\\ &\quad-(h_n-h_m)^2 d\tilde{\mu}^c_{\langle R_\alpha g\rangle}\,, \end{aligned}$$

we have

$$(3.13)\qquad \tilde{E}^{(c)}(h_n-h_m,\, h_n-h_m)=\frac{1}{2}\int_X d\tilde{\mu}^c_{\langle R_\alpha^{(n)}g-R_\alpha^{(m)}g\rangle}+\frac{1}{2}\int_X (h_n-h_m)^2 d\tilde{\mu}^c_{\langle R_\alpha g\rangle}$$
$$-\int_X (h_n-h_m)d\tilde{\mu}^c_{\langle R_\alpha^{(n)}g-R_\alpha^{(m)}g,R_\alpha g\rangle}.$$

The first and second terms of (3.13) clearly tend to zero by letting n, m to infinite. Since

$$\left|\int_X (h_n-h_m)d\tilde{\mu}^c_{\langle R_\alpha^{(n)}g-R_\alpha^{(m)}g,R_\alpha g\rangle}\right|\leqq\left(\int_X (h_n-h_m)^2 d\tilde{\mu}^c_{\langle R_\alpha g\rangle}\right)^{1/2}\left(\int_X d\tilde{\mu}^c_{\langle R_\alpha^{(n)}g-R_\alpha^{(m)}g\rangle}\right)^{1/2}$$

by Schwarz inequality, the third term also tends to zero.

On the other hand, we get

$$(3.14)\quad \tilde{E}^{(j)}(h_n-h_m,\, h_n-h_m)=\int_{X\times X-d}\frac{(R_\alpha g(x)R_\alpha^{n,m}g(y)-R_\alpha g(y)R_\alpha^{n,m}g(x))^2}{R_\alpha g(x)R_\alpha g(y)}J(dx,dy)$$

where $R_\alpha^{n,m}g=R_\alpha^{(n)}g-R_\alpha^{(m)}g$. Since $|R_\alpha g(x)R_\alpha^{n,m}g(y)-R_\alpha g(y)R_\alpha^{n,m}g(x)|\leqq R_\alpha g(x)$ $|R_\alpha^{n,m}g(y)-R_\alpha^{n,m}g(x)|+|R_\alpha^{n,m}g(x)|\,|R_\alpha g(x)-R_\alpha g(y)|$ and the left hand side is also dominated by $R_\alpha g(y)|R_\alpha^{n,m}g(x)-R_\alpha^{n,m}g(y)|+|R_\alpha^{n,m}g(y)|\,|R_\alpha g(y)-R_\alpha g(x)|$, it holds that

$$\tilde{E}^{(j)}(h_n-h_m,\, h_n-h_m)\leqq\int_{X\times X-d}(R_\alpha^{n,m}g(x)-R_\alpha^{n,m}g(y))^2 J(dx,\, dy)$$
$$+2\int_{X\times X-d}|h_n(x)-h_m(x)|\,|R_\alpha^{n,m}g(x)-R_\alpha^{n,m}g(y)|\,|R_\alpha g(x)-R_\alpha g(y)|\,J(dx,\, dy)$$
$$+\int_{X\times X-d}|h_n(x)-h_m(x)|\,|h_n(y)-h_m(y)|(R_\alpha g(x)-R_\alpha g(y))^2 J(dx,\, dy).$$

The first term of the right hand side clearly tends to zero by letting n, m to infinite and the third term tends to zero by the bounded convergence theorem. Finally we see the second term also tends to zero since it is dominated by

$$2\left(\int_{X\times X-d}(h_n(x)-h_m(x))^2(R_\alpha g(x)-R_\alpha g(y))^2 J(dx,\, dy)\right)^{1/2}$$
$$\times\left(\int_{X\times X-d}(R_\alpha^{n,m}g(x)-R_\alpha^{n,m}g(y))^2 J(dx,\, dy)\right)^{1/2},$$

and this means $\tilde{E}^{(j)}(h_n-h_m,\, h_n-h_m)\to 0$ $(n,\, m\to\infty)$. Therefore, it holds that $\tilde{E}_1(h_n-h_m,\, h_n-h_m)\to 0$ $(n,\, m\to\infty)$. Moreover, because $h_n\to 1$ $(n\to\infty)$, $1\in\tilde{F}$.

Next we get $\lim_{n\to\infty}\tilde{E}(h_n,\, h_n)=0$, because

$$\int_X \rho^2 d\tilde{\mu}^c_{\langle h_n\rangle}=\int_X d\tilde{\mu}^c_{\langle R_\alpha^{(n)}g\rangle}+\int_X h_n^2 d\tilde{\mu}^c_{\langle R_\alpha g\rangle}-2\int_X h_n d\tilde{\mu}^c_{\langle R_\alpha^{(n)}g,R_\alpha g\rangle}$$
$$\xrightarrow[n\to\infty]{} 2\int_X d\tilde{\mu}^c_{\langle R_\alpha g\rangle}-2\int_X d\tilde{\mu}^c_{\langle R_\alpha g\rangle}=0,$$

and

$$\int_{X\times X-d}(h_n(x)-h_n(y))^2 J(dx, dy) = \int_{X\times X-d}(R_\alpha^{(n)}g(x)-R_\alpha^{(n)}g(y))^2 J(dx, dy)$$
$$-2\int_{X\times X-d}h_n(x)(R_\alpha g(x)-R_\alpha g(y))(R_\alpha^{(n)}g(x)-R_\alpha^{(n)}g(y))J(dx, dy)$$
$$+\int_{X\times X-d}h_n(x)h_n(y)(R_\alpha g(x)-R_\alpha g(y))^2 J(dx, dy)$$
$$\xrightarrow[n\to\infty]{} 2\int_{X\times X-d}(R_\alpha g(x)-R_\alpha g(y))^2 J(dx, dy)$$
$$-2\int_{X\times X-d}(R_\alpha g(x)-R_\alpha g(y))^2 J(dx, dy) = 0\,.$$

This means $\tilde{E}(1, 1)=0$. Therefore, we conclude that $\tilde{\boldsymbol{M}}$ is conservative. q.e.d.

By virtue of (3.11) and Theorem 3.2, it holds that

$$\text{(3.15)}\quad \boldsymbol{P}_x(\Lambda;\, t<\zeta) = \tilde{\boldsymbol{E}}_x\Big(\frac{\rho(X_0)}{\rho(X_t)}\exp\Big(\int_0^t \frac{A\rho}{\rho}(X_s)ds\Big);\, \Lambda\Big) \qquad \text{q.e. } x\in X\ \ \Lambda\in\mathcal{B}_t\,.$$

This formula will be used in the next section.

4. A lower estimate related to the Donsker-Varadhan theory

The Dirichlet space (E, F) is called *irreducible* if any P_t-invariant set is trivial, namely, a Borel set $A\subset X$ satisfies either $m(A)=0$ or $m(X-A)=0$ whenever $P_t(I_A u)=I_A P_t u$ for any $u\in\mathcal{B}_b^+$ and $t>0$. Denote by $\mathcal{M}$ the space of probability measures equipped with the weak topology and let $L(t, \omega, A)=\int_0^t I_A(X_s)ds$.

Proposition 4.1. *If (E, F) is irreducible, then for $\rho\in\mathcal{X}$*

$$\text{(4.1)}\qquad \varliminf_{t\to\infty}\frac{1}{t}\log \boldsymbol{P}_x(L(t, \omega, \cdot)\in N_{\rho^2},\, t<\zeta)\geqq -E(\rho, \rho)\quad \text{q.e. } x\in X$$

where N_{ρ^2} is any neighborhood of $\rho^2 m\in\mathcal{M}$.

Proof. By (3.15),

$$\text{(4.2)}\quad \boldsymbol{P}_x(L(t, \omega, \cdot)\in N_{\rho^2},\, t<\zeta) = \tilde{\boldsymbol{E}}_x\Big(\frac{\rho(X_0)}{\rho(X_t)}\exp\Big(\int_0^t\frac{A\rho}{\rho}(X_s)ds\Big);\, L(t, \omega, \cdot)\in N_{\rho^2}\Big)\quad \text{q.e. } x\in X\,.$$

We set

$$S(t, \varepsilon) = \Big\{\omega\in\Omega;\, \Big|\int_X \frac{A\rho}{\rho}(y)L(t, \omega, dy)-\int_X \rho A\rho dm\Big|<\varepsilon\Big\}$$

$$S'(t, \varepsilon) = \{\omega \in \Omega;\ L(t, \omega, \cdot) \in N_{\rho^2}\} \cap S(t, \varepsilon)$$

then, the right hand side is greater than

$$\exp\Big(t\Big(\int_X \rho A \rho dm - \varepsilon\Big)\Big) \times \tilde{\boldsymbol{E}}_x\Big(\frac{\rho(X_0)}{\rho(X_t)};\ S'(t, \varepsilon)\Big) \geqq \exp\Big(t\Big(\int_X \rho A \rho dm - \varepsilon\Big)\Big) \times \frac{\rho(x)}{\|\rho\|_\infty}\big(1 - \tilde{\boldsymbol{P}}_x(\Omega - S'(t, \varepsilon))\big)\,.$$

Now, the irreducibility of (E, F) implies the same property of $(\tilde{E}, \tilde{F})$ because L_t is positive until the life time ζ. Since $\tilde{\boldsymbol{M}}$ is conservative with invariant measure $\rho^2 dm$ by Theorem 3.2, $\tilde{\boldsymbol{M}}$ is ergodic ([9]) and

$$\tilde{\boldsymbol{P}}_{\rho^2 m}\Big(\lim_{t\to\infty} \frac{1}{t} \int_0^t \frac{A\rho}{\rho}(X_s) ds = \int_X \rho A \rho dm\Big) = 1\,. \tag{4.3}$$

If we denote by Λ the event inside the braces of (4.3), we get $\tilde{\boldsymbol{P}}_x(\Lambda) = 1$, $\tilde{\boldsymbol{M}}$-q.e. by noticing that $\tilde{\boldsymbol{P}}_x(\Lambda)$ is an excessive function. By virtue of the remark made before Theorem 3.2, we then have $\lim_{t\to\infty} \tilde{\boldsymbol{P}}_x(\Omega - S'(t, \varepsilon)) = 0$, q.e. Therefore

$$\varliminf_{t\to\infty} \frac{1}{t} \log \boldsymbol{P}_x(L(t, \omega, \cdot) \in N_{\rho^2};\ t < \zeta) \geqq \int_X \rho A \rho dm - \varepsilon, \text{ q.e.} \tag{4.4}$$

As ε is arbitrary and $\int_X \rho A \rho dm = -E(\rho, \rho)$, we arrive at Proposition 4.1. q.e.d.

From now we require the following assumption.

ASSUMPTION 4.1. There exists a sequence $\mathcal{G} = \{G_n\}_{n=1}^\infty$ of increasing relatively compact open sets satisfying: 1) $\bigcup_n G_n = X$, 2) (E_{G_n}, F_{G_n}) is irreducible.

A comparison theorem concerning the irreducibility of local Dirichlet forms has been given in [8]; if $E^{(1)}$ and $E^{(2)}$ are such forms, $E^{(1)}$ is dominating $E^{(2)}$ on a common core and $E^{(2)}$ is irreducible, then $E^{(1)}$ is also irreducible. From this, we can show that the first two examples in § 1 satisfy Assumption 4.1. The irreducibility of the third example in § 1 follows from the relation between J and the Lévy system stated in § 2.

Let us introduce the following space

$$\tilde{\mathcal{X}} = \{\rho;\ \rho = R_\alpha^{G_n} g,\ \alpha > 0,\ g > 0,\ g \in C_b,\ \int_X \rho^2 dm = 1,\ n = 1, 2, \cdots\}$$

and let Φ be a function on $\mathcal{M}$ such that $\Phi(\mu_n) \to \Phi(\mu)$ whenever $\mu_n \in \mathcal{M}$ converge weakly to μ and the support of μ_n is contained in some compact set. We further assume that $\Phi(\mu) \neq -\infty$ for any $\mu \in \mathcal{M}$.

Lemma 4.1. *It holds that*

(4.5) $$\varliminf_{t\to\infty} \frac{1}{t} \log \boldsymbol{E}_x(e^{-t\Phi(L(t,\omega,\cdot))}) \geqq -\inf_{\rho\in\tilde{\mathfrak{X}}} [\Phi(\rho^2)+E(\rho,\rho)] \quad \text{q.e.}$$

Proof. Take $\rho=R_\alpha^{G_n}g\in\tilde{\mathfrak{X}}$. Since (E_{G_n}, F_{G_n}) is an irreducible C_0-regular Dirichlet space, it follows from Proposition 4.1

$$\varliminf_{t\to\infty} \frac{1}{t} \log \boldsymbol{P}_x(L(t,\omega,\cdot)\in N_{\rho^2},\ t<\zeta) \geqq -E(\rho,\rho) \quad \text{q.e.}$$

Therefore, this theorem can be shown by the same method as [3]. q.e.d.

Lemma 4.2. *It holds that*

(4.6) $$\inf_{\rho\in\tilde{\mathfrak{X}}} [\Phi(\rho^2)+E(\rho,\rho)] \leqq \inf_{\rho\in U} [\Phi(\rho^2)+E(\rho,\rho)]$$

where $U=\{\rho\in F\cap C_0^+;\ \int_X \rho^2 dm=1\}$.

Proof. For any $\rho\in U$, take $G_n\in\mathcal{G}$ such that $\mathrm{Supp}[\rho m]\subset G_n$. Since $\alpha R_\alpha^{G_n}\rho\to\rho(\alpha\to\infty)$ in E_1 and $R_\alpha^{G_n}(\rho\vee\varepsilon)\to R_\alpha^{G_n}\rho(\varepsilon\to 0)$ in E_1, there exists a sequence $\{\rho_p=R_\alpha^{G_n}g_p;\ \alpha>0,\ g_p>0,\ g_p\in C_b\}_{p=1}^\infty$ such that $\rho_p\to\rho$ $(p\to\infty)$ in E_1. Therefore, $\frac{\rho_p}{\|\rho_p\|_{L^2}}\in\tilde{\mathfrak{X}}\to\rho$ $(p\to\infty)$ in E_1, and we get (4.6) in view of the property of Φ. q.e.d.

Lemma 4.3. *It holds that*

(4.7) $$\inf_{\rho\in U} [\Phi(\rho^2)+E(\rho,\rho)] = \inf_{\rho\in V} [\Phi(\rho^2)+E(\rho,\rho)]$$

where $V=\{\rho\in F;\ \mathrm{Supp}[\rho m]$ *is compact.* $\int_X \rho^2 dm=1\}$.

Proof. For $\rho\in V$, we take the relatively compact open set such that $\mathrm{Supp}[\rho m]\subset G$. Since (E_G, F_G) is a regular Dirichlet space, there exists a sequence $\{\rho_n\}\subset F_G\cap C_0$ such that $\rho_n\to\rho$ in E_1. Therefore, noting that $E(|\rho_n|, |\rho_n|)\leqq E(\rho_n,\rho_n)$, we see that the left hand side is not greater than the right in the same way as Lemma 4.2. q.e.d.

Theorem 4.1. *It holds that*

(4.8) $$\varliminf_{t\to\infty} \frac{1}{t} \log \boldsymbol{E}_x(e^{-t\Phi(L(t,\omega,\cdot))}) \geqq -\inf_{\rho\in V} [\Phi(\rho^2)+E(\rho,\rho)] \quad \text{q.e.}$$

Proof. By virtue of Lemma 4.2 and Lemma 4.3, we have

(4.9) $$\inf_{\rho\in\tilde{\mathfrak{X}}} [\Phi(\rho^2)+E(\rho,\rho)] = \inf_{\rho\in V} [\Phi(\rho^2)+E(\rho,\rho)] .$$

Hence this theorem is clear by Lemma 4.1. q.e.d.

The above proof shows that, in equality (4.9), we can replace the space V by a smaller one $V \cap D$ where D is any core of F satisfying the conditions of [8; pp. 198].

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(r, p)-Capacities for general Markovian semigroups

1. INTRODUCTION

Consider the Gamma transform

$$V_r(x) = \frac{1}{\Gamma(r/2)} \int_0^\infty t^{\frac{r}{2}-1} e^{-t} p_t(x)dt, \qquad r > 0,$$

of the transition density $p_t(x) = (2\pi t)^{-d/2} e^{-|x|^2/2t}$ of the d-dimensional standard Brownian motion. V_r is then the Bessel convolution kernel $(\hat{V}_r(\xi) = (1 + \frac{|\xi|^2}{2})^{-r/2})$ and the space of Bessel potentials $V_r * f$ L^p-functions f coincides with the Sobolev space W_r^p when r is an integer ([9]). This observation suggests a way of introducing an analogue $F_{r,p}$ of the Sobolev space and a relevant capacity $C_{r,p}$ for a much more general Markovian semigroup $\{T_t, t > 0\}$.

Already Meyers [7] has made a general approach to this kind of capacity starting with a lower semicontinuous kernel k(x,y). But we wish to avoid the assumption that the operator V_r admits such a smooth kernal k, because we are particularly interested in the application to cases where the underlying spaces are infinite dimensional ([4], [6], [10]).

Accordingly the proof of the continuity

$$A_n \uparrow \Rightarrow C_{r,p}(\cup_n A_n) = \sup_n C_{r,p}(A_n)$$

of the set function $C_{r,p}$ becomes more difficult in our setting. When the underlying space is locally compact, this continuity of the outer capacity implies the capacitability of the analytic set. Even in non-locally-compact cases such as the Wiener space case, this property is essentially important in carrying out the relevant analysis ([4], [10]). In section 3 we prove this property by assuming that the space $F_{r,p}$ is regular, namely, that it densely contains continuous functions. We can then appeal to Deny's principle [2] that the regularity of a function space implies the continuity of the capacity.

2. MARKOVIAN SEMIGROUP AND (r,p)-CAPACITY

Let X be a metric space, m be a measure on X and T_t, $t > 0$ be a strongly continuous contraction semigroup of linear operators on $L^p = L^p(X,m)$, $p > 1$ being fixed. We suppose that T_t, $t > 0$, are *Markovian:*

$$f \in L^p, \quad 0 \leqq f \leqq 1 \quad \text{m-a.e} \Rightarrow 0 \leqq T_t f \leqq 1 \quad \text{m-a.e.} \tag{2.1}$$

We then let, for each $r > 0$,

$$V_r = \frac{1}{\Gamma(r/2)} \int_0^\infty t^{\frac{r}{2} - 1} e^{-t} T_t \, dt. \tag{2.2}$$

V_r is Markovian. It is contractive:

$$\| V_r f \|_{L^p} \leqq \| f \|_{L^p}, \qquad f \in L^p. \tag{2.3}$$

Furthermore,

$$V_r V_{r'} = V_{r+r'}, \qquad r, r' > 0. \tag{2.4}$$

To see this, it suffices to express the left-hand side as a double integral of two variables, say, t and s and changes them by $t = \xi \cos^2\theta$, $s = \xi \sin^2\theta$. Suppose $V_r f = 0$, then $V_{r'} f = 0$ for all $r' \geqq r$ by (2.4) and consequently $T_t f = 0$, $t > 0$, which implies $f = 0$. Thus, each V_r is injective on L^p and hence the space $(F_{r,p}, \| \ \|_{r,p})$ defined below gives us a Banach space:

$$\begin{cases} F_{r,p} = V_r(L^p) \\ \| u \|_{r,p} = \| f \|_{L^p} \quad \text{for } u = V_r f, \quad f \in L^p. \end{cases} \tag{2.5}$$

The associated set function $C_{r,p}$ is then defined, for open set A, by

$$C_{r,p}(A) = \inf \{ \| u \|_{r,p}^p : u \in F_{r,p}, \ u \geqq 1 \ \text{m-a.e. on } A \}$$

and, for any set A, by

$$C_{r,p}(A) = \inf \{ C_{r,p}(B) : A \subset B, \ B \text{ is open} \}.$$

We call $C_{r,p}$ the (r,p)-*capacity* for $\{T_t, t > 0\}$.

Theorem 1. (i) $m(A) \leqq C_{r,p}(A)$

(ii) $r < r' \Rightarrow C_{r,p}(A) \leqq C_{r',p}(A)$

(iii) $A \subset B \Rightarrow C_{r,p}(A) \leqq C_{r,p}(B)$

(iv) $C_{r,p}(\bigcup_n A_n) \leqq \sum_n C_{r,p}(A_n)$.

(i) of Theorem 1 is immediate from (2.3). (ii) follows from (2.3) and (2.4). (iii) is trivial. (iv) can be proved easily ([7]) but we give its proof by using the next lemma for later convenience.

Lemma 1. For any open set A with finite (r,p)-capacity, there exists a unique element u_A in $F_{r,p}$ such that $u_A \geqq 1$ m-a.e. on A and $\| u_A \|^p_{r,p} = C_{r,p}(A)$. u_A admits an expression $u_A = V_r f$ for some non-negative $f \in L^p$.

Proof. Since $(F_{r,p}, \| \ \|_{r,p})$ is isometric to $L^p(X;m)$, it is uniformly convex: for any $\varepsilon > 0$ and $M > 0$, there is a $\delta > 0$ such that $\| u \|_{r,p} \leqq M$, $\| v \|_{r,p} \leqq M$ and $\| u-v \|_{r,p} > \varepsilon$ imply $\| u+v \|_{r,p} \leqq 2M-\delta$. From this follows the uniqueness of u_A. The existence follows from the Banach-Saks theorem for L^p space. If $u_A = V_r f$, $f \in L^p$, then $V_r f \leqq V_r f^+$, $\| f^+ \|_{L^p} \leqq \| f \|_{L^p}$ and consequently $f = f^+$ by the uniqueness.

Proof of Theorem 1 (iv) It suffices to assume in the desired inequality (iv) that A_n are open and the right-hand side is finite. Let $u_{A_n} = V_r f_n$, $f_n \in L^p_+$, and $f = \sup_n f_n$. Since $f^p \leqq \sum_n f_n^p$, we have $f \in L^p$ and $\| V_r f \|^p_{r,p} \leqq \sum_n C_{r,p}(A_n)$. On the other hand, $C_{r,p}(\bigcup_n A_n) \leqq \| V_r f \|^p_{r,p}$ because $V_r f \geqq 1$ m-a.e. on $\bigcup_n A_n$.

Remark 1. We see in the proof of Lemma 1 that, for an open set A,

$$C_{r,p}(A) = \inf\{ \| f \|^p_{L^p} : V_r f \geqq 1 \text{ m-a.e. on } A, \ f \in L^p_+ \}, \tag{2.6}$$

where L^p_+ denotes the set of non-negative functions in L^p. In the case of the Bessel potentials on R^d mentioned at the beginning of section 1, it further holds that

$$C_{r,p}(A) = \inf \{ \| f \|^p_{L^p} : V_r f(x) \geqq 1 \text{ for any } x \in A,\ f \in L^p_+ \}, \qquad (2.7)$$

because, for the Bessel potential $u = V_r f$, $f \in L^p_+$, we have $\lim_{r \to 0} M_r u(x) = u(x)$ at each $x \in R^d$, where $M_r u(x)$ denotes the volume average of u on the ball of radius r, centred at x. In this case, we also know from [7] that the identity (2.7) holds for any set $A \subset R^d$.

Remark 2. We have started with a strongly continuous contraction semigroup of Markovian linear operators on L^p for a fixed $p \geqq 1$. Suppose that we are first given such a semigroup on L^1. Then it uniquely determines a semigroup of the same properties on L^p for every $p \geqq 1$ ([5]) and hence there correspond (r,p)-capacities $C_{r,p}$ indexed by all $r > 0$ and all $p > 1$. In accordance with Malliavin [6], we say in this case that a set $A \subset X$ is *slim* if $C_{r,p}(A) = 0$ for all $r > 0$, $p > 1$.

In the case of the Bessel potentials on R^d discussed in Remark 1, no non-empty slim set exists since each one point set of R^d has a positive (r,p)-capacity whenever $rp > d$ ([1]). The situation may be quite different when the underlying space X is infinite-dimensional. In fact, there are many non-empty slim sets for the Ornstein-Uhlenbeck semigroup $\{T_t, t > 0\}$ on the Wiener space. In particular, each one-point set is slim ([10]).

Remark 3. Suppose that we are first given a strongly continuous contraction semigroup $\{T_t, t > 0\}$ of Markovian linear operators on L^2. In addition, we assume that T_t is symmetric and the measure m is σ-finite (this amounts to assuming that we are first given a Dirichlet space on L^2 ([3])). Then $\{T_t, t > 0\}$ uniquely decides a semigroup of the same properties on L^1 ([8]) and consequently on L^p for every $p \geqq 1$.

Denote by L the infinitesimal generator of $\{T_t, t > 0\}$ on L^2. Substituting the spectral representation of L into (2.2), we easily see that

$$V_r = (I-L)^{-r/2} \quad \text{on} \quad L^2 \qquad (2.8)$$

$$F_{r,2} = \mathcal{D}((I-L)^{r/2}),\ \|u\|_{r,2} = \|(I-L)^{r/2}u\|_{L^2}, \qquad (2.9)$$

and consequently $F_{1,2}$ coincides with the associated Dirichlet space on L^2 and $\|u\|_{1,2}$ is the 1-order Dirichlet norm. $C_{1,2}$ is exactly the same as the capacity Cap defined in [3] for this Dirichlet space. Hence the set of

(1,2)-capacity zero admits a probabilistic characterization related to the Markovian semigroup $\{T_t,\ t > 0\}$ ([3]), but no such interpretation seems to be possible for (r,p)-capacity when $(r,p) \neq (1,2)$.

In view of expression (2.9) and its formal extension

$$\|u\|_{r,p} = \|(I-L)^{r/2}u\|_{L^p}, \quad u \in F_{r,p}$$

we may well say that the space $F_{r,p}$ is a right analogue to the Sobolev space $W_{r,p}$ for the present general semigroup $\{T_t,\ t > 0\}$.

3. CONTINUITY OF CAPACITIES

As in section 2, we treat a strongly continuous contraction semigroup $\{T_t,\ t > 0\}$ of Markovian linear operators on $L^p = L^p(X;m)$. In this section, we assume that X is a separable metric space and $m(A) > 0$ for any non-empty open set A. We further assume that the space $F_{r,p}$ is regular:

$$F_{r,p} \cap C(X) \text{ is dense in } (F_{r,p}, \|\ \|_{r,p}), \tag{3.1}$$

where C(X) denotes the space of (not necessarily bounded) continuous functions on X.

We fix the index (r,p). "Quasi-everywhere" or "q.e" will mean "except on a set of (r,p)-capacity zero". A function u on X is said to be *quasi-continuous* if, for any $\varepsilon > 0$, there exists an open set A with $C_{r,p}(A) < \varepsilon$ such that the restriction of u to X-A is continuous. Just as in the case of the Dirichlet space ([3]), we can show the following:

(a) If u is quasi-continuous and $u \geq 0$ m-a.e. on an open set G, then $u \geq 0$ q.e. on G.

(b) Each $u \in F_{r,p}$ admits a quasi-continuous modification (denoted by $\tilde{u}$) and

$$C_{r,p}(|\tilde{u}|>\lambda) \leq \frac{1}{\lambda^p}\|u\|_{r,p}^p, \qquad \lambda > 0. \tag{3.2}$$

(c) If a sequence of quasi-continuous functions $u_n \in F_{r,p}$ converges to $u \in F_{r,p}$ in metric $\|\ \|_{r,p}$, then a subsequence of u_n converges q.e. to a quasi-continuous modification of u.

<u>Lemma 2.</u> For any set $B \subset X$ with finite (r,p)-capacity, there exists a unique element e_B in the set $L_B = \{u \in F_{r,p} : \tilde{u} \geq 1 \text{ q.e. on } B\}$ minimizing

the norm $\| \ \|_{r,p}$. e_B is non-negative and

$$C_{r,p}(B) = \| e_B \|^p_{r,p} . \tag{3.3}$$

Proof. The unique existence of e_B and its non-negativity can be shown in the same way as in the proof of Lemma 1. For any $\varepsilon > 0$, there exists an open set $A \supset B$ such that $C_{r,p}(B) > C_{r,p}(A) - \varepsilon$. Since u_A of Lemma 1 belongs to L_B by (a), $C_{r,p}(A) = \| u_A \|^p_{r,p} \geqq \| e_B \|^p_{r,p}$ and we get the inequality "$\geqq$" in (3.3).

To prove the converse inequality, we adopt the method of Deny [2]. Take a quasi-continuous version $\tilde{e}_B$ of e_B. For any $\varepsilon > 0$, choose an open set A_ε such that $C_{r,p}(A_\varepsilon) < \varepsilon$, $\tilde{e}_B|_{X-A_\varepsilon}$ is continuous and $\tilde{e}_B \geqq 1$ on $B \cap (X-A_\varepsilon)$. Denote by u_ε the function of Lemma 1 for the open set A_ε. Now the set

$$G_\varepsilon = \{x \in X-A_\varepsilon : \tilde{e}_B(x) > 1 - \varepsilon\} \cup A_\varepsilon$$

is open and $B \subset G_\varepsilon$. Moreover, $e_B + u_\varepsilon \geqq 1 - \varepsilon$ m-a.e. on G_ε. Therefore $C_{r,p}(B) \leqq C_{r,p}(G_\varepsilon) \leqq (1-\varepsilon)^{-p} \| e_B + u_\varepsilon \|^p_{r,p} \leqq (1-\varepsilon)^{-p} (\| e_B \|_{r,p} + \varepsilon)^p$. By letting $\varepsilon \downarrow 0$, we arrive at (3.3).

Theorem 2. $A_n \uparrow \Rightarrow C_{r,p}(\bigcup_n A_n) = \sup_n C_{r,p}(A_n)$.

Proof. We may assume that the right-hand side is finite. We put $A = \bigcup_n A_n$. Let e_n be the function of Lemma 2 for the set A_n. Since $\| e_n \|^p_{r,p} = C_{r,p}(A_n)$ are bounded, a Cesaro mean u_m of a subsequence of e_n converges strongly to some $u \in F_{r,p}$ by virtue of the Banach-Saks theorem. Take quasi-continuous modifications $\tilde{u}_m$ of u_m. Then $\lim_{m\to\infty} \tilde{u}_m(x) \geqq 1$ q.e. on A_n for each n and consequently q.e. on A. By (c), a subsequence of $\tilde{u}_m$ converges q.e. to a quasi-continuous modification $\tilde{u}$ of u. Hence $u \in L_A$ and

$$C_{r,p}(A) \leqq \| u \|^p_{r,p} = \lim_{m\to\infty} \| u_m \|^p_{r,p} = \lim_{n\to\infty} \| e_n \|^p_{r,p} = \sup_n C_{r,p}(A_n).$$

Remark 4. When $r = 1$ and $p = 2$, Theorem 2 holds without the regularity assumption (3.1) because $C_{1,2}$ is strongly subadditive, which follows from the characteristic property of the Dirichlet space that every normal contraction operates on $F_{1,2}$ ([3]).

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ON THE CONTINUITY OF PLURISUBHARMONIC FUNCTIONS ALONG CONFORMAL DIFFUSIONS

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1. Introduction

A stochastic process $Z_t=(Z_t^1, \cdots, Z_t^n)$ taking values in C^n is called a *conformal martingale* if Z_t^α and $Z_t^\alpha Z_t^\beta$, $1\leqq\alpha, \beta\leqq n$, are continuous local martingales. When Z_t is defined only on a time interval $[0, \eta)$ for some predictable stopping time η, Z_t is said to be a conformal martingale if so is the stopped process $Z_{t\wedge\eta'}$ for any stopping time η' strictly less than η.

Let M be a complex manifold of complex dimension n. By a diffusion process $\boldsymbol{D}=(Z_t, P_z)$ on M, we mean a strong Markov process on M with continuous sample paths defined on $[0, \zeta)$, ζ being the life time. In this paper, we assume without specific mention that the diffusion $\boldsymbol{D}$ admits no killing inside M in the sense that $P_z(\tau_U<\zeta<+\infty)=P_z(\zeta<+\infty)$, $z\in U$, for any relatively compact open set $U\subset M$, where τ_U denotes the first exit time from U: $\tau_U=\inf\{t\geqq0: Z_t\notin U\}$. We see then that, for any open set $U\subset M$, τ_U is a predictable stopping time with respect to P_z for $z\in U$.

We call a diffusion process $\boldsymbol{D}=(Z_t, P_z)$ on M a *conformal diffusion* on M if, for any holomorphic coordinate neighbourhood (U, ϕ), the C^n-valued process $\phi(Z_t)$ defined on $[0, \tau_U)$ is a conformal martingale with respect to P_z for each $z\in U$. We occasionally assume that the transition function p_t of $\boldsymbol{D}$ is absolutely continuous with respect to a volume element V on M:

$$p_t(z, \cdot)<V, \qquad z\in M. \tag{1.1}$$

We aim at proving the following theorem.

Theorem. *Let $\boldsymbol{D}=(Z_t, P_z)$ be a conformal diffusion on M satisfying the condition* (1.1). *Then, for any plurisubharmonic function u on M,*
$P_z(u(Z_t)$ *is continuous in* $t\in[0, \zeta)$ *and finite for* $t\in(0, \zeta))=1$, $z\in M$.

This is a generalization of a theorem of Doob [2] to the cases of higher complex dimension and our proof is also similar to the one given in [2] in the sense that we utilize the quasi-continuity of plurisubharmonic functions with respect to a specific capacity related to the extremal function.

As we shall see, any plurisubharmonic function u on M is $\boldsymbol{D}$-subharmonic in Dynkin's sense and consequently $u(Z_t)$ is right continuous. Therefore its continuity would follow from a Hunt's theorem on the regularity of excessive functions provided that

(1.2) every semi-polar set is polar

for the diffusion $\boldsymbol{D}$. However, it seems to be unknown whether (1.2) is fulfilled for all the conformal diffusions being considered.

Indeed, a typical conformal diffusion is a diffusion $\boldsymbol{D}$ on M whose infinitesimal generator is expressible on a local chart as

$$L=\frac{1}{2}\sum g^{\alpha\bar{\beta}}\frac{\partial^2}{\partial z^\alpha\partial\bar{z}^\beta} \tag{1.3}$$

with a continuous non-negative Hermitian tensor field g on M ([6]). If g is sufficiently smooth and non-degenerate, (1.2) is known to be correct for $\boldsymbol{D}$. But, if g is merely continuous positive or degenerate ($\boldsymbol{D}$ may still satisfy (1.1) in the latter case), we do not know to what extent (1.2) is true. (1.2) becomes true under the additional condition of the symmetrizability. But the latter condition might be false either in general in view of a Fujita's result [4] saying that there exists a manifold M where no diffusion with generator (1.3) is symmetrizable (although only smooth and non-degenerate cases are treated in [4]).

We add a remark that there are many conformal diffusions whose generators are not expressible by the usual differential operator like (1.3). It was shown in [5] that fairly general class of symmetrizable conformal diffusions can be characterized by closed positive currents of type $(n-1, n-1)$. The first two propositions of the present paper have been proven in [5] for this class of diffusions on a domain of C^n.

This work was motivated by the lectures of Professor Laurent Schwartz delivered at Kyoto University (cf. [7]). I am grateful to him for his kind guidance to the problem.

2. $\boldsymbol{D}$-subharmonicity of plurisubharmonic functions

A function u on an open set $E\subset M$ taking values in $[-\infty, +\infty)$ is said to be *plurisubharmonic* on E if, on each holomorphic coordinate neighbourhood $U\subset E$, u is locally integrable, $\sum\frac{\partial^2 u}{\partial z^\alpha\partial\bar{z}^\beta}\xi^\alpha\bar{\xi}^\beta$ is a non-negative distribution for any $\xi=(\xi^1, \cdots, \xi^n)\in C^n$ and $\operatorname{ess}\limsup_{z'\to z} u(z')=u(z)$, $z\in U$. Those properties are intrinsic because they are preserved under holomorphic transformations. Any plurisubharmonic function is upper semicontinuous.

Let $\boldsymbol{D}=(Z_t, P_z)$ be a conformal diffusion on M with transition function p_t.

A Borel function u on M taking values in $[0, +\infty]$ is called *p_t-excessive* if $p_t u(Z) \uparrow u(z)$ as $t \downarrow 0$ for each $z \in M$. A Borel function u on an open set $E \subset M$ taking values in $[-\infty, +\infty)$ is called ***D**-subharmonic* on E if u is $\boldsymbol{D}$-finely upper semicontinuous, locally bounded from above and, for any open set V with compact closure $\bar{V} \subset E$,

$$u(z) \leqq E_z(u(Z_{\tau_V}); \tau_V < \zeta), \qquad z \in V. \tag{2.1}$$

Any $\boldsymbol{D}$-subharmonic function is $\boldsymbol{D}$-finely continuous and hence right continuous along Z_t in $t \in [0, \zeta)$ P_z-almost surely ([3]). The negative of a $\boldsymbol{D}$-subharmonic function is said to be $\boldsymbol{D}$-superharmonic. A Dynkin's theorem [3; Theorem 12.4] says that a non-negative Borel function on M is p_t-excessive if and only if it is $\boldsymbol{D}$-superharmonic.

Proposition 1. *Any plurisubharmonic function on M is $\boldsymbol{D}$-subharmonic. The negative of a non-positive plurisubharmonic function on M is p_t-excessive.*

Proof. Let u be a plurisubharmonic function on M. u is then locally bounded from above on each coordinate neighbourhood. Besides the subharmonicity for the diffusion process is a local property according to a Sur's theorem ([3; Theorem 12.11]). Hence we may only prove the $\boldsymbol{D}$-subharmonicity of u by assuming that M is a bounded domain $D \subset C^n$ and u is non-positive.

Take any open set V with compact closure in D and denote by τ the first exit time of Z_t from V. Since (Z_t, P_z) is a conformal martingale, we see, by virtue of Schwartz [6; Proposition (5.10)], that $(u(Z_{t\wedge\tau}), P_z)$ is a generalized submartingale for $z \in V$, and consequently,

$$-\infty \leqq u(z) \leqq E_z(u(Z_{t\wedge\tau})), \qquad z \in V.$$

The right hand side is not greater than $E_z(u(Z_{t\wedge\tau}); \tau < \zeta)$ and we get the inequality (2.1) by letting $t \to +\infty$. Since u is upper semicontinuous, we conclude that u is $\boldsymbol{D}$-subharmonic.

In the remainder of this section, we only consider a bounded domain D of C^n. For $E \subset D$, the extremal function u_E^* is defined by $u_E(z) = \sup\{v(z): v$ plurisubharmonic on D, $-1 \leqq v \leqq 0$ on D, $v = -1$ on $E\}$, $u_E^*(z) = \overline{\lim}_{z' \to z} u_E(z')$, $z \in D$. We further introduce a set function C_* by

$$C_*(E) = -\int_D u_E^*(z) dV(z) \quad \left(= -\int_D u_E(z) dV(z)\right) \tag{2.2}$$

where V denotes the Lebesgue measure on D. C_* is known to be a Choquet capacity ([1; Proposition 8.4]). Moreover $C_*(N) = 0$ if and only if N is pluripolar, namely, there exists a plurisubharmonic function v on D with $N \subset v^{-1}(-\infty)$.

Let $\boldsymbol{D} = (Z_t, P_z)$ be a conformal diffusion on D. Denote by σ_E the hitting time of a set $E\subset D$ after $0+$: $\sigma_E=\inf\{t>0: Z_t\in E\}$. We let $\sigma_E=+\infty$ if the event in the braces is empty.

Proposition 2. *For any Borel set $E\subset D$,*

$$\int_D P_z(\sigma_E<\zeta)dV(z)\leqq C_*(E).$$

Proof. By Choquet's lemma, there is a non-decreasing sequence of plurisubharmonic functions v_k such that $-1\leqq v_k\leqq 0$, $v_k=-1$ on E and $u_E^*(z)=\overline{\lim_{z'\to z}}\, v_0(z')$ for $v_0=\lim_{k\to\infty} v_k$. By Proposition 1, $\{-v_k(Z_t), P_z\}$ is supermartingale and $-v_k(z)\geqq -E_z(v_k(Z_{\sigma_K}): \sigma_K<\zeta)\geqq P_z(\sigma_K<\zeta)$ for any compact set $K\subset E$ and $z\in D$, on account of the optional sampling theorem. Letting $k\to\infty$ and integrating by dV, we have

$$\int_D P_z(\sigma_K<\zeta)dV(z)\leqq C_*(E)$$

since $v_0=u_E^*$ V-a.e. Taking then an increasing sequence of compact sets $K_m\subset E$ such that $\sigma_{K_m}\downarrow\sigma_E$ as $m\to\infty$, P_V-a.e., we get the desired inequality.

Corollary 1

(i) *If $N\subset D$ is pluripolar, then there exists a Borel set $N'\supset N$ and*

$$P_z(\sigma_{N'}<\zeta)=0 \qquad V\text{-a.e.} \quad z\in D. \tag{2.3}$$

(ii) *If $O_k\subset D$ are decreasing open sets such that $\lim_{k\to\infty} C_*(O_k)=0$, then*

$$P_z(\lim_{k\to\infty}\sigma_{O_k}<\zeta)=0 \qquad V\text{-a.e.} \quad z\in D. \tag{2.4}$$

Proof. (ii) is a stronger assertion than (i). (ii) is immediate from Proposition 2.

We denote by θ_s the usual shift operator defined by $Z_t(\theta_s\omega)=Z_{s+t}(\omega)$. In particular we have

$$s+\sigma_E\circ\theta_s(\omega)=\inf\{t>s: Z_t(\omega)\in E\}, \qquad s\geqq 0. \tag{2.5}$$

Corollary 2. *Suppose that the transition function p_t of $\boldsymbol{D}$ satisfies the absolute continuity condition* (1.1).

(i) *If $N\subset D$ is pluripolar, then N is $\boldsymbol{D}$-polar: there exists a Borel set $N'\supset N$ and* (2.3) *holds for every $z\in D$.*

(ii) *If $O_k\subset D$ are decreasing open sets such that $\lim_{k\to\infty} C_*(O_k)=0$, then*

$$P_z(\lim_{k\to\infty}(s+\sigma_{O_k}\circ\theta_s)<\zeta,\ s<\zeta)=0 \tag{2.6}$$

for every $s>0$ and $z\in D$.

Proof. (i) Denote by $f(z)$ the left hand side of (2.3). Then $f(z)=$

$\lim_{s\downarrow 0} p_s f(z)=0$, $z\in D$, by the assumption and Corollary 1 (i). (ii) The left hand side of (2.6) equals $p_s f(z)$ for the function f defined by the left hand side of (2.4).

3. $C_\sharp$-quasi-continuity of plurisubharmonic functions

We continue to consider a bounded domain $D\subset C^n$ and the capacity $C_\sharp$ defined by (2.2).

Proposition 3. *Suppose that the domain D is strongly pseudo-convex. Any plurisubharmonic function u on D is then $C_\sharp$-quasi-continuous. More specifically, for any $\varepsilon>0$, there exists an open set $O\subset D$ with $C_\sharp(O)<\varepsilon$ such that u is finite valued and continuous on $D-O$ with respect to the relative topology.*

Proof. We deduce this from several results of Bedford-Taylor [1]. First, according to [1; Theorem 3.5], any plurisubharmonic function on a bounded domain D is quasi-continuous in the above sense but with respect to another capacity which we shall denote by C_{BT}. C_{BT} admits the expression

$$C_{BT}(O)=\int_D (dd^c u_O^*)^n, \tag{3.1}$$

for open set O with compact closure $\bar{O}\subset D$. Therefore it suffices to show the implication

$$C_{BT}(O_k)\underset{k\to\infty}{\longrightarrow} 0 \Rightarrow C_\sharp(O_k)\underset{k\to\infty}{\longrightarrow} 0 \tag{3.2}$$

for decreasing sequence of open sets $O_k\subset D$.

A function on D is quasi-continuous relative to a capacity if and only if it is so on each open set E with compact closure $\bar{E}\subset D$. Hence, in proving (3.2), we may assume that O_1 has compact closure $\bar{O}_1\subset D$. Set $v=\lim_{k\to\infty} u_{O_k}^*$, $v^*(z)=\lim_{z'\to z} v(z')$, and assume now the strong pseudo-convexity of D. We then easily see that $v^*(z)\to 0$ as $z\to\partial D$. Moreover by the continuity of the Bedford-Taylor measures [1; Proposition 5.2], $(dd^c u_{O_k}^*)^n\to(dd^c v^*)^n$, $k\to\infty$. Hence we get, from (3.1) and the hyposesis in (3.2), $\int_D (dd^c v^*)^n=0$ and consequently $(dd^c v^*)^n$ is the zero measure. We can finally use a comparison theorem [1, Corollary 4.4] to obtain $v^*=0$ and $v=0$ V-a.e. We arrive at the conclusion in (3.2): $\lim_{k\to\infty} C_\sharp(O_k)=\int_D v(z)dV(z)=0$.

4. Proof of Theorem

The right continuity of $u(Z_t)$ at $t=0$

$$P_z(\lim_{t\downarrow 0} u(Z_t)=u(z))=1, \qquad z\in M, \tag{4.1}$$

follows from Proposition 1.

For a stopping time $\eta\in[0, +\infty]$, let us consider the event $\Lambda_\eta=\{u(Z_t)$ is finite and continuous for $t\in(0, \eta)\}$. We aim at proving

$$P_z(\Lambda_\zeta) = 1\,, \qquad z\in M\,. \tag{4.2}$$

We assume the condition (1.1). Choose a system of holomorphic coordinate neighbourhoods (U_α, ϕ_α) of M such that $\phi_\alpha(U_\alpha)$ is a strongly pseudo-convex bounded domain of C^n. Proposition 3 and Corollary 2 (ii) to Proposition 2 are applicable to the function $u|_{U_\alpha}$ and to the part $\boldsymbol{D}_\alpha$ of $\boldsymbol{D}$ on U_α respectively. In view of (2.5), we then readily see that $P_z(u(Z_t)$ is finite and continuous for $t\in(s, \tau_{U_\alpha})$, $s'<\tau_{U_\alpha})=P_z(s'<\tau_{U_\alpha})$, $0<s\leqq s'$, $z\in U_\alpha$. By letting $s\downarrow 0$ and then $s'\downarrow 0$, we get

$$P_z(\Lambda_{\tau_{U_\alpha}}) = 1\,, \qquad z\in U_\alpha\,. \tag{4.3}$$

We now use a Sur's method. Take two members, say, U_0 and U_1 from the chart system and let V be an arbitrary open set with $\bar V\subset U_0\cup U_1$. By [3; Lemma 12.6], we can find open sets V_0 and V_1 such that $V=V_0\cup V_1$, $\bar V_0\subset U_0$, $\bar V_1\subset U_1$ and $\overline{V_0\cap(M-V_1)}\cap\overline{V_1\cap(M-V_0)}=\phi$. Denote by τ^0 and τ^1 the exit time from V_0 and V_1 respectively, and let $\gamma_0=0$, $\gamma_{k+1}=\gamma_k+\tau^{\varepsilon_k}\circ\theta_{\gamma_k}$, $k\geqq 1$, where $\varepsilon_k=k$ mod 2. By virtue of [3; Lemma 12.4], it holds then that

$$\gamma_k = \tau_V \qquad \text{from some } k \text{ on.} \tag{4.4}$$

Together with the event Λ_η for the stopping time η, we also consider the event $\tilde\Lambda_\eta=\{u(Z_t)$ is finite continuous at each $t\in(0, \eta)$ and also at $t=\eta$ if $\eta<+\infty\}$. In view of (4.3), we have

$$P_z(\tilde\Lambda_{\tau^i}) = 1\,, \qquad z\in V_i\,,\ i = 0, 1\,. \tag{4.5}$$

Since (4.5) is trivially true for $z\in D-V_i$, we obtain from the strong Markov property and (4.5),
$P_z(\tilde\Lambda_{\gamma_2})=P_z(u(Z_t)$ is finite continuous for $t>0$, $\tau^0=+\infty)+E_z(u(Z_t)$ is finite continuous for $t\in(0, \tau^0]$, $\tau^0<+\infty$; $P_{z_{\tau^0}}(\tilde\Lambda_{\tau^1}))=P_z(\tilde\Lambda_{\tau^0})=1$, $z\in V$. By induction and (4.4), we get

$$P_z(\tilde\Lambda_{\tau_V}) = 1\,, \qquad z\in V\,. \tag{4.6}$$

By letting $V\downarrow U_0\cup U_1$, we are led from (4.6) to

$$P_z(\Lambda_{\tau_G}) = 1\,, \qquad z\in G\,, \tag{4.7}$$

for $G=U_0\cup U_1$. Repeating the same argument, (4.7) can be seen to be true for the union G of finite number of U_α's. Now (4.7) holds for any relatively compact open set $G\subset M$. We finally let $G\uparrow M$ to get the desired identity (4.2).

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On Dirichlet forms for plurisubharmonic functions

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§ 1. Introduction

We give a new proof of several basic properties of plurisubharmonic functions on $\mathbf{C}^n$ by making a systematic use of the notion of Dirichlet forms associated with closed positive currents of bidegree $(n-1, n-1)$. We further extend some of the properties stochastically and also exhibit some specific sample path behaviours of the related conformal diffusions. In the classical case that $n=1$, there are notions of the Laplace operator, the Green function, the Dirichlet integral and the Brownian motion, each of which is known to play an equivalent role to the subharmonic function in classical potential theory. In higher complex dimensions, we may think of the family of the above mentioned Dirichlet forms and the family of the conformal diffusions as the counterparts of the Dirichlet integral and Brownian motion respectively. Thus we may well expect that the Dirichlet space theory initiated by Beurling and Deny ([4], [8]) should work intrinsically in understanding and developing the theory related to the plurisubharmonic function.

First of all we describe the preliminary notions and notations. Let D be a bounded open set in the complex n-space $\mathbf{C}^n$. A function u on D taking values in $[-\infty, +\infty)$ is called *plurisubharmonic* (psh in abbreviation) if u is locally integrable on D with respect to the Lebesgue measure (denoted by V),

$$\sum_{\alpha,\beta=1}^{n} \frac{\partial^2 u}{\partial z_\alpha \partial \bar{z}_\beta} \xi_\alpha \bar{\xi}_\beta$$

is a positive distribution for any $\xi \in \mathbf{C}^n$ and

$$u(z) = \inf_{U(z)} V\text{-ess}\sup_{z' \in U(z)} u(z'), \quad z \in D,$$

$U(z)$ ranging over all neighbourhoods of z. The real L^p space based on the Lebesgue measure V is denoted by $L^p(D)$. $\mathcal{P}(D)$ will stand for the set of all psh functions on D and we let $\mathcal{P}_b(D)=\mathcal{P}(D)\cap L^\infty(D)$. We use the notations $d=\partial+\bar\partial$ and $d^c=i(\bar\partial-\partial)$. Thus

$$dd^c u = 2\sum_{\alpha,\beta=1}^{n} \frac{\partial^2 u}{\partial z_\alpha \partial \bar z_\beta} i dz_\alpha \wedge d\bar z_\beta$$

sends $u\in\mathcal{P}(D)$ into the space of closed positive currents of bidegree $(1,1)$ (see Lelong [16] for the latter notion). Given $u\in\mathcal{P}(D)\cap L^\infty_{\mathrm{loc}}(D)$ and a closed positive current θ of bidegree (r,r) $(1\leq r\leq n-1)$, the formula

$$\int dd^c u\wedge\theta\wedge\psi = \int u\theta\wedge dd^c\psi$$

for test forms ψ of bidegree $(n-r-1, n-r-1)$ defines a closed positive current $dd^c u\wedge\theta$ of bidegree $(r+1,r+1)$. Thus $dd^c\wedge dd^c u_2\wedge\ldots\wedge dd^c u_r$ is well defined as a closed positive current of bidegree (r,r) for any $u_1,u_2,\ldots,u_r\in\mathcal{P}(D)\cap L^\infty_{\mathrm{loc}}(D)$, $1\leq r\leq n$. For $E\subset D$, we denote by u_E the upper envelope

$$u_E(z)=\sup\{v(z)\colon v\in\mathcal{P}(D),\ v\leq 0 \text{ on } D,\ v\leq -1 \text{ on } E\}$$

and by u_E^* its upper regularization:

$$u_E^*(z)=\overline{\lim_{z'\to z}}\, u_E(z'),\quad z\in D.$$

We then introduce the set function $C_\#$ on D by

$$C_\#(E)=-\int_D u_E^*(z)\,dV(z),\quad E\subset D.$$

This type of set function was considered by Cegrell [5], Bedford-Taylor [3] and also in [10], [11]. A set $N\subset D$ is said to be *pluri-negligible* if there exists a locally uniformly bounded family of psh functions such that, denoting the upper envelope of the family by u and the upper regularization of u by u^*, N is contained in the set $\{z\in D\colon u(z)<u^*(z)\}$. A set $N\subset \mathbf{C}^n$ is called *pluripolar* if each $z\in N$ admits a neighbourhood $U(z)$ and a function $p\in\mathcal{P}(U(z))$ such that $N\cap U(z)\subset p^{-1}(-\infty)$. We shall write $E\subset\subset D$ to indicate that $\bar E$ is a compact subset of D.

Given a closed positive current θ of bidegree $(n-1,n-1)$, we let

$$\mathscr{E}^\theta(\varphi, \psi) = \int_D d\varphi \wedge d^c\psi \wedge \theta, \quad \varphi, \psi \in C_0^\infty(D). \tag{1.1}$$

$\mathscr{E}^\theta$ is then a non-negative definite real symmetric bilinear form on $C_0^\infty(D)$ satisfying a specific Markovian property and local property (see Appendix (§9) for these properties). In accordance with the authors' previous paper [12], we use the following term. If $\mathscr{E}^\theta$ is closable on $L^2(D; m)$ for some positive Radon measure m on D with $\mathrm{supp}[m]=D$, then we say that (θ, m) is an *admissible pair*. In this case the closure of $\mathscr{E}^\theta$ is denoted by $\mathscr{E}^\theta$ again and the domain of the closure is designated by $\mathscr{F}^\theta$. $\mathscr{F}^\theta$ (resp. $\mathscr{E}^\theta$) is then a Dirichlet space (resp. Dirichlet form) on $L^2(D; m)$ possessing $C_0^\infty(D)$ as its core. The terms "$\mathscr{E}^\theta$-polar" and "$\mathscr{E}^\theta$-quasi-continuous" will be used in relation to the capacity defined by the metric $\mathscr{E}_1^\theta(\varphi, \varphi)=\mathscr{E}^\theta(\varphi, \varphi)+(\varphi, \varphi)_{L^2(D;m)}$ ([9]).

From §3 to §6 of the present paper, we make use of the Dirichlet forms $\mathscr{E}^\theta$ for suitably chosen currents θ to prove the five properties of psh functions listed below which have played quite important roles in resolving complex Monge-Ampère equations ([1], [3], [7], [11]), in developing the relevant potential theory ([3], [18], [19], [20], [21]) and also in the study of conformal diffusions ([10], [11], [12]):

(P.1) continuity of the measure $v^{(0)} dd^c v^{(1)} \wedge \ldots \wedge dd^c v^{(n)}$ under monotone (increasing or decreasing) limits of $v^{(0)}, v^{(1)}, \ldots, v^{(n)} \in \mathscr{P}(D) \cap L^\infty_{\mathrm{loc}}(D)$.

(P.2) comparison theorem:

$$u, v \in \mathscr{P}_b(D), \quad \liminf_{\zeta \to \partial D} (u(\zeta) - v(\zeta)) \geq 0 \quad \Rightarrow \quad \int_{\{u<v\}} (dd^c u)^n \geq \int_{\{u<v\}} (dd^c v)^n.$$

(P.3) minimum principle:

$$u, v \in \mathscr{P}_b(D), \quad (dd^c u)^n \leq (dd^c v)^n, \quad \liminf_{\zeta \to \partial D} (u(\zeta) - v(\zeta)) \geq 0 \quad \Rightarrow \quad u(z) \geq v(z), \quad z \in D.$$

(P.4) pluripolarity of pluri-negligible sets.

(P.5) $C_\#$-quasi-continuity of psh functions.

The first four properties were obtained in Bedford-Taylor [3] by using the property

(P.6) C_{BT}-quasi-continuity of psh functions,

where the set function C_{BT} (which we call the Bedford-Taylor capacity) is defined by

$$C_{\mathrm{BT}}(E) = \sup\left\{ \int_E (dd^c p)^n : p \in \mathscr{P}(D),\ 0 \leq p \leq 1 \right\}$$

for compact sets $E\subset D$. The property (P.5) was derived in [10] also from (P.6). But our present method of proving the listed five properties does not require (P.6). As we see in §4 the notions of quasi-continuity in C_{BT} and $C_{\#}$ are actually equivalent.

Given an admissible pair (θ, m), we show in §2 that any $u\in\mathscr{P}(D)\cap L^{\infty}_{\mathrm{loc}}(D)$ is an $\mathscr{E}^{\theta}$-quasi-continuous function in $\mathscr{F}^{\theta}_{\mathrm{loc}}$ and that this correspondence into $\mathscr{F}^{\theta}_{\mathrm{loc}}$ is continuous under the monotone (increasing or decreasing) limit. We next consider the family

$$\mathscr{P}_{+}(D)=\{p\in\mathscr{P}_{\mathrm{b}}(D):\ p(z)=q(z)+\delta|z|^2,\ z\in D \text{ for } q\in\mathscr{P}_{\mathrm{b}}(D) \text{ and } \delta>0\}. \quad (1.2)$$

It will be seen in §2 that, if θ is given by

$$\theta=dd^{c}p^{(1)}\wedge\ldots\wedge dd^{c}p^{(n-1)},\quad p^{(1)},\ldots,p^{(n-1)}\in\mathscr{P}_{+}(D), \quad (1.3)$$

then the pair (θ, V) is admissible. This choice of the underlying measure (rather than $\theta\wedge dd^{c}|z|^2$) will be crucial in later applications.

In §3, we give a new proof of the property (P.1) as a straightforward application of the analytical results of §2. We further present a variant of (P.1) by generalizing the factor $v^{(0)}$ and thereby prove the right directedness of the space of subsolutions of a Monge-Ampère equation.

From §4 however, we add probabilistic considerations in terms of the diffusion process $\mathbf{M}^{\theta}=(Z_t,\zeta,P^{\theta}_z)$ associated with the Dirichlet space $(\mathscr{F}^{\theta},\mathscr{E}^{\theta})$. $\mathbf{M}^{\theta}$ is called *conformal* because the stochastic process (Z_t, P^{θ}_z) is a conformal martingale for each z ([10, 11, 12]).

A stochastic extension of the minimum principle (P.3) has been given in [11] by employing the stochastic boundary limits along sample paths of the conformal diffusion $\mathbf{M}^{\theta}$ for θ given by

$$\theta=(dd^{c}u)^{n-1}+(dd^{c}u)^{n-2}\wedge dd^{c}v+\ldots+(dd^{c}v)^{n-1},\quad u\in\mathscr{P}_{\mathrm{b}}(D),\ v\in\mathscr{P}_{+}(D). \quad (1.4)$$

We show in §4 that an analogous extension of the comparison theorem (P.2) is possible. But this time we take, as an underlying measure, the Lebesgue measure V rather than $\theta\wedge dd^{c}|z|^2$. Our result of §4 will imply yet another version of Theorem 2 of [11].

Turning to the proof of (P.4) and (P.5), we denote by $\mathscr{E}^{(p)}$ the Dirichlet form on $L^2(D)$ for $\theta=(dd^{c}p)^{n-1}$, $p\in\mathscr{P}_{+}(D)$. The associated conformal diffusion is denoted by $\mathbf{M}^{(p)}=(Z_t,\zeta,P^{(p)}_z)$. In §6, we shall prove the properties (P.4) and (P.5) along with the expression

$$C_{\#}(E) = \sup_{p \in \mathscr{P}_+(D)} \int_D P_z^{(p)}(\sigma_E < +\infty)\, V(dz) \tag{1.5}$$

holding for a strongly pseudo-convex domain D and any Borel $E \subset D$. A key step is in § 5, where we prove (1.5) for compact E by estimating $C_{\#}(E)$ from above probabilistically. At this stage, we use the fact that $(dd^c u_K^*)^n = 0$ on $D - K$ for compact K (Lemma 4.4). Lemma 4.4 was proven in [3; Proposition 5.3(i)] and the proof required three things: properties (P.1), (P.3) and an existence theorem for the Monge-Ampère equation $(dd^c u)^n = 0$ on a ball with smooth boundary data [1: Theorem 8.1]. The last theorem is the only fact we need to employ which is not directly linked to our Dirichlet forms $\mathscr{E}^\theta$.

In § 6, we prove (P.4) and (P.5) as direct applications of the upper estimate of § 5 and the continuity property (P.1). $C_{\#}$ then becomes a Choquet capacity by (P.4), and (1.5) extends from compact sets to Borel sets. The expression (1.5) for compact E has been shown in [12] for a slightly different family of diffusions $\mathbf{M}^{(p)}$ but the very validity of the property (P.4) was presupposed in [12]. We emphasize that the probabilistic argument in § 5 involves only an elementary principle in the diffusion theory—the stochastic super-mean-valued property of $\mathscr{E}$-superharmonic functions, which is formulated in the appendix (§ 9) in the framework of the general Dirichlet space theory for the sake of convenience for reference. This principle was also utilized in the direct proof of (P.3) in [11]. The probabilistic argument in § 4 to prove (P.2) is even simpler in that it only involves computations of resolvents.

The expression (1.5) means that a set is pluripolar iff it is unattainable by the diffusion $\mathbf{M}^{(p)}$ for any $p \in \mathscr{P}_+(D)$. In § 7, we consider a simple example where

$$\begin{aligned} D &= \{z \in \mathbf{C}^2 \colon |z_1|^2 + |z_2|^2 < 3\} \\ E &= \{z \in \mathbf{C}^2 \colon y_1 = y_2 = 0,\ |x_1| < 1,\ |x_2| < 1\}. \end{aligned} \tag{1.6}$$

E is not pluripolar but polar with respect to the Newtonian capacity of $\mathbf{R}^4$. We can see that the conformal diffusion $\mathbf{M}^{(p)}$ associated with $p(z) = \frac{1}{2}(|y_1| + |y_2|) + \frac{1}{4}|z|^2$ $(\in \mathscr{P}_+(D))$ is actually attainable to the set E. We explain how the typical sample path of $\mathbf{M}^{(p)}$ behaves differently from $\mathbf{R}^4$-Brownian motion.

Up to § 7, we deal with the minimum Dirichlet space $\mathscr{F}^\theta$ in the sense that $C_0^\infty(D)$ is a core of $\mathscr{F}^\theta$. § 8 concerns the behaviours of $\mathscr{E}^\theta(\varphi, \psi)$ for $\varphi, \psi \in C^\infty(\bar{D})$. We study a related closability and related inequalities involving a surface integral.

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§ 2. Basic relations between $\mathscr{E}^\theta$ and $\mathscr{P}(D)$

Given a closed positive current θ of bidegree $(n-1, n-1)$, we define the bilinear form $\mathscr{E}^\theta$ on $C_0^\infty(D)$ by (1.1). An integration by parts yields

$$\mathscr{E}^\theta(\varphi, \psi) = -\int_D \varphi dd^c\psi \wedge \theta, \quad \varphi, \psi \in C_0^\infty(D). \tag{2.1}$$

which implies (cf. [12; Lemma 1])

$$\mathscr{E}^\theta \text{ is closable on } L^2(D;\ \theta \wedge dd^c|z|^2). \tag{2.2}$$

Together with this, we shall utilize the following Schwarz inequality and Poincaré type inequality, which are found in [18]:

$$\left(\int_D \xi\eta d\varphi \wedge d^c\psi \wedge \theta\right)^2 \leq \int_D \xi^2 d\varphi \wedge d^c\varphi \wedge \theta \int_D \eta^2 d\psi \wedge d^c\psi \wedge \theta, \quad \xi, \eta \in C_0^\infty(D), \quad \varphi, \psi \in C^\infty(D), \tag{2.3}$$

$$\int_D \varphi^2 dd^c q \wedge \theta \leq 8\|q\|_\infty \mathscr{E}^\theta(\varphi, \varphi), \quad q \in \mathscr{P}_b(D), \quad \varphi \in C_0^\infty(D), \tag{2.4}$$

(2.3) particularly implies

$$\mathscr{E}^\theta(\eta\varphi, \eta\varphi) \leq 2\int_D \eta^2 d\varphi \wedge d^c\varphi \wedge \theta + 2\int_D \varphi^2 d\eta \wedge d^c\eta \wedge \theta, \quad \eta \in C_0^\infty(D), \quad \varphi \in C^\infty(D). \tag{2.5}$$

LEMMA 2.1. *Consider an open set $D_1 \subset\subset D$ and a sequence $\{v_k\}$ of functions on D such that $v_k \in \mathscr{P}(D_1) \cap C^\infty(D_1)$ and $\{v_k\}$ is decreasing and locally uniformly bounded on D_1. Then, for any $\eta \in C_0^\infty(D)$ with* supp$[\eta] \subset D_1$. *$\{\eta v_k\}$ is a Cauchy sequence with respect to $\mathscr{E}^\theta$.*

Proof. We may assume $0 \leq v_k \leq M$ on $K = \text{supp}[\eta]$ for some positive constant M. Since

$$dd^c(v_k^2) \wedge \theta = 2dv_k \wedge d^c v_k \wedge \theta + 2v_k dd^c v_k \wedge \theta \geq 2dv_k \wedge d^c v_k \wedge \theta$$

on K, we have

$$\int_D \eta^2 dv_k \wedge d^c v_k \wedge \theta \leq \int_D \eta^2 dd^c(v_k^2) \wedge \theta = \int_D v_k^2 dd^c(\eta^2) \wedge \theta \leq M^2 C \int_K dd^c|z|^2 \wedge \theta$$

where C depends only on η and n. Hence, if we let $a_k = \mathscr{E}_\theta(\eta v_k, \eta v_k)$, then we have from (2.5) and the Schwarz inequality (2.3)

$$\sup_k a_k < \infty, \quad \lim_{k,l\to\infty} \int_D \eta(v_l - v_k)\, d\eta \wedge d^c v_k \wedge \theta = 0. \tag{2.6}$$

Now, for $k<l$,

$$\mathscr{E}^\theta(\eta v_k - \eta v_l, \eta v_k - \eta v_l) = a_l - a_k + 2\{a_k - \mathscr{E}^\theta(\eta v_l, \eta v_k)\}.$$

Since

$$\int_D \eta^2 (v_l - v_k)\, dd^c v_k \wedge \theta \leq 0,$$

we have

$$a_k - \mathscr{E}^\theta(\eta v_l, \eta v_k) = \int_D \eta(v_l - v_k)\, dd^c(\eta v_k) \wedge \theta \leq b_{k,l}$$

where

$$b_{k,l} = \int_D \eta(v_l - v_k)\, v_k\, dd^c \eta \wedge \theta + 2\int_D \eta(v_l - v_k)\, d\eta \wedge d^c v_k \wedge \theta.$$

Hence

$$\mathscr{E}^\theta(\eta v_k - \eta v_l, \eta v_k - \eta v_l) \leq a_l - a_k + 2b_{k,l}, \quad k<l. \tag{2.7}$$

In view of (2.6), $\{a_k\}$ is bounded and $b_{k,l} \to 0$ as $k, l \to \infty$. Therefore (2.7) means first that the finite limit $\lim_{k\to\infty} a_k$ exists and secondly that $\{\eta v_k\}$ is $\mathscr{E}^\theta$-Cauchy. q.e.d.

For $v \in \mathscr{P}(D)$, denote by v^δ the function $u * \alpha_\delta$ with the mollifier α_δ. On each open set $D_1 \subset\subset D$, v^δ then belongs to $\mathscr{P}(D_1) \cap C^\infty(D_1)$ and decreases to v on D_1 as $\delta \downarrow 0$ ([16]). Therefore the next theorem is immediate from Lemma 2.1 and (2.1).

THEOREM 2.2. *Let $(\mathscr{F}^\theta, \mathscr{E}^\theta)$ be the Dirichlet space for an admissible pair (θ, m). Then any $v \in \mathscr{P}(D) \cap L^\infty_{\mathrm{loc}}(D)$ is an $\mathscr{E}^\theta$-quasicontinuous function in $\mathscr{F}^\theta_{\mathrm{loc}}$. ηv^δ is $\mathscr{E}^\theta$-convergent to ηv as $\delta \downarrow 0$ for any $\eta \in C_0^\infty(D)$. Further*

$$\mathscr{E}^\theta(\xi v, \varphi) = -\int_D \varphi dd^c v \wedge \theta, \quad \varphi \in C_0^\infty(D) \tag{2.8}$$

whenever $\xi \in C_0^\infty(D)$ *and* $\xi = 1$ *on* $\mathrm{supp}[\varphi]$.

We now turn to the proof of the following theorem.

THEOREM 2.3. *Let* $(\mathscr{F}^\theta, \mathscr{E}^\theta)$ *be the Dirichlet space for an admissible pair* (θ, m). *Consider* $v_k \in \mathscr{P}(D) \cap L^\infty_{\mathrm{loc}}(D)$, $k = 1, 2, \ldots,$ *and* $v \in \mathscr{P}(D) \cap L^\infty_{\mathrm{loc}}(D)$ *such that* $\{v_k\}$ *is monotone (increasing or decreasing), locally uniformly bounded and* $\lim_{k\to\infty} v_k(z) = v(z)$ *for* m-a.e. $z \in D$. *Then* ηv_k *is* $\mathscr{E}^\theta$*-convergent to* ηv *as* $k \to \infty$ *for any* $\eta \in C_0^\infty(D)$.

Proof. The assertion being local, it suffices to prove that each point $z \in D$ admits a neighbourhood U such that ηv_k is an $\mathscr{E}^\theta$-Cauchy sequence for any $\eta \in C_0^\infty(D)$ with $\mathrm{supp}[\eta] \subset U$. Therefore the following reduction is possible. Take open sets U and D_1 such that $U \subset\subset D_1 \subset\subset D$ and D_1 is a strongly pseudo-convex domain with a determining smooth psh function ϱ: $D_1 = \{\varrho < 0\}$. According to [3; p. 5], there exist then a compact set K with $U \subset K \subset D_1$, constants $A \geq 0$, B and a sequence $\{\hat{v}_k\}$ of uniformly bounded monotone (increasing or decreasing) psh functions on D_1 such that $\hat{v}_k = v_k$ on U and $\hat{v}_k = A\varrho + B$ on $D_1 - K$. Take next an open set E with $K \subset E \subset\subset D_1$ and a function $\xi \in C_0^\infty(D)$ such that $\xi = 1$ on E and $\mathrm{supp}[\xi] \subset D_1$.

We see from (2.5) and Theorem 2.2 that, for $\eta \in C_0^\infty(D)$ with $0 \leq \eta \leq 1$ and $\mathrm{supp}[\eta] \subset U$,

$$\begin{aligned}\mathscr{E}^\theta(\eta(v_k - v_l), \eta(v_k - v_l)) &= \mathscr{E}^\theta(\eta\xi(\hat{v}_k - \hat{v}_l), \eta\xi(\hat{v}_k - \hat{v}_l)) \\ &\leq 2\int_D \xi^2 (v_k - v_l)^2 d\eta \wedge d^c\eta \wedge \theta + 2\mathscr{E}^\theta(\xi(\hat{v}_k - \hat{v}_l), \xi(\hat{v}_k - \hat{v}_l)).\end{aligned}$$

Hence it is enough to show that $\{\xi \hat{v}_k\}$ is $\mathscr{E}^\theta$-Cauchy. To this end, we first assume that $\hat{v}_k$ is increasing.

By (2.1), we have for $k < l$

$$-\mathscr{E}^\theta(\xi\hat{v}_k^\delta, \xi\hat{v}_l^\delta) = \int_D \xi\hat{v}_k^\delta dd^c(\xi\hat{v}_l^\delta) \wedge \theta = \int_E \hat{v}_k^\delta dd^c\hat{v}_l^\delta \wedge \theta + \int_{D-E} \xi\hat{v}_k^\delta dd^c(\xi\hat{v}_l^\delta) \wedge \theta$$

which is not greater than

$$\int_E \hat{v}_l^\delta dd^c\hat{v}_l^\delta \wedge \theta + \int_{D-E} \xi\hat{v}_l^\delta dd^c(\xi\hat{v}_l^\delta) \wedge \theta = -\mathscr{E}^\theta(\xi\hat{v}_l^\delta, \xi\hat{v}_l^\delta)$$

because $\hat{v}_k^\delta$ is increasing in k and $\xi\hat{v}_k^\delta$ is independent of k on $D-E$ for sufficiently small $\delta>0$. Consequently

$$\mathscr{E}^\theta(\xi\hat{v}_k^\delta-\xi\hat{v}_l^\delta,\xi\hat{v}_k^\delta-\xi\hat{v}_l^\delta)\leq\mathscr{E}^\theta(\xi\hat{v}_k^\delta,\xi\hat{v}_k^\delta)-\mathscr{E}^\theta(\xi\hat{v}_l^\delta,\xi\hat{v}_l^\delta).$$

By letting $\delta\downarrow 0$, we get on account of Theorem 2.2,

$$\mathscr{E}^\theta(\xi\hat{v}_k-\xi\hat{v}_l,\xi\hat{v}_k-\xi\hat{v}_l)\leq\mathscr{E}^\theta(\xi\hat{v}_k,\xi\hat{v}_k)-\mathscr{E}^\theta(\xi\hat{v}_l,\xi\hat{v}_l),\quad k<l,$$

which means first that $\mathscr{E}^\theta(\xi\hat{v}_k,\xi\hat{v}_k)$ is decreasing and secondly that $\{\xi\hat{v}_k\}$ is $\mathscr{E}^\theta$-Cauchy.

Assume next that $\hat{v}_k$ is decreasing. Then

$$\mathscr{E}^\theta(\xi\hat{v}_k-\xi\hat{v}_l,\xi\hat{v}_k-\xi\hat{v}_l)\leq\mathscr{E}^\theta(\xi\hat{v}_l,\xi\hat{v}_l)-\mathscr{E}^\theta(\xi\hat{v}_k,\xi\hat{v}_k),\quad k<l,$$

and $\mathscr{E}^\theta(\xi\hat{v}_k,\xi\hat{v}_k)$ is increasing this time. Let $\hat{v}=\lim_{k\to\infty}\hat{v}_k$, then $\hat{v}\in\mathscr{P}_b(D_1)$ and we can see as above

$$\mathscr{E}^\theta(\xi\hat{v}_k^\delta,\xi\hat{v}_k^\delta)\leq\mathscr{E}^\theta(\xi\hat{v}^\delta,\xi\hat{v}_k^\delta)\leq\mathscr{E}^\theta(\xi\hat{v}^\delta,\xi\hat{v}^\delta)\quad\text{and}\quad\mathscr{E}^\theta(\xi\hat{v}_k,\xi\hat{v}_k)\leq\mathscr{E}^\theta(\xi\hat{v},\xi\hat{v})$$

by letting $\delta\downarrow 0$. Therefore $\{\xi\hat{v}_k\}$ is $\mathscr{E}^\theta$-Cauchy again. q.e.d.

By virtue of Theorem 2.2, the functions v_k and v in Theorem 2.3 are $\mathscr{E}^\theta$-quasi-continuous. Further the function v_0 defined by $v_0(z)=\lim_{k\to\infty}v_k(z)$, $z\in D$, is also $\mathscr{E}^\theta$-quasi-continuous because v_k converges to v_0 in the topology of $\mathscr{F}^\theta_{\mathrm{loc}}$. But $v=v_0$ m-a.e. by assumption and consequently $v=v_0$ up to an $\mathscr{E}^\theta$-polar set.

THEOREM 2.4. *Let $(\mathscr{F}^\theta,\mathscr{E}^\theta)$ be the Dirichlet space for an admissible pair (θ,m). If m is absolutely continuous with respect to the Lebesgue measure V, then any pluri-negligible set is $\mathscr{E}^\theta$-polar.*

Proof. By Choquet's lemma, any pluri-negligible set is contained in the set $N=\{v_0<v\}$, where v_0 is the limit function of some increasing sequence $\{v_k\}$ of locally uniformly bounded psh functions and v is the upper regularization of v_0. In particular, v is psh. Since $V(N)=0$, we have $m(N)=0$ by the assumption. Hence we are in the situation of Theorem 2.3 and we get $\mathrm{Cap}^\theta(N)=0$ by the preceding observation. q.e.d.

As our next task in this section, we consider the family $\mathscr{P}_+(D)$ of psh functions and the closed positive current θ of bidegree $(n-1,n-1)$ defined by (1.2) and (1.3) respectively.

THEOREM 2.5. *If θ is defined by* (1.3), *then* (θ, V) *is admissible. In other words,* $\mathscr{E}^{\theta}$ *is closable on* $L^2(D)$.

Proof. We can write $p^{(1)}, \ldots, p^{(n-1)} \in \mathscr{P}_+(D)$ as

$$p^{(1)} = q^{(1)} + \delta|z|^2, \ldots, \quad p^{(n-1)} = q^{(n-1)} + \delta|z|^2, \quad q^{(1)}, \ldots, q^{(n-1)} \in \mathscr{P}_b(D), \quad \delta > 0.$$

We then have

$$dd^c p^{(1)} \wedge \ldots \wedge dd^c p^{(n-1)} = \delta^{n-1}(dd^c|z|^2)^{n-1} + \sum_{l=1}^{n-1} \delta^{n-l-1} \sum_{1 \leq i_1 < \ldots < i_l \leq n-1} \theta^{i_1 \cdots i_l}$$

where

$$\theta^{i_1 \cdots i_l} = dd^c q^{(i_1)} \wedge \ldots \wedge dd^c q^{(i_l)} \wedge (dd^c|z|^2)^{n-l-1}$$

and accordingly

$$\mathscr{E}^{\theta}(\varphi, \psi) = \delta^{n-1}\mathscr{E}^{(0)}(\varphi, \psi) + \sum_{l=1}^{n-1} \delta^{n-l-1} \sum_{1 \leq i_1 < \ldots < i_l \leq n-1} \mathscr{E}^{i_1 \cdots i_l}(\varphi, \psi) \tag{2.9}$$

with

$$\begin{aligned} \mathscr{E}^{(0)}(\varphi, \psi) &= \int_D d\varphi \wedge d^c\psi \wedge (dd^c|z|^2)^{n-1} \\ \mathscr{E}^{i_1 \cdots i_l}(\varphi, \psi) &= \mathscr{E}^{\theta^{i_1 \cdots i_l}}(\varphi, \psi). \end{aligned} \tag{2.10}$$

Suppose that a sequence $\varphi_k \in C_0^{\infty}(D)$ constitutes an $\mathscr{E}^{\theta}$-Cauchy sequence and converges to zero in $L^2(D)$. Then $\{\varphi_k\}$ is $\mathscr{E}^{(0)}$-Cauchy and $\mathscr{E}^{i_1 \cdots i_l}$-Cauchy as well. But $\mathscr{E}^{(0)}$ is a constant multiple of the usual Dirichlet integral which is closable on $L^2(D)$. Hence we have $\mathscr{E}^{(0)}(\varphi_k, \varphi_k) \to 0$, $k \to \infty$. On the other hand, the Poincaré type inequality (2.4) with $q = q^{(i_1)}$ and $\theta = (dd^c|z|^2)^{n-1}$ reads

$$\int_D \varphi^2 \theta^{i_1} \wedge dd^c|z|^2 \leq 8\|q^{(i_1)}\|_{\infty} \mathscr{E}^{(0)}(\varphi, \varphi), \quad 1 \leq i_1 \leq n-1.$$

Therefore $\varphi_k \to 0$ in $L^2(D; \theta^{i_1} \wedge dd^c|z|^2)$ and consequently $\mathscr{E}^{i_1}(\varphi_k, \varphi_k) \to 0$ as $k \to +\infty$ in view of (2.2) for $\theta = \theta^{i_1}$, $1 \leq i_1 \leq n-1$. Inequality (2.4) again reads

$$\int_D \varphi^2 \theta^{i_1 i_2} \wedge dd^c|z|^2 \leq 8\|q^{(i_1)}\|_{\infty} \mathscr{E}^{i_2}(\varphi, \varphi), \quad 1 \leq i_1 < i_2 \leq n-1,$$

which, together with (2.2) for $\theta=\theta^{i_1 i_2}$, means

$$\lim_{k\to\infty} \mathscr{E}^{i_1 i_2}(\varphi_k, \varphi_k)=0.$$

Using (2.2) and (2.4) repeatedly this way, we see that

$$\lim_{k\to\infty} \mathscr{E}^{i_1\cdots i_l}(\varphi_k, \varphi_k)=0 \quad \text{for } 1\leqslant i_1<\ldots<i_l\leqslant n-1,\ 1\leqslant l\leqslant n-1,$$

as was to be proved. q.e.d.

As a corollary of Theorem 2.4 and Theorem 2.5, we get the following property apparently weaker than (P.4):

COROLLARY 2.6. *If θ is given by* (1.3), *then any plurinegligible set is $\mathscr{E}^\theta$-polar, $\mathscr{E}^\theta$ being considered on $L^2(D)$.*

If θ of (1.3) is expressed as in the proof of Theorem 2.5, then inequality (2.4) for $q=|z|^2$ and $\theta=(dd^c|z|^2)^{n-1}$ implies

$$\int_D \varphi^2 dV \leqslant \frac{2\gamma}{4^{n-1}n!\delta^{n-1}} \mathscr{E}^\theta(\varphi, \varphi), \quad \varphi \in C_0^\infty(D), \tag{2.11}$$

where $\gamma=\sup_{z\in D}|z|^2$. The same inequality for $q=q^{(0)}\in \mathscr{P}_b(D)$ and present θ give

$$\int_D \varphi^2 dd^c q^{(0)}\wedge dd^c q^{(1)}\wedge\ldots\wedge dd^c q^{(n-1)} \leqslant 8\|q^{(0)}\|_\infty \mathscr{E}^\theta(\varphi, \varphi), \quad \varphi\in C_0^\infty(D). \tag{2.12}$$

§ 3. Continuity of the measure $v^{(0)}dd^c v^{(1)}\wedge\ldots\wedge dd^c v^{(n)}$ under monotone limits

The first half of this section is devoted to the proof of the property (P.1). We only use Lemma 2.1, Theorem 2.2, Theorem 2.3 and Theorem 2.5 of § 2.

LEMMA 3.1. *For any $v^{(1)}, \ldots, v^{(n)} \in \mathscr{P}_b(D)$*

$$dd^c v^{(1)}\wedge dd^c v^{(2)}\wedge\ldots\wedge dd^c v^{(n)} = dd^c v^{(i_1)}\wedge dd^c v^{(i_2)}\wedge\ldots\wedge dd^c v^{(i_n)}$$

where $(i_1, i_2, \ldots, i_n)$ is any permutation of $(1, 2, \ldots, n)$.

Proof. We first show

$$dd^c v^{(1)}\wedge dd^c v^{(2)}\wedge\theta = dd^c v^{(2)}\wedge dd^c v^{(1)}\wedge\theta \tag{3.1}$$

where θ is a closed positive current of bidegree $(n-2, n-2)$. It suffices to prove

$$\lim_{\delta\downarrow 0}\lim_{\delta'\downarrow 0}\int_D \eta dd^c v^{(1),\delta}\wedge dd^c v^{(2),\delta'}\wedge\theta = \lim_{\delta\downarrow 0}\int_D \eta dd^c v^{(1),\delta}\wedge dd^c v^{(2),\delta}\wedge\theta \tag{3.2}$$

for any $\eta\in C_0^\infty(D)$, because the left hand side of (3.2) equals the integral of η against the left hand side of (3.1) and, on the other hand, we can interchange $v^{(i),\delta}$ in the integral of the right hand side of (3.2).

Take a function $\xi\in C_0^\infty(D)$ such that $\xi=1$ on $\mathrm{supp}[\eta]$. Take A large enough so that $\varrho=A|z|^2-\eta\in\mathscr{P}_b(D)$. Rewriting η as $\eta=A|z|^2-\varrho$ and introducing the closed positive currents θ_1, θ_2 of bidegree $(n-1, n-1)$ by $\theta_1=Add^c|z|^2\wedge\theta$, $\theta_2=dd^c\varrho\wedge\theta$, we have

$$\begin{aligned}
&\left|\int_D \eta dd^c v^{(1),\delta}\wedge dd^c(v^{(2),\delta'}-v^{(2),\delta})\wedge\theta\right| \\
&\quad=\left|\int_D d(\xi v^{(1),\delta})\wedge d^c(\xi v^{(2),\delta'}-\xi v^{(2),\delta})\wedge dd^c\eta\wedge\theta\right| \\
&\quad\leqslant|\mathscr{E}^{\theta_1}(\xi v^{(1),\delta}, \xi v^{(2),\delta'}-\xi v^{(2),\delta})|+|\mathscr{E}^{\theta_2}(\xi v^{(1),\delta}, \xi v^{(2),\delta'}-\xi v^{(2),\delta})|
\end{aligned}$$

which tends to zero as δ', $\delta\downarrow 0$ by virtue of Lemma 2.1, proving (3.2). Lemma 3.1 is a consequence of (3.1). For instance, denoting by θ a closed positive current of bidegree $(n-3, n-3)$, we have

$$\begin{aligned}
\int_D \eta dd^c v^{(1)}\wedge dd^c v^{(2)}\wedge dd^c v^{(3)}\wedge\theta &= \int_D \eta dd^c v^{(2)}\wedge dd^c v^{(1)}\wedge dd^c v^{(3)}\wedge\theta \\
&=\lim_{\delta\downarrow 0}\int_D \eta dd^c v^{(1)}\wedge dd^c v^{(3)}\wedge\theta\wedge dd^c v^{(2),\delta} \\
&=\lim_{\delta\downarrow 0}\int_D \eta dd^c v^{(3)}\wedge dd^c v^{(1)}\wedge\theta\wedge dd^c v^{(2),\delta} \\
&=\int_D \eta dd^c v^{(2)}\wedge dd^c v^{(3)}\wedge dd^c v^{(1)}\wedge\theta.
\end{aligned}$$

q.e.d.

Now, for each choice of $q^{(1)}, \ldots, q^{(n-1)}\in\mathscr{P}_b(D)$, we may consider the associated $(n-1, n-1)$ current

$$\theta = dd^c q^{(1)}\wedge\ldots\wedge dd^c q^{(n-1)} \tag{3.3}$$

and the associated bilinear form $\mathscr{E}^\theta$ on $C_0^\infty(D)$. However $q^{(j)}$'s are not necessarily belonging to $\mathscr{P}_+(D)$ and it is convenient to consider a perturbed current

$$\tilde{\theta} = dd^c(q^{(1)}+\delta_0|z|^2)\wedge\ldots\wedge dd^c(q^{(n-1)}+\delta_0|z|^2) \tag{3.4}$$

for a fixed $\delta_0>0$. On account of Theorem 2.5, we may consider the corresponding Dirichlet space $(\mathscr{F}^{\tilde{\theta}}, \mathscr{E}^{\tilde{\theta}})$ on $L^2(D)$. Since

$$\mathscr{E}^{\theta}(\varphi,\varphi) \leqslant \mathscr{E}^{\tilde{\theta}}(\varphi,\varphi), \quad \varphi \in C_0^\infty(D), \tag{3.5}$$

$\mathscr{E}^{\theta}$ also extends to $\mathscr{F}^{\tilde{\theta}}$ and, in the remainder of this section, $\mathscr{E}^{\theta}$ will denote this specific extension. Applying Theorem 2.2 to $(\tilde{\theta}, V)$, $\mathscr{E}^{\theta}$ is seen to be well defined on the linear span of

$$\{\eta v;\ \eta \in C_0^\infty(D),\ v \in \mathscr{P}(D) \cap L^\infty_{\mathrm{loc}}(D)\}$$

and

$$\mathscr{E}^{\theta}(\eta v, \eta v) = \lim_{\delta \downarrow 0} \mathscr{E}^{\theta}(\eta v^{\delta}, \eta v^{\delta}). \tag{3.6}$$

Further, if ξ, $\eta \in C_0^\infty(D)$ and $\xi=1$ on supp$[\eta]$, then we have for u, $v \in \mathscr{P}(D) \cap L^\infty_{\mathrm{loc}}(D)$

$$\mathscr{E}^{\theta}(\eta u, \xi v) = -\int_D \eta u\, dd^c v \wedge \theta, \quad \eta \in C_0^\infty(D) \tag{3.7}$$

because the left side equals

$$\lim_{\delta \downarrow 0} \lim_{\delta' \downarrow 0} \mathscr{E}^{\theta}(\eta u^{\delta}, \xi v^{\delta'})$$

and (2.1) applies. We next apply Theorem 2.3 to $(\tilde{\theta}, V)$ in getting

$$\lim_{k\to\infty} \mathscr{E}^{\theta}(\eta v_k - \eta v, \eta v_k - \eta v) = 0, \quad \eta \in C_0^\infty(D), \tag{3.8}$$

whenever v_k, $k=1,2,\ldots$, and v belong to $\mathscr{P}(D) \cap L^\infty_{\mathrm{loc}}(D)$, $\{v_k\}$ is monotone (increasing or decreasing) and $\lim_{k\to\infty} v_k(z)=v(z)$ for V-a.e. $z \in D$. Keeping these in mind, let us proceed to the proof of property (P.1).

THEOREM 3.2. *Suppose that* $v_k^{(i)}$, $v^{(i)} \in \mathscr{P}(D) \cap L^\infty_{\mathrm{loc}}(D)$, $0 \leqslant i \leqslant n$, $k=1,2,\ldots$, *satisfy the following conditions:*

(i) *either* $v_k^{(0)}$ *are locally uniformly bounded in k with* $\lim_{k\to\infty} v_k^{(0)}(z)=v^{(0)}(z)$, $z \in D$, *or* $v_k^{(0)}$ *are increasing in k with* $\lim_{k\to\infty} v_k^{(0)}=v^{(0)}$ V-a.e.

(ii) *either* $v_k^{(i)}$ $1 \leqslant i \leqslant n$, *are simultaneously decreasing in k with* $\lim_{k\to\infty} v_k^{(i)}(z)=v^{(i)}(z)$,

$z\in D$, or $v_k^{(i)}$, $1\le i\le n$, are simultaneously increasing in k with $\lim_{k\to\infty} v_k^{(i)}=v^{(i)}$ V-a.e. *Then we have*

$$v_k^{(0)}dd^cv_k^{(1)}\wedge dd^cv_k^{(2)}\wedge\ldots\wedge dd^cv_k^{(n)} \underset{k\to\infty}{\to} v^{(0)}dd^cv^{(1)}\wedge dd^cv^{(2)}\wedge\ldots\wedge dd^cv^{(n)}$$

as the vague limit of Radon measures.

Proof. It is enough to show for $\eta\in C_0^\infty(D)$

$$\lim_{k\to\infty}\int_D \eta v_k^{(0)}dd^cv^{(1)}\wedge\ldots\wedge dd^cv^{(n)}=\int_D \eta v^{(0)}dd^cv^{(1)}\wedge\ldots\wedge dd^cv^{(n)} \tag{3.9}$$

and, for each $m=0,1,\ldots,n-1$ and $\eta\in C_0^\infty(D)$,

$$\begin{aligned}&\lim_{k\to\infty}\int_D \eta v_k^{(0)}dd^cv_k^{(1)}\wedge\ldots\wedge dd^cv_k^{(m)}\wedge dd^cv_k^{(m+1)}\wedge dd^cv^{(m+2)}\wedge\ldots\wedge dd^cv^{(n)}\\ &=\lim_{k\to\infty}\int_D \eta v_k^{(0)}dd^cv_k^{(1)}\wedge\ldots\wedge dd^cv_k^{(m)}\wedge dd^cv^{(m+1)}\wedge dd^cv^{(m+2)}\wedge\ldots\wedge dd^cv^{(n)}.\end{aligned} \tag{3.10}$$

(3.9) is evident when $v_k^{(0)}$ and $v^{(0)}$ satisfy the former condition of (i) and it is also clear in the latter case because the difference of the integrals of the both sides of (3.9) equals, in view of (3.7),

$$\mathscr{E}^\theta(\eta v_k^{(0)}-\eta v^{(0)},\ \xi v^{(1)})$$

for $\theta=dd^cv^{(2)}\wedge\ldots\wedge dd^cv^{(n)}$, $\xi\in C_0^\infty$; $\xi=1$ on supp$[\eta]$, and then (3.8) applies.

As for the proof of (3.10), we can make the same reduction as in the proof of Theorem 2.3, since the assertion is local. In particular, we may assume that D is a strongly pseudo-convex domain with a determining smooth strictly psh function ϱ: $D=\{\varrho<0\}$. For open $U\subset\subset D$, we can choose compact K with $U\subset K\subset D$, constants $A>0$, B and $\hat{v}_k^{(i)}$, $\hat{v}^{(i)}\in\mathscr{P}_b(D)$, $1\le i\le n$, $k=1,2,\ldots$, such that $\{\hat{v}_k^{(i)}\}$ is monotone (increasing or decreasing) in k, $\lim_{k\to\infty}\hat{v}_k^{(i)}=\hat{v}^{(i)}$ V-a.e., $1\le i\le n$, and

$$\hat{v}_k^{(i)}=v_k^{(i)},\quad \hat{v}^{(i)}=v^{(i)}\quad\text{on } U,\quad 1\le i\le n,\ k=1,2,\ldots,$$

$$\hat{v}_k^{(i)}=A\varrho+B,\quad \hat{v}^{(i)}=A\varrho+B\quad\text{on } D-K,\quad 1\le i\le n,\ k=1,2,\ldots.$$

Take an open set E with $K\subset E\subset\subset D$ and a non-negative function $\xi\in C_0^\infty(D)$ with $\xi=1$ on E. Fix any $\eta\in C_0^\infty(D)$ with supp$[\eta]\subset U$. We then see from Lemma 3.1 and the identities (3.7) and (3.8) that the difference of the integrals of the both sides of (3.10) is equal to

$$-\mathscr{E}^{\theta_{m,k}}(\eta v_k^{(0)},\ \xi\hat{v}_k^{(m+1)}-\xi\hat{v}^{(m+1)}) = -\lim_{l\to\infty}\mathscr{E}^{\theta_{m,k}}(\eta v_k^{(0)},\ \xi\hat{v}_k^{(m+1)}-\xi\hat{v}_l^{(m+1)})$$

where

$$\theta_{m,k} = dd^c\hat{v}_k^{(1)}\wedge\ldots\wedge dd^c\hat{v}_k^{(m)}\wedge dd^c\hat{v}^{(m+2)}\wedge\ldots\wedge dd^c\hat{v}^{(n)}. \tag{3.11}$$

For simplicity we denote $\hat{v}_k^{(m+1)}$ and $\hat{v}^{(m+1)}$ by $\hat{v}_k$ and $\hat{v}$ respectively. Thus what we must show is

$$\lim_{k\to\infty}\lim_{l\to\infty}\mathscr{E}^{\theta_{m,k}}(\eta v_k^{(0)},\ \xi\hat{v}_k-\xi\hat{v}_l) = 0. \tag{3.12}$$

By Schwarz' inequality, we have

$$\mathscr{E}^{\theta_{m,k}}(\eta v_k^{(0)},\ \xi\hat{v}_k-\xi\hat{v}_l)^2 \leqslant b_{m,k}\cdot\mathscr{E}^{\theta_{m,k}}(\xi\hat{v}_k-\xi\hat{v}_l,\ \xi\hat{v}_k-\xi\hat{v}_l), \tag{3.13}$$

where $b_{m,k}=\mathscr{E}^{\theta_{m,k}}(\eta v_k^{(0)}, \eta v_k^{(0)})$. In order to estimate the right hand side of (3.13), we proceed essentially on the same line as in the proof of Theorem 2.3. To get a bound of $b_{m,k}$, we first note from the proof of Lemma 2.1 that

$$b_{m,k} = \lim_{\delta\downarrow 0}\mathscr{E}^{\theta_{m,k}}(\eta v_k^{(0),\delta}, \eta v_k^{(0),\delta}) \leqslant CM^2\int_U \theta_{m,k}\wedge dd^c|z|^2$$

for $M=\sup_k\sup_{z\in U}|v_k^{(0)}(z)|$ and $C=C(\eta)>0$. Let $\varphi_0=\varrho+B'$ for sufficiently large B' so that $\varphi_0\geqslant 1$ on D. By assuming that $\{\hat{v}_k^{(i)}\}$ is increasing in k for each i, it then holds that

$$\int_U \theta_{m,k}\wedge dd^c|z|^2 \leqslant \int_D \xi\varphi_0\,\theta_{m,k}\wedge dd^c|z|^2$$

$$= \int_E \hat{v}_k^{(1)}dd^c\varphi_0\wedge dd^c\hat{v}_k^{(2)}\wedge\ldots\wedge dd^c|z|^2 + \int_{D-E}\hat{v}_k^{(1)}dd^c(\xi\varphi_0)\wedge dd^c\hat{v}_k^{(2)}\wedge\ldots\wedge dd^c|z|^2$$

which is not greater than

$$\int_E \hat{v}^{(1)}dd^c\varphi_0\wedge dd^c\hat{v}_k^{(2)}\wedge\ldots\wedge dd^c|z|^2 + \int_{D-E}\hat{v}^{(1)}dd^c(\xi\varphi_0)\wedge dd^c\hat{v}_k^{(2)}\wedge\ldots\wedge dd^c|z|^2$$

$$= \int_D \xi\varphi_0 dd^c\hat{v}^{(1)}\wedge dd^c\hat{v}_k^{(2)}\wedge\ldots\wedge dd^c|z|^2$$

because $\hat{v}_k^{(1)} \leq \hat{v}^{(1)}$ everywhere and $\hat{v}_k^{(1)} = \hat{v}^{(1)} = A\varrho + B$ on $D-E$. Using Lemma 3.1 and the above argument repeatedly, we get

$$\sup_k b_{m,k} \leq CM^2 \int_D \xi\varphi_0 dd^c\hat{v}^{(1)} \wedge \dots \wedge dd^c\hat{v}^{(m)} \wedge dd^c\hat{v}^{(m+2)} \wedge \dots \wedge dd^c|z|^2. \tag{3.14}$$

In case that $\{\hat{v}_k^{(i)}\}$ is decreasing in k, we have in the same way

$$\sup_k b_{m,k} \leq CM^2 \int_D \xi\varphi_0 dd^c\hat{v}_1^{(1)} \wedge \dots \wedge dd^c\hat{v}_1^{(m)} \wedge dd^c\hat{v}^{(m+2)} \wedge \dots \wedge dd^c|z|^2. \tag{3.14)'$$

In view of (3.13), (3.14) and (3.14)', it only remains to show

$$\lim_{k,l\to\infty} \mathscr{E}^{\theta_{m,k}}(\xi v_k - \xi v_l, \xi v_k - \xi v_l) = 0. \tag{3.15}$$

Assume first that $\{v_k^{(i)}\}$ is increasing in k. Then we have already seen in the proof of Theorem 2.3 the inequality

$$\mathscr{E}^{\theta_{m,k}}(\xi v_k^\delta - \xi v_l^\delta, \xi v_k^\delta - \xi v_l^\delta) \leq \mathscr{E}^{\theta_{m,k}}(\xi v_k^\delta, \xi v_k^\delta) - \mathscr{E}^{\theta_{m,k}}(\xi v_l^\delta, \xi v_l^\delta), \quad k<l. \tag{3.16}$$

But this time we go on further in performing a similar computation:

$$\begin{aligned} -\mathscr{E}^{\theta_{m,k}}(\xi\hat{v}_l^\delta, \xi\hat{v}_l^\delta) &= -\int_D d(\xi\hat{v}_l^\delta) \wedge d^c(\xi\hat{v}_l^\delta) \wedge dd^c\hat{v}_k^{(1)} \wedge \dots \wedge dd^c\hat{v}^{(n)} \\ &= \int_E \hat{v}_k^{(1)} (dd^c\hat{v}_l^\delta)^2 \wedge dd^c\hat{v}_k^{(2)} \wedge \dots \wedge dd^c\hat{v}^{(n)} \\ &\quad + \int_{D-E} \hat{v}_k^{(1)} \{dd^c(\xi\hat{v}_l^\delta)\}^2 \wedge dd^c\hat{v}_k^{(2)} \wedge \dots \wedge dd^c\hat{v}^{(n)}, \end{aligned}$$

which is not greater than

$$\int_D \hat{v}_l^{(1)} (dd^c(\xi\hat{v}_l^\delta))^2 \wedge dd^c\hat{v}_k^{(2)} \wedge \dots \wedge dd^c\hat{v}^{(n)} = -\int_D d(\xi\hat{v}_l^\delta) \wedge d^c(\xi\hat{v}_l^\delta) \wedge dd^c\hat{v}_l^{(1)} \wedge dd^c\hat{v}_k^{(2)} \wedge \dots \wedge dd^c\hat{v}^{(n)}$$

by the same reason as in the preceding computation. Using Lemma 3.1 and the above argument repeatedly, we see that the right hand side of (3.16) is dominated by

$$\mathscr{E}^{\theta_{m,k}}(\xi\hat{v}_k^\delta, \xi\hat{v}_k^\delta) - \mathscr{E}^{\theta_{m,l}}(\xi\hat{v}_l^\delta, \xi\hat{v}_l^\delta), \quad k<l.$$

Now let $\delta \downarrow 0$. Then by (3.6), we get

$$\mathscr{E}^{\theta_{m,k}}(\xi\hat{v}_k - \xi\hat{v}_l, \xi\hat{v}_k - \xi\hat{v}_l) \leq \mathscr{E}^{\theta_{m,k}}(\xi\hat{v}_k, \xi\hat{v}_k) - \mathscr{E}^{\theta_{m,l}}(\xi\hat{v}_l, \xi\hat{v}_l), \quad k<l,$$

which means first that $\mathscr{E}^{\theta_{m,k}}(\xi\hat{v}_k, \xi\hat{v}_k)$ is decreasing in k and secondly that (3.15) is valid.

When $\{\hat{v}_k^{(i)}\}$ is decreasing, it holds on the contrary that

$$\mathscr{E}^{\theta_{m,k}}(\xi\hat{v}_k-\xi\hat{v}_l, \xi\hat{v}_k-\xi\hat{v}_l) \leq \mathscr{E}^{\theta_{m,l}}(\xi\hat{v}_l, \xi\hat{v}_l)-\mathscr{E}^{\theta_{m,k}}(\xi\hat{v}_k, \xi\hat{v}_k), \quad k<l,$$

which means that $\mathscr{E}^{\theta_{m,k}}(\xi\hat{v}_k, \xi\hat{v}_k)$ is increasing. But the same computation as above gives

$$\mathscr{E}^{\theta_{m,k}}(\xi\hat{v}_k, \xi\hat{v}_k) \leq \mathscr{E}^{\theta}(\xi\hat{v}_k, \xi\hat{v}_k)$$

for

$$\theta = dd^c\hat{v}^{(1)} \wedge \ldots \wedge dd^c\hat{v}^{(m)} \wedge dd^c\hat{v}^{(m+2)} \wedge \ldots \wedge dd^c\hat{v}^{(n)},$$

and we have also $\mathscr{E}^{\theta}(\xi\hat{v}_k, \xi\hat{v}_k) \leq \mathscr{E}^{\theta}(\xi\hat{v}, \xi\hat{v})$ from the proof of Theorem 2.3 and equality (3.6). Hence we get (3.15) in this case too. q.e.d.

Although we do not state it explicitly, our method of the proof of Lemma 2.1, Theorem 2.2, Theorem 2.3 and Theorem 3.2 suggests the possibility of extending these assertions by replacing the local boundedness condition for psh functions with certain local integrability conditions. However the monotonicity assumption for the sequences of psh functions in these statements is essential. See Cegrell [6] in this connection.

It is also possible to extend Theorem 3.2 by generalizing the factors $v_k^{(0)}$ and $v^{(0)}$ in the following manner:

PROPOSITION 3.3. *Let* $v_k^{(i)}, v^{(i)}$, $1 \leq i \leq n$, $k=1,2,\ldots$, *be as in Theorem* 3.2. *Consider further functions* $u_k^{(l)}$, $u^{(l)} \in \mathscr{P}(D) \cap L^{\infty}_{\mathrm{loc}}(D)$, $1 \leq l \leq r$, $k=1,2,\ldots$, *such that* $u_k^{(l)}$ *are locally uniformly bounded in k and* $\lim_{k\to\infty} u_k^{(l)}(z)=u^{(l)}(z)$, $z \in D$. *Then, for any bounded continuously differentiable function f* on $\mathbf{R}^r$ with bounded derivatives,

$$f(u_k^{(1)}, \ldots, u_k^{(r)})\, dd^c\hat{v}_k^{(1)} \wedge \ldots \wedge dd^c\hat{v}_k^{(n)} \underset{k\to\infty}{\rightarrow} f(u^{(1)}, \ldots, u^{(r)}) dd^c v^{(1)} \ldots dd^c v^{(n)}$$

as the vague limit of Radon measures.

Proof. Consider θ (resp. $\tilde{\theta}$) of the type (3.3) (resp. (3.4)) and η, $\eta_1 \in C^{\infty}(D)$ with $\eta_1=1$ on $\mathrm{supp}[\eta]$. For any $w^{(1)}, \ldots, w^{(r)} \in \mathscr{F}^{\tilde{\theta}}_{\mathrm{loc}}$, we see from (2.3), (2.5) and (3.5) that $\eta f(w^{(1)}, \ldots, w^{(r)}) \in \mathscr{F}^{\tilde{\theta}}_{\mathrm{loc}}$ and

$$\begin{aligned} &\mathscr{E}^{\theta}(\eta f(w^{(1)}, \ldots, w^{(r)}), \eta f(w^{(1)}, \ldots, w^{(r)})) \\ &\leq 2r\|\eta\|^2_{L^{\infty}(D)} \sum_{l=1}^{r} \|f_{x_l}\|^2_{L^{\infty}(R^r)}\, \mathscr{E}^{\theta}(\eta_1 w^{(l)}, \eta_1 w^{(l)}) + 2\|f\|_{L^{\infty}(\mathbf{R}^r)} \mathscr{E}^{\theta}(\eta, \eta). \end{aligned} \tag{3.17}$$

Now, in order to prove Proposition 3.3, it suffices to show (3.10) with $v_k^{(0)}$ being replaced by $f(u_k^{(1)}, \dots, u_k^{(r)})$. Hence, in view of the proof of Theorem 3.2, it is enough to show the bound

$$\mathscr{E}^{\theta_{m,k}}(\eta f(u_k^{(1)}, \dots, u_k^{(r)}), \eta f(u_k^{(1)}, \dots, u_k^{(r)})) \leqslant C \int_U \theta_{m,k} \wedge dd^c|z|^2,$$

for $0 \leqslant m \leqslant n-1$, $\eta \in C_0^\infty(D)$ with $\operatorname{supp}[\eta] \subset U$ and for some constant C independent of k. Here $\theta_{m,k}$ is given by (3.11). But this bound can be achieved by virtue of the inequality (3.17) holding for $\theta = \theta_{m,k}$, $w^{(l)} = u_k^{(l)}$, $1 \leqslant l \leqslant r$, and for $\eta_1 \in C_0^\infty(D)$ such that $\operatorname{supp}[\eta_1] \subset U$ and $\eta_1 = 1$ on $\operatorname{supp}[\eta]$. q.e.d.

Proposition 3.3 enables us to establish the next theorem.

THEOREM 3.4. *Let μ be a positive Radon measure on D. For $u, v \in \mathscr{P}(D) \cap L_{\mathrm{loc}}^\infty(D)$, we let $w = u \vee v$. If $(dd^c u)^n \geqslant \mu$ and $(dd^c v)^n \geqslant \mu$, then $(dd^c w)^n \geqslant \mu$.*

Proof. Take any $\eta \in C_0^\infty(D)$, $\eta \geqslant 0$ and a sequence $\{u_k\}$ of continuous psh functions decreasing to u on an open set G such that $\operatorname{supp}[\eta] \subset G \subset\subset D$. We let $w_k = u_k \vee v$. Then, by virtue of Theorem 3.2, the measures $\eta(dd^c w_k)^n$ converge weakly to $\eta(dd^c w)^n$ on G as $k \to \infty$. In particular, $\eta(dd^c w_k)^n$ are uniformly bounded on G.

On the other hand, we have from Proposition 3.3 that, for any $f \in C_0^\infty(\mathbf{R}^1)$,

$$\lim_{k\to\infty} \int_G f(u-v)\,\eta(dd^c w_k)^n = \int_G f(u-v)\,\eta(dd^c w)^n,$$

which can be written as

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} f(x)\,dF_k(x) = \int_{-\infty}^{\infty} f(x)\,dF(x) \tag{3.18}$$

with

$$F_k(x) = \int_G I_{(-\infty,x]}(u-v)\,\eta(dd^c w_k)^n$$

and

$$F(x) = \int_G I_{(-\infty,x]}(u-v)\,\eta(dd^c w)^n.$$

Since $u-v$ is bounded on G, the supports of the one dimensional measures dF_k and dF are concentrated on a common finite interval. Hence we see from (3.18) that dF_k converge weakly to dF as $k\to\infty$.

Choose positive $\varepsilon_l \downarrow 0$ so that each ε_l is a continuous point of the measure dF. Then $\lim_{k\to\infty} F_k(\varepsilon_l)=F(\varepsilon_l)$ and hence

$$\lim_{k\to\infty}\int_{\Gamma_l}\eta(dd^c w_k)^n=\int_{\Gamma_l}\eta(dd^c w)^n, \tag{3.19}$$

where $\Gamma_l=\{z\in G: u>v+\varepsilon_l\}$. Applying the same argument to u_k and u, we may further assume

$$\lim_{k\to\infty}\int_{\Gamma_l}\eta(dd^c u_k)^n=\int_{\Gamma_l}\eta(dd^c u)^n. \tag{3.20}$$

Now Γ_l is contained in each open set $\{z\in G: u_k>v+\varepsilon_l\}$ on which $w_k=u_k$ and $(dd^c w_k)^n=(dd^c u_k)^n$. We get therefore from (3.19) and (3.20),

$$\int_{\Gamma_l}\eta(dd^c w)^n=\int_{\Gamma_l}\eta(dd^c u)^n\geqslant\int_{\Gamma_l}\eta\mu.$$

By letting $l\to\infty$, we have $\int_{\{u>v\}}\eta(dd^c w)^n\geqslant\int_{\{u>v\}}\eta\mu$, and by symmetry,

$$\int_{\{u<v\}}\eta(dd^c w)^n\leqslant\int_{\{u<v\}}\eta\mu.$$

Thus the desired inequality $\int_D\eta(dd^c w)^n\geqslant\int_D\eta\mu$ is achieved provided that $\mu(u=v)=0$.

In general, choose positive $\delta_k\downarrow 0$, $k=1,2,\ldots$, such that $\mu(u=v+\delta_k)=0$, $k=1,2,\ldots$. Since $(dd^c u)^n\geqslant\mu$ and $[dd^c(v+\delta_k)]^n=(dd^c v)^n\geqslant\mu$, we have $[dd^c\{u\vee(v+\delta_k)\}]^n\geqslant\mu$ by the preceding observation. Now let $k\to\infty$ and use Theorem 3.2. q.e.d.

As far as bounded continuous psh functions u, v are concerned, this theorem was proven in [1; Proposition 2.9]. [1] also contains a counterexample for locally unbounded psh functions.

To illustrate a use of Theorem 3.4, let us consider a strongly pseudo-convex domain D and a Monge-Ampère equation

$$u\in\mathscr{P}_b(D),\quad (dd^c u)^n=f\,dV\quad\text{on } D \tag{3.21}$$

with boundary condition

$$\lim_{\zeta \to z, \zeta \in D} u(\zeta) = \varphi(z), \quad z \in \partial D, \tag{3.22}$$

for given data $f \in L^\infty(D)$, $f \geq 0$, $\varphi \in C(\partial D)$. The associated Perron Bremermann family is

$$\mathcal{B}(f, \varphi) = \{v \in \mathcal{P}_b(D): (dd^c v)^n \geq f dV \text{ on } D, \overline{\lim_{\zeta \to z}} v(\zeta) \leq \varphi(z), z \in \partial D\}.$$

This family is non-empty, uniformly upper bounded and, by virtue of Theorem 3.4, right directed. Therefore the upper regularization of the upper envelope of the family $\mathcal{B}(f, \varphi)$ satisfies the equation (3.21) by [11; Theorem 6]. A similar statement holds for a weakly pseudo-convex domain with a boundary data being assigned on the Silov boundary.

§ 4. Stochastic extensions of the comparison theorem and the minimum principle

LEMMA 4.1. *Let θ be given by* (1.4) *for $u \in \mathcal{P}_b(D)$ and $v \in \mathcal{P}_+(D)$. Then (θ, V) is admissible.*

Proof. Put $\hat{\theta} = dd^c(u+v))^{n-1}$. Then

$$C_1 \mathcal{E}^{\hat{\theta}}(\varphi, \varphi) \leq \mathcal{E}^{\theta}(\varphi, \varphi) \leq C_2 \mathcal{E}^{\hat{\theta}}(\varphi, \varphi), \quad \varphi \in C_0^\infty(D), \tag{4.1}$$

for constants C_1, $C_2 > 0$. But $(\hat{\theta}, V)$ is admissible by Theorem 2.5. q.e.d.

Given $u \in \mathcal{P}_b(D)$ and $v \in \mathcal{P}_+(D)$, we define $\theta = \theta^{u,v}$ by (1.4) and denote the associated Dirichlet space $(\mathcal{F}^\theta, \mathcal{E}^\theta)$ on $L^2(D)$ and the conformal diffusion $\mathbf{M}^\theta = (Z_t, \zeta, P_z^\theta)$ by $(\mathcal{F}^{u,v}, \mathcal{E}^{u,v})$ and $\mathbf{M}^{u,v} = (Z_t, \zeta, P_z^{u,v})$ respectively. Because of the inequalities (4.1) and (2.11), the life time of $\mathbf{M}^{u,v}$ has a finite expectation

$$E_z^{u,v}(\zeta) < \infty, \quad \mathcal{E}^{u,v}\text{-q.e. } z \in D \tag{4.2}$$

in view of Theorem 9.4 of the appendix. It also follows from Theorem 2.2 and Lemma 9.2 that, for any $q \in \mathcal{P}(D) \cap L_{\text{loc}}^\infty(D)$, $-q$ is $\mathcal{E}^{u,v}$-quasi-continuous, $dd^c q \wedge \theta^{u,v}$ charges no $\mathcal{E}^{u,v}$-polar set and

$$\mathcal{E}^{u,v}(q, \varphi) = -\int_D \tilde{\varphi} dd^c q \wedge \theta^{u,v}, \quad \varphi \in \mathcal{F}_G^{u,v}, \tag{4.3}$$

for any open $G \subset\subset D$, where $\mathscr{F}_G^{u,v}$ is defined by (9.1) for $\mathscr{F}^{u,v}$ and $\tilde{\varphi}$ is an $\mathscr{E}^{u,v}$-quasi-continuous version of φ.

Since $\mathbf{M}^{u,v}$ is a conformal diffusion and D is bounded, we have that $Z_{\zeta-}=\lim_{t\uparrow\zeta} Z_t$ exists and $Z_{\zeta-}\in\partial D$, $P_z^{u,v}$-a.s. on $\{\zeta<\infty\}$ for each $z\in D$. Moreover for any $w\in\mathscr{P}_b(D)$, $w(Z_t)$ is a $P_z^{u,v}$-submartingale and hence $\lim_{t\uparrow\zeta} w(Z_t)$ exists $P_z^{u,v}$-a.s. for $z\in D$. See [11; §2] for more details. Accordingly the following theorem immediately implies the comparison theorem (P.2).

THEOREM 4.2. *Suppose that u, $v\in\mathscr{P}_b(D)$ satisfy for any $\delta>0$*

$$\lim_{t\uparrow\zeta} u(z_t) \geq \lim_{t\uparrow\zeta} v(z_t) \quad P_z^{u,v+\delta|z|^2}\text{-a.s. } V\text{-a.e. } z\in D. \tag{4.4}$$

Then

$$\int_{\{u<v\}} (dd^c v)^n \leq \int_{\{u<v\}} (dd^c u)^n. \tag{4.5}$$

Proof. Fix $\delta>0$ and set $\hat{v}(z)=v(z)+\delta|z|^2-\delta\gamma$, $z\in D$, where $\gamma=\sup_{z\in D}|z|^2$. By (4.4), the function $w=u-\hat{v}$ satisfies

$$\lim_{t\uparrow\zeta} w(Z_t) \geq 0, \quad P_z^{u,v+\delta|z|^2}\text{-a.s. } V\text{-a.e. } z\in D. \tag{4.6}$$

Note that $P_z^{u,v+\delta|z|^2}=P_z^{u,\hat{v}}$ because $\theta^{u,v}$ does not change if we add a constant to v. On the other hand, (4.3) and the identity

$$dd^c w\wedge\theta^{u,\hat{v}} = (dd^c u)^n-(dd^c\hat{v})^n$$

implies that w is $\mathscr{E}^{u,\hat{v}}$-quasi-continuous and

$$-\mathscr{E}^{u,\hat{v}}(w,\varphi) = \int_D \tilde{\varphi}\{(dd^c u)^n-(dd^c\hat{v})^n\}, \quad \varphi\in\mathscr{F}_G^{u,\hat{v}}, \tag{4.7}$$

for any open $G\subset\subset D$.

Consider now the set $S=\{z\in D: w(z)<0\}$. Choose a sequence $\{G_k\}$ of open sets such that $G_k\subset\subset G_{k+1}\subset D$ and $G_k\uparrow D$, $k\to\infty$. We let $S_k=S\cap G_k$ and denote by R_α^k and H_α^k the resolvent and the α-order hitting measure defined as (9.2) for the Borel set $D-S_k$. Then we have from (9.3) that, for non-negative $f\in L^2(D)$, $R_\alpha^k f$ is an $\mathscr{E}^{u,\hat{v}}$-quasi-continuous element of $\mathscr{F}_{S_k}^{u,\hat{v}}$ $(\subset\mathscr{F}_{G_k}^{u,\hat{v}})$ and

$$\mathscr{E}_\alpha^{u,\hat{v}}(w, R_\alpha^k f) = (w - H_\alpha^k w, f)_{S_k}, \quad \alpha > 0, \tag{4.8}$$

for any $\mathscr{E}^{u,\hat{v}}$-quasi-continuous $w \in \mathscr{F}_{\mathrm{loc}}^{u,\hat{v}}$. Here $(\ ,\ )_{S_k}$ denotes the inner product of $L^2(S_k, V)$.

Since $R_\alpha^k 1(z) = 0$ for $\mathscr{E}^{u,\hat{v}}$-q.e. $z \in D - S_k$ by [9; Theorem 4.2.3], we have from (4.7) and (4.8)

$$\begin{aligned} \alpha \int_{S_k} R_\alpha^k 1\{(dd^c u)^n - (dd^c \hat{v})^n\} &= -\alpha \mathscr{E}^{u,\hat{v}}(w, R_\alpha^k 1) \\ &= -\alpha \mathscr{E}_\alpha^{u,\hat{v}}(w, R_\alpha^k 1) + \alpha^2 (w, R_\alpha^k 1)_{S_k} \\ &= -\alpha(1 - \alpha R_\alpha^k 1, w)_{S_k} + \alpha \int_{S_k} H_\alpha^k w(z)\, dV(z) \end{aligned}$$

and

$$\int_S (dd^c u)^n \geqslant \alpha \int_{S_k} R_\alpha^k 1 (dd^c v)^n + \alpha \int_{S_k} H_\alpha^k w(z)\, dV(z). \tag{4.9}$$

Now

$$\begin{aligned} \lim_{k\to\infty} H_\alpha^k w(z) &= E_z^{u,\hat{v}}(e^{-\alpha(\tau_S \wedge \zeta)} \lim_{k\to\infty} w(Z_{\tau_k})) \\ &= E_z^{u,\hat{v}}(e^{-\alpha \tau_S} w(Z_{\tau_S}); \tau_S < \zeta) + E_z^{u,\hat{v}}(e^{-\alpha\zeta} \lim_{t\uparrow\zeta} w(Z_t); \tau_S = \infty). \end{aligned}$$

The first term of the last expression vanishes for V-a.e. $z \in D$ because w is $\mathscr{E}^{u,\hat{v}}$-quasi-continuous and so $w(Z_t)$ is continuous at τ_S $P_z^{u,\hat{v}}$-a.s. on $\{\tau_S < \zeta\}$ for V-a.e. $z \in D$. The second term also vanishes for V-a.e. $z \in D$ on account of (4.6). We let $k \to \infty$ in (4.9) and use the monotone convergence theorem to the first member of the right hand side and the bounded convergence theorem to the second one (at this stage the finiteness of $V(D)$ is used), we arrive at

$$\int_S (dd^c u)^n \geqslant \alpha \int_S R_\alpha^S 1 (dd^c v)^n.$$

w being $\mathbf{M}^{u,\hat{v}}$-finely continuous, S is $\mathbf{M}^{u,\hat{v}}$-fine open and $\alpha R_\alpha^S 1(z) \uparrow 1$, $\alpha \uparrow \infty$, for any $z \in S$. Moreover S increases to $\{u < v\}$ as $\delta \downarrow 0$. (4.5) is proven. q.e.d.

Just as (P.3) follows from (P.2) (see Corollary 4.4 of [3]), the next theorem can be derived from Theorem 4.2.

THEOREM 4.3. *Suppose that* $u, v \in \mathscr{P}_b(D)$ *satisfy* $(dd^c u)^n \leq (dd^c v)^n$ *on D and* (4.4) *for any* $\delta > 0$. *Then* $u(z) \geq v(z)$ *for any* $z \in D$.

This is essentially the same as Theorem 2 of [11] except that the underlying measure $\theta \wedge dd^c|z|^2$ for the Dirichlet forms $\mathscr{E}^\theta$ in [11] is now replaced by the Lebesgue measure V. Of course Theorem 4.3 implies the minimum principle (P.3).

In the remainder of this section, we state some applications of properties (P.2) and (P.3). The minimum principle (P.3) is useful not only in proving the uniqueness of the solutions of Monge-Ampère equations but also in constructing them by the method of the spherical modification ([1]). Especially the following lemma was proven in [3; Proposition 5.3] by using (P.1), (P.3) and the existence theorem of the solution of $(dd^c u)^n = 0$ on a ball with a smooth boundary function [1, Theorem 8.1].

LEMMA 4.4. *Suppose D be a bounded strongly pseudo-convex domain, then for any compact* $K \subset D$, $(dd^c u_K^*)^n = 0$ on $D - K$.

In the same proposition of [3], Bedford and Taylor made use of Lemma 4.4 and the properties (P.1) and (P.2) to get the representation of the Bedford-Taylor capacity

$$C_{\mathrm{BT}}(E) = \int_D (dd^c u_E^*)^n \tag{4.10}$$

holding for a bounded strongly pseudo-convex domain D and any compact or open set $E \subset\subset D$.

We now mention an application of (4.10) and (P.3). A function u defined on an open set $E \subset D$ is said to be C_{BT}-*quasi-continuous* on E if, for any $\varepsilon > 0$, there exists an open set $O \subset D$ with $C_{\mathrm{BT}}(O) < \varepsilon$ such that the restriction of u to the set $E - O$ is a continuous function. Since C_{BT} is countably subadditive, the C_{BT}-quasi-continuity is a local property: a function u on D is C_{BT}-quasi-continuous on D iff, for any point $z \in D$, there exists an open set E with $z \in E \subset D$ and u is C_{BT}-quasi-continuous on E.

The $C_\#$-quasi-continuity is defined analogously. It is also a local property owing to Lemma 6.1 of §6. Let us denote C_{BT} and $C_\#$ by C_{BT}^D and $C_\#^D$ respectively to indicate their dependence on the bounded open set D.

PROPOSITION 4.5. *Let D be bounded open and f be a function defined on an open subset of D. Then the following conditions are equivalent:*

(i) *f is* C_{BT}^D-*quasi-continuous*

(ii) *f is* $C_\#^D$-*quasi-continuous.*

Furthermore each of these conditions is independent of the choice of the reference set D.

Proof. Suppose first that D is a bounded strongly pseudo-convex domain. Let $\{O_k\}$ be a sequence of decreasing open sets with $O_1 \subset\subset D$. Then, by (4.10), (P.1) and (P.3), we readily see the equivalence

$$\lim_{k\to\infty} C^D_{\mathrm{BT}}(O_k) = 0 \quad \Leftrightarrow \quad \lim_{k\to\infty} C^D_{\#}(O_k) = 0. \tag{4.11}$$

Since the quasi-continuity is a local property, we get the equivalence of (i) and (ii) from this.

Observe now that the set function C^D_{BT} is decreasing in D, while $C^D_{\#}$ is increasing. Hence we can conclude that (i) and (ii) are equivalent for any bounded open D and that the conditions are irrelevant to the choice of D. q.e.d.

From the next section, we shall be exclusively concerned with the set function $C_{\#}$ because it is more directly linked to the probabilistic notion than C_{BT}. Lemma 4.4 will be used in the next section. But otherwise no result of this section will be utilized in the rest of this paper.

§5. Upper estimates of $C_{\#}$

For $p \in \mathscr{P}_+(D)$, the pair $((dd^c p)^{n-1}, V)$ is admissible by Theorem 2.5. The corresponding Dirichlet space on $L^2(D)$ and the conformal diffusion are denoted by $(\mathscr{F}^{(p)}, \mathscr{E}^{(p)})$ and $\mathbf{M}^{(p)} = (Z_t, \zeta, P^{(p)}_z)$ respectively. The next proposition has been shown in [10; Proposition 2] for any conformal diffusion (cf. [12; Lemma 4]).

PROPOSITION 5.1. *For a Borel set $E \subset\subset D$*

$$C_{\#}(E) \geqslant \int_D P^{(p)}_z(\sigma_E < \infty)\, dV(z), \quad p \in \mathscr{P}_+(D),$$

where σ_E denotes the first hitting time of E.

In order to get an estimate of $C_{\#}$ in the opposite direction, we now consider a compact set $K \subset D$ and a function

$$p(z) = u^*_K(z) + \delta |z|^2, \quad z \in D, \tag{5.1}$$

for $\delta>0$. The associated objects $\mathscr{E}^{(p)}$, $\mathscr{F}^{(p)}$, $\mathrm{Cap}^{(p)}$, and $P_z^{(p)}$ with this specific $p\in\mathscr{P}_+(D)$ will be denoted by $\mathscr{E}^{(K,\delta)}$, $\mathscr{F}^{(K,\delta)}$, $\mathrm{Cap}^{(K,\delta)}$, and $P_z^{(K,\delta)}$ respectively.

THEOREM 5.2. *If D is a bounded strongly pseudo-convex domain and K is a compact subset of D, then*

$$C_\#(K)\leqslant\int_D P_z^{(K,\delta)}(\sigma_K<\infty)\,dV(z)+2(n-1)\gamma\delta V(D), \tag{5.2}$$

where $\gamma=\gamma(D)=\sup_{z\in D}|z|^2$.

Proof. By Theorem 2.2, u_K^* is an $\mathscr{E}^{(K,\delta)}$-quasi-continuous element of $\mathscr{F}_{\mathrm{loc}}^{(K,\delta)}$ and

$$\mathscr{E}^{(K,\delta)}(u_K^*,\varphi)=-\int_D \varphi dd^c u_K^*\wedge(dd^c p)^{n-1},\quad \text{for } \varphi\in C_0^\infty(D).$$

Since

$$dd^c u_K^*\wedge(dd^c p)^{n-1}\leqslant(dd^c u_K^*)^n+(n-1)\,\delta dd^c|z|^2\wedge(dd^c p)^{n-1},$$

we conclude using Lemma 4.4 that the function

$$w=u_K^*-(n-1)\,\delta|z|^2 \tag{5.3}$$

is $\mathscr{E}^{(K,\delta)}$-superharmonic on $D-K$ in the sense of the Appendix. Since $\mathscr{E}^{(K,\delta)}$ has the property (2.11), Theorem 9.4 applies and

$$w(z)\geqslant E_z^{(K,\delta)}(w(Z_{\tau_G\wedge\sigma_K}))\quad \text{for } V\text{-a.e. } z\in D,$$

and hence we have

$$-u_K^*(z)\leqslant -E_z^{(K,\delta)}(u_K^*(Z_{\tau_G\wedge\sigma_K}))+2(n-1)\gamma\delta\quad \text{for } V\text{-a.e. } z\in D. \tag{5.4}$$

Because of the strong pseudo-convexity of D, $\lim_{z\to\partial D}u_K^*(z)=0$ and consequently $E_z^{(K,\delta)}(u_K^*(Z_{\tau_G\wedge\sigma_K}))$ approaches $E_z^{(K,\delta)}(u_K^*(Z_{\sigma_K});\,\sigma_K<\infty)$ as we let G increase to D. Obviously $-E_z^{(K,\delta)}(u_K^*(Z_{\sigma_K});\,\sigma_K<\infty)\leqslant P_z^{(K,\delta)}(\sigma_K<\infty)$, V-a.e. $z\in D$, and (5.2) follows from (5.4). q.e.d.

Since Theorem 5.2 holds for any $\delta>0$, we get from it and Proposition 5.1 the following.

PROPOSITION 5.3. *The equality* (1.5) *holds if D is a bounded strongly pseudo-convex domain and E is compact.*

This proposition implies the validity of property (P.4) for compact sets. In fact, if E is bounded, $D \supset E$ is bounded open and $C_{\#}(E)=0$ relative to D, then $u_E^*=0$ on D and, as is easily seen, E is pluripolar. On the other hand, a Borel set $E \subset D$ is $\mathscr{E}^{(p)}$-polar iff $P_z^{(p)}(\sigma_E<\infty)=0$ V-a.e. $z \in D$. Hence we get from the above proposition and Corollary 2.6

Proposition 5.4. *If E is a compact pluri-negligible subset of a strongly peudo-convex domain D, then E is pluripolar.*

Finally we rewrite Theorem 5.2 in a way convenient for the next section.

Proposition 5.5. *Under the hypothesis of Theorem* 5.2,

$$C_{\#}(K) \leq \left(\frac{2\gamma V(D)}{4^{n-1} n! \delta^{n-1}}\right)^{1/2} \mathrm{Cap}^{(K,\delta)}(K)^{1/2} + 2(n-1)\gamma\delta V(D)$$

for any $\delta>0$ with $\gamma\delta<1$.

Proof. Let $e_K(z)=P_z^{(K,\delta)}(\sigma_K<\infty)$, $z \in D$. Then Theorem 5.2 reads

$$C_{\#}(K) \leq (e_K, 1)_{L^2} + 2(n-1)\gamma\delta\, V(D).$$

Since $\mathscr{E}^{(K,\delta)}$ satisfies the bound (2.11), we can use formulae (9.7) and (9.11) of the Appendix to get

$$(e_K, 1)_{L^2} = \mathscr{E}^{(K,\delta)}(e_K, G1) \leq \sqrt{\mathrm{Cap}^{(K,\delta)}(K)}\ \sqrt{\mathscr{E}^{(K,\delta)}(G1, G1)}.$$

But the second factor of the last expression is dominated by

$$\left(\frac{2\gamma V(D)}{4^{n-1} n! \delta^{n-1}}\right)^{1/2}$$

according to the bound (2.11). q.e.d.

§ 6. $C_{\#}$-quasi-continuity, pluri-negligibility and pluripolarity

The next lemma follows easily from the definition of the set function $C_{\#}$ on D. We shall denote $C_{\#}$ by $C_{\#}^D$ whenever it is necessary to indicate its relevance to the bounded open set D.

Lemma 6.1. (i) *For any open set $E \subset D$,*

$$C_{\#}(E)=\sup\{C_{\#}(K)\colon K \textit{ is compact}, K\subset E\}.$$

(ii) *For any set* $E\subset D$,

$$C_{\#}(E)=\inf\{C_{\#}(O)\colon O \textit{ is open}, O\supset E).$$

(iii) *For* $E_1, E_2, \ldots \subset D$, $C_{\#}(\bigcup_{l=1}^{\infty} E_l)\leq \Sigma_{l=1}^{\infty} C_{\#}(E_l)$.

(iv) *If* D_1, D_2 *are bounded open and* $D_1\subset D_2$, *then* $C_{\#}^{D_1}(E)\leq C_{\#}^{D_2}(E)$ *for any* $E\subset D_1$.

Proof. (i) For an open set $E\subset D$, choose compact sets K_j increasing to E. Then $u_{K_j}^*$ decreases to a psh function v. Since $v=-1$ V-a.e. on the open set E, $v=-1$ identically on E. Hence $v\leq u_E\leq u_E^*$ and we have $v=u_E^*$.

(ii) It suffices to find, for any set $E\subset D$, a decreasing sequence of open sets $O_j\supset E$ such that u_{O_j} converges to u_E V-a.e. By Choquet's lemma, there is an increasing sequence of $\varphi_j\in\mathscr{P}(D)\cap L^{\infty}(D)$ such that $\varphi_j(z)\leq u_E(z)$, $z\in D$, and $\lim_{j\to\infty}\varphi_j(z)=u_E(z)$ for V-a.e. $z\in D$. We let $O_j=\{z\in D\colon (1+1/j)\,\varphi_j<-1\}$. Then the O_j's are open sets containing E and $(1+1/j)\,\varphi_j\leq u_{O_j}\leq u_E$.

(iii) When the E_l's are open, $u_{E_l}=-1$ identically on E_l and

$$u_{\cup E_l}^*\geq u_{\cup E_l}\geq \sum u_{E_l}^*.$$

(iv) trivial. q.e.d.

We are now ready to prove properties (P.4) and (P.5). The proof is based on the upper estimate of $C_{\#}$ in Proposition 5.5 and the continuity property (P.1). Let us first give the proof of (P.5). See the latter half of §4 for the precise definition of quasi-continuity. Because of the countable subadditivity of $C_{\#}$ shown above, the $C_{\#}$-quasi-continuity is a local property.

THEOREM 6.2. *Any function of* $\mathscr{P}(D)$ *is* $C_{\#}^{D}$*-quasi-continuous on* D.

Proof. For any $u\in\mathscr{P}(D)$, consider open sets $O_j=\{z\in D\colon u(z)<-j\}$. Then $C_{\#}(O_j)\to 0$ as $j\to\infty$ because $0\geq u_{O_j}\geq(u/j)\vee(-1)$ and $u_{O_j}\to 0$ V-a.e. as $j\to\infty$. Therefore, replacing u by $u\vee(-j)$ if necessary, we may assume that u is locally bounded.

Take any $v\in\mathcal{P}(D)\cap L^{\infty}_{\rm loc}(D)$ and open $E\subset\subset D$. In proving the $C^D_{\#}$-quasi-continuity of v on E, we may assume that D is a strongly pseudo-convex domain $\{\varrho<0\}$ with a smooth psh function ϱ by replacing D with a larger ball if necessary owing to Lemma 6.1 (iv). We may further assume that there exist a compact set K with $E\subset K\subset D$ and functions $v_k\in\mathcal{P}(D)\cap C(D)$ such that $v_k\downarrow v$ as $k\to\infty$ and $v_k=v=A\varrho+B$ on $D-K$, $k=1,2,\dots,$ for some constants $A>0$ and B (cf. [3; p. 5]).

We then let, for $\lambda>0$ and $k<j$,

$$O_k=\{v_k-v>\lambda\},\quad K_{k,j}=\{v_k-v_j\geq\lambda+\frac{1}{j}\}.$$

$K_{k,j}$ is compact and increasing to the open set O_k as $j\to\infty$. Hence $C_{\#}(K_{k,j})\to C_{\#}(O_k)$, $j\to\infty$ and $u^*_{K_{k,j}}$ decreases to $u^*_{O_k}$ as $j\to\infty$ by virtue of Lemma 6.1 (i).

Denote $u^*_{K_{k,j}}+\delta|z|^2$ by $p_{k,j}$. Choose a non-negative $\xi\in C_0^{\infty}(D)$ with $\xi=1$ on K. Then $v_k-v_j=\xi v_k-\xi v_j$ belongs to the Dirichlet space $\mathcal{F}^{(p_{k,j})}$ by Theorem 2.2. Therefore we can combine Proposition 5.5 with the identity (9.10) of the Appendix to get the bound, for $k<j$,

$$C_{\#}(K_{k,j})\leq C(\delta)\frac{1}{\lambda+(1/j)}\mathcal{E}^{(p_{k,j})}(v_k-v_j,v_k-v_j)^{1/2}+R(\delta),\tag{6.1}$$

where

$$C(\delta)=\left(\frac{2\gamma V(D)}{4^{n-1}n!\delta^{n-1}}\right)^{1/2}\quad\text{and}\quad R(\delta)=2(n-1)\gamma\delta V(D).$$

By Theorem 2.2. we have

$$\mathcal{E}^{(p_{k,j})}(v_k-v_j,v_k-v_j)\leq\int_D(\xi v_k-\xi v_j)\,dd^c v_j\wedge(dd^c P_{k,j})^{n-1},\quad k<j,$$

and the right hand side converges as $j\to\infty$ to $\int_D(\xi v_k-\xi v)\,dd^c v\wedge(dd^c p_k)^{n-1}$ by virtue of Theorem 3.2 where $p_k=u^*_{O_k}+\delta|z|^2$. Hence we have from (6.1)

$$C_{\#}(O_k)\leq\frac{C(\delta)}{\lambda}\left(\int_D(\xi v_k-\xi v)dd^c v\wedge(dd^c p_k)^{n-1}\right)^{1/2}+R(\delta).\tag{6.2}$$

Let p^* be the upper regularization of the function $p=\lim_{k\to\infty}p_k$. Since p_k is increasing in k, $p=p^*$ V-a.e. and v_k decreases to v. Theorem 3.2 implies that the integral in (6.2) tends to zero as $k\to\infty$. Therefore by letting $k\to\infty$ in (6.2) and then $\delta\downarrow 0$, we arrive at

$$\lim_{k\to\infty} C_{\#}(O_k) = 0. \tag{6.3}$$

which means that v is a $C_{\#}$-quasi-uniform limit of continuous functions v_k and hence v is $C_{\#}$-quasi-continuous on D. q.e.d.

Only a slight modification of the above proof leads us also to the property (P.4). Let $N\subset D$ be pluri-negligible. In order to get the pluripolarity of N, it suffices to show $C_{\#}(N)=0$. By Lemma 6.1, we may assume that $D=\{\varrho<0\}$ with a smooth strict psh function ϱ and $N\subset\subset D$. Take $u_k\in\mathscr{P}_b(D)$ such that u_k increases as $k\to\infty$ to a bounded function u and $N\subset\{u<u^*\}$. We can then choose $u_{k,j}, v_j\in\mathscr{P}_b(D)\cap C(D), j=1,2,\ldots,$ such that $u_{k,j}\leqslant v_j$ and $u_{k,j}$ (resp. v_j) decreases to u_k (resp. u^*) as $j\to\infty$. As in the preceding proof, we may further assume that $u_{k,j}$ and v_j are equal to $A\varrho+B$ outside some common compact set.

Now let $O_k=\{v_k-u_k>\lambda\}$ for $\lambda>0$. Since $\{u^*-u>\lambda\}\subset O_k$, it is enough to show (6.3) for the present open sets O_k. Since $K_{k,j}=\{v_k-u_{k,j}\geqslant\lambda+1/j\}$, $j>k$, is compact and increases to O_k as $j\to\infty$, we proceed exactly in the same way as in the preceding proof to obtain the inequality (6.2) with v in the right hand side being replaced now by u_k. We then let $k\to\infty$. Since u_k increases to u^* V-a.e. and v_k decreases to u^*, we again achieve (6.3) by Theorem 3.2. Thus we have proven (P.4):

THEOREM 6.3. *Any pluri-negligible set is pluripolar.*

Remark. (P.4) can also be derived from Proposition 5.4 (validity of (P.4) for compact sets) and Theorem 6.2 (property (P.5)) in the same way as in the proof of Proposition 5.1 of Bedford-Taylor [3].

$C_{\#}$ is an outer capacity by Lemma 6.1 (ii). To prove that $C_{\#}$ is a Choquet capacity, it is therefore enough to show

$$E_j\uparrow E \quad\Rightarrow\quad C_{\#}(E)=\sup_j C_{\#}(E_j). \tag{6.4}$$

THEOREM 6.4. *$C_{\#}$ is a Choquet capacity. In particular. $C_{\#}(E)=\sup\{C_{\#}(K)$: K compact $\subset E\}$ for any Borel set $E\subset D$.*

Proof. This theorem is contained in Proposition 8.4 of [3]. Indeed, as was pointed out in [3], we get (6.4) from Theorem 6.3 as follows: we let $v=\lim_{j\to\infty} u^*_{E_j}$. Then $v\in\mathscr{P}(D)$

because v is a decreasing limit of functions in $\mathscr{P}(D)$. If we put $E'=\{z\in E: v(z)=-1\}$, then $E-E'$ is pluri-negligible and $C_{\#}(E-E')=0$ by Theorem 6.3. Hence $C_{\#}(E)\leqslant C_{\#}(E')$ by Lemma 6.1 (iii). On the other hand, $v\leqslant u_{E'}$ and consequently,

$$\lim_{j\to\infty} C_{\#}(E_j)\geqslant C_{\#}(E')\geqslant C_{\#}(E).$$

The converse inequality is clear. q.e.d.

From Proposition 5.3 and Theorem 6.4, we have

THEOREM 6.5. *Suppose D be bounded strongly pseudo-convex, then the identity* (1.5) *holds for any Borel set $E\subset D$:*

$$C_{\#}(E)=\sup_{p\in\mathscr{P}_+(D)}\int_D P_z^{(p)}(\sigma_E<\infty)\,dV(z).$$

Remark. Obviously this is valid in a more general form: for any non-negative bounded Borel f on D, we have

$$-\int_D u_E^*(z)f(z)\,dV(z)=\sup_{p\in\mathscr{P}_+(D)}\int_D P_z^{(p)}(\sigma_E<\infty)f(z)\,dV(z)$$

which is the present version of our previous result [12; Lemma 8].

THEOREM 6.6. *Let E be a bounded set. Take a bounded open set $D\supset E$. Then the following conditions are equivalent for E:*

(i) $C_{\#}^{D}(E)=0$.

(ii) $\mathrm{Cap}^{(p)}(E)=0$ *for any* $p\in\mathscr{P}_+(D)$.

(iii) *There exists a Boret set $E'\supset E$ such that*

$$P_z^{(p)}(\sigma_{E'}<\infty)=0 \quad V\text{-a.e. } z\in D \quad \text{for any } p\in\mathscr{P}_+(D).$$

Moreover, each of the above conditions for E is independent of the choice of the reference set D which is bounded open.

Proof. Denote the above three conditions by $(\mathrm{i})_D$, $(\mathrm{ii})_D$ and $(\mathrm{iii})_D$ respectively to indicate their relevance to D. By Theorem 6.5, $(\mathrm{i})_D$ and $(\mathrm{iii})_D$ are equivalent when D is a bounded strongly pseudo-convex domain. $(\mathrm{ii})_D$ and $(\mathrm{iii})_D$ are equivalent for any bounded open $D\supset E$ ([9]).

Consider two bounded open sets $D, \tilde{D}$ with $E \subset D \subset \tilde{D}$. If $p \in \mathscr{P}_+(\tilde{D})$, then $p \in \mathscr{P}_+(D)$ and $\mathscr{E}_D^{(p)}$ equals the part of $\mathscr{E}_{\tilde{D}}^{(p)}$ on D (see Appendix), and consequently, $\mathrm{Cap}_D^{(p)}(E) \geq \mathrm{Cap}_{\tilde{D}}^{(p)}(E)$. This means the implication $(\mathrm{ii})_D \Rightarrow (\mathrm{ii})_{\tilde{D}}$. By Lemma 6.1(iv), we have the converse implication for (i): $(\mathrm{i})_{\tilde{D}} \Rightarrow (\mathrm{i})_D$. Therefore, taking the countable subadditivity of $\mathrm{Cap}^{(p)}$ also into account, we conclude that $(\mathrm{i})_D$ and $(\mathrm{ii})_D$ are equivalent for any bounded open $D \supset E$ and they are independent of the choice of D. q.e.d.

Remark. The independence on D of the condition (i) of Theorem 6.6 enables us to prove the following fact due to Josefson [15] exactly in the same way as the proof of [3; Theorem 6.8]: A set $E \subset \mathbf{C}^n$ is pluripolar if and only if there exists a psh function p on $\mathbf{C}^n$ such that $E \subset p^{-1}(-\infty)$. In particular, each of the conditions of Theorem 6.6 is equivalent to the pluripolarity of E.

The next lemma will be refered to in §7.

LEMMA 6.7. *If E is a Borel subset of D and $(dd^c q)^n(E) > 0$ for some $q \in \mathscr{P}_b(D)$, then E is not pluripolar and for $p = q + \delta|z|^2$, $\delta > 0$, we have* $\mathrm{Cap}^{(p)}(E) > 0$ *and $P_z^{(p)}(\sigma_E < \infty) > 0$ for $z \in D$ of positive Lebesque measure.*

Proof. This is a consequence of Theorem 6.6 and the bound $\int_E (dd^c q)^n \leq 8\|q\|_\infty \mathrm{Cap}^{(p)}(E)$, which follows from (2.12) and (9.9). q.e.d.

§7. An example

Let us consider the domain $D \subset \mathbf{C}^2$ and its subset E defined by (1.6), where the coordinates of $z \in \mathbf{C}^2$ are denoted by $z = (z_1, z_2)$, $z_j = x_j + iy_j$, $j = 1, 2$. The 4-dimensional Newtonian capacity of E is zero because the codimension of E is 2. We consider the function $q(z_1, z_2) = \frac{1}{2}(|y_1| + |y_2|)$, $(z_1, z_2) \in D$. Then

$$dd^c q = \frac{\partial^2 |y_1|}{\partial z_1 \partial \bar{z}_1} i dz_1 \wedge d\bar{z}_1 + \frac{\partial^2 |y_2|}{\partial z_1 \partial \bar{z}_2} i dz_2 \wedge d\bar{z}_2$$

and hence $q \in \mathscr{P}_b(D)$. Moreover we see that

$$(dd^c q)^2 = 2 dx_1 \, \delta_0(dy_1) \, dx_2 \, \delta_0(dy_2), \quad (dd^c q)^2(E) = 8. \tag{7.1}$$

E is therefore not pluripolar by Lemma 6.7.

By virtue of Lemma 6.7, $\mathrm{Cap}^{(p)}(E)>0$ for

$$p(z_1, z_2) = \frac{1}{2}(|y_1|+|y_2|)+\frac{1}{4}|z|^2, \quad z\in D,$$

and E is attainable by the associated conformal diffusion $(Z_r, P_z^{(p)})$ on D. The Dirichlet form

$$\mathscr{E}^{(p)}(u, u) = \frac{1}{2}\int_D du\wedge d^c u\wedge dd^c p, \quad u\in C_0^\infty(D),$$

has the expression

$$\mathscr{E}^{(p)}(u, u) = \frac{1}{2}\mathbf{D}(u, u)+\frac{1}{2}\int_{\Gamma_2}(u_{x_1}^2+u_{y_1}^2)\, dx_1\, dy_1\, dx_2+\int_{\Gamma_1}(u_{x_2}^2+u_{y_2}^2)\, dx_1\, dx_2\, dy_2 \tag{7.2}$$

where $\mathbf{D}$ denotes the usual 4-dimensional Dirichlet integral on D and

$$\Gamma_j = \{z\in D: y_j=0\}, \quad j=1,2.$$

This expression gives us an intuitive picture how the sample paths Z_t under the law $P_z^{(p)}$ are attainable to the set E: Z_t starting at $z\in D-\Gamma_1-\Gamma_2$ is governed by the form $\frac{1}{2}\mathbf{D}(u, u)$ and behaves as a 4-dimensional Brownian motion. It can not attain directly the 2-dimensional set E but can hit any non-empty open subset of the 3-dimensional set Γ_1 (and of Γ_2). Upon the arrival of Z_t at Γ_2 at some point $(x_1^0, y_1^0, x_2^0, 0)\in\Gamma_2$ with $|x_2^0|<1$, an additional diffusion on Γ_2 governed by the second term of the right hand side of (7.2) is superposed to the 4-dimensional Brownian motion. The typical sample path of this diffusion on Γ_2 behaves as the 2-dimensional Brownian motion on the plane domain $\{x_2=x_2^0, y_2=0\}\cap D$ starting at $(x_1^0, y_1^0, x_2^0, 0)$. Therefore it can attain the one-dimensional segment $\{|x_1|<1, y_1=0, x_2=x_2^0, y_2=0\}$ which is a part of E.

The above intuitive description could be made rigorous if one constructs the diffusion $(Z_t, P_z^{(p)})$ by the method of skew products as in Ikeda-Watanabe [14].

§8. $\mathscr{E}^\theta(\varphi, \psi)$ for $\varphi, \psi\in C^\infty(\bar{D})$

In this section, we deal with the symmetric form $\mathscr{E}^\theta(\varphi, \psi)$ for functions φ, ψ belonging to the space $C^\infty(\bar{D})$ instead of $C_0^\infty(D)$. We study the closability of $\mathscr{E}^\theta$ and give some formulae involving $\mathscr{E}^\theta$ and a surface integral on ∂D. A Hartogs' type property of θ will be presented as an application.

We first show a natural extension of the property (2.2) from $C_0^\infty(D)$ to $C^\infty(\bar{D})$. Given a closed positive current θ of bidegree $(n-1, n-1)$, we consider $\mathscr{E}^\theta(\varphi, \psi) = \int_D d\varphi \wedge d^c\psi \wedge \theta$ for $\varphi, \psi \in \mathscr{C}$ where

$$\mathscr{C} = \{\varphi \in C^\infty(\bar{D}) : \varphi \in L^2(D; \theta \wedge dd^c|z|^2), \int_D d\varphi \wedge d^c\varphi \wedge \theta < \infty\}. \tag{8.1}$$

THEOREM 8.1. *Suppose m is a positive Radon measure satisfying $m \geq f \cdot \theta \wedge dd^c|z|^2$ for some strictly positive continuous function f on D, then E^θ defined on C is closable on $L^2(D; m)$.*

Proof. Let $\varphi_k \in \mathscr{C}$ be an $\mathscr{E}^\theta$-Cauchy sequence such that $\varphi_k \to 0$, $k \to \infty$, in $L^2(D; m)$. First we note $\lim \mathscr{E}^\theta(\eta\varphi_k, \eta\varphi_k) = 0$ for any $\eta \in C_0^\infty(D)$ with $0 \leq \eta \leq 1$. To see this, it suffices to show, on account of (2.2), that $\{\eta\varphi_k\}$ is $\mathscr{E}^\theta$-Cauchy, which is however a consequence of (2.5):

$$\mathscr{E}^\theta(\eta\varphi_k - \eta\varphi_l, \eta\varphi_k - \eta\varphi_l) \leq 2\,\mathscr{E}^\theta(\varphi_k - \varphi_l, \varphi_k - \varphi_l) + 2C(\eta)\|\varphi_k - \varphi_l\|^2_{L^2(D;m)} \to 0, \quad k, l \to \infty.$$

For any $\varepsilon > 0$, choose N such that $\mathscr{E}^\theta(\varphi_k - \varphi_N, \varphi_k - \varphi_N) < \varepsilon$, $k \geq N$, and a compact set K such that $\int_{D-K} d\varphi_N \wedge d^c\varphi_N \wedge \theta < \varepsilon$. Take $\eta \in C_0^\infty(D)$ with $0 \leq \eta \leq 1$ on D and $\eta = 1$ on K. Then

$$\mathscr{E}^\theta(\varphi_k, \varphi_k) \leq 2\mathscr{E}^\theta(\eta\varphi_k, \eta\varphi_k) + 2\mathscr{E}^\theta((1-\eta)\varphi_k, (1-\eta)\varphi_k)$$

and the second term of the right hand side is dominated by

$$4\int_{D-K} (1-\eta)^2 d\varphi_k \wedge d^c\varphi_k \wedge \theta + 4\int_D \varphi_k^2 d\eta \wedge d^c\eta \wedge \theta$$

$$\leq 8\int_{D-K} d\varphi_N \wedge d^c\varphi_N \wedge \theta + 8\,\mathscr{E}^\theta(\varphi_k - \varphi_N, \varphi_k - \varphi_N) + 4\,C(\eta)\,\|\varphi_k\|^2_{L^2(D;m)}$$

$$\leq 16\,\varepsilon + 4C(\eta)\,\|\varphi_k\|^2_{L^2(D;m)}.$$

Hence

$$\overline{\lim_{k\to\infty}}\, \mathscr{E}^\theta(\varphi_k, \varphi_k) \leq 16\varepsilon. \qquad \text{q.e.d.}$$

Suppose m in Theorem 8.1 further satisfies $m(D)<\infty$ and $\mathrm{supp}[m]=D$, then $\mathscr{C}=C^\infty(\bar{D})$ and the $\mathscr{E}^\theta$-closure of $C^\infty(\bar{D})$ on $L^2(D;m)$ gives rise to a regular Dirichlet form on $L^2(\bar{D};m)$ (we set $m(\partial D)=0$), whose part on D is identical, on account of Proposition 9.1 of the Appendix, just with the $\mathscr{E}^\theta$-closure of $C_0^\infty(D)$.

As an application, let us consider an open set $D_1\subset\subset D$ and a closed positive current θ of bidegree $(n-1, n-1)$ defined on a neighbourhood of $\bar{D}_1$. We then let

$$\tilde{\theta}=\begin{cases}\theta & \text{on } D_1\\ 0 & \text{on } D-D_1\end{cases} \tag{8.2}$$

so that

$$\int_D d\varphi\wedge d^c\psi\wedge\tilde{\theta}=\int_{D_1} d\varphi\wedge d^c\psi\wedge\theta,\quad \varphi,\psi\in C_0^\infty(D). \tag{8.3}$$

Thus $\tilde{\theta}$ is a positive current on D but not closed as we shall see presently.

Nevertheless $\tilde{\theta}$ gives us a Dirichlet form. Denote the left (resp. right) hand side of (8.3) by $\mathscr{E}_D^{\tilde{\theta}}(\varphi,\psi)$ (resp. $\mathscr{E}_{D_1}^\theta(\varphi,\psi)$) indicating the domain of integration. Let

$$m=\tilde{\theta}\wedge dd^c|z|^2+\delta_0(dd^c|z|^2)^n$$

for a fixed $\delta_0>0$. Then we see from Theorem 8.1 applied to $\mathscr{E}_{D_1}^\theta$ on $C_\infty(\bar{D}_1)$ that $\mathscr{E}_D^{\tilde{\theta}}$ on $C_0^\infty(D)$ is closable on $L^2(D;m)$. The resulting Dirichlet space $(\mathscr{F}^\theta, \mathscr{E}^\theta)$ on $L^2(D;m)$ has a special property that associated semigroup $\{T_t^{\tilde{\theta}},\ t\geq 0\}$ makes the set $D-D_1$ invariant:

$$T_t^{\tilde{\theta}}(I_{D-D_1}u)=I_{D-D_1}\cdot T_t^{\tilde{\theta}}u,\quad u\in L^2(D;m)$$

(actually $T_t^{\tilde{\theta}}u(x)=u(x)$ m-a.e. $x\in D-D_1$ in the present case). By symmetry, the set D_1 is also $T_t^{\tilde{\theta}}$-invariant.

We next establish a Poincaré type inequality and a trace inequality involving $\mathscr{E}^\theta$ and an integral on ∂D. First of all, we assume that D is a bounded domain defined as $D=\{r<1\}$ by a non-negative smooth psh function r on a neighbourhood of $\bar{D}$ such that $dr\neq 0$ on ∂D and θ is a smooth closed positive differential form on $\bar{D}$ of bidegree $(n-1, n-1)$. $d^c r\wedge\theta$ then induces a non-negative surface element on ∂D (denoted by $d^c r\wedge\theta$ again) and we may consider the following three integrals of φ for $\varphi\in C^\infty(\bar{D})$ with respect to three non-negative measures:

$$S(\varphi)=\int_{\partial D}\varphi\, d^c r\wedge\theta,\quad I(\varphi)=\int_D \varphi\, dd^c r\wedge\theta,\quad A(\varphi)=\int_D \varphi\, dr\wedge d^c r\wedge\theta.$$

LEMMA 8.2. *Suppose* $S(\varphi^2)$, $I(\varphi^2)$ and $A(\varphi^2)$ *are finite for* $\varphi\in C^\infty(\bar{D})$, then

$$S(\varphi^2)\leq 8\,\mathscr{E}^\theta(\varphi,\varphi)+\frac{3}{2}I(\varphi^2) \tag{8.4}$$

$$I(\varphi^2)\leq 4\,\mathscr{E}^\theta(\varphi,\varphi)+S(\varphi^2) \tag{8.5}$$

Proof. Let us write S, I, A and E for the above three integrals of φ^2 and $\mathscr{E}^\theta(\varphi,\varphi)$ respectively. Using the positivity of $dd^c r\wedge\theta$, Stokes' theorem and finally Schwarz inequality (2.3), we then have

$$\begin{aligned}A&=\int_D \varphi^2\, d(rd^c r\wedge\theta)=\int_D (d\varphi^2 rd^c r\wedge\theta)-\int_D rd\varphi^2\wedge d^c r\wedge\theta\\&=\int_D \varphi^2\, rd^c r\wedge\theta-2\int_D \varphi r d\varphi\wedge d^c r\wedge\theta\leq S+2\sqrt{EA}.\end{aligned}$$

Hence $\sqrt{A}\leq\sqrt{E}+\sqrt{E+S}$. On the other hand, we have analogously

$$S=\int_D d(\varphi^2\, d^c r\wedge\theta)=\int_D d\varphi^2\wedge d^c r\wedge\theta+\int_D \varphi^2\, dd^c r\wedge\theta\leq 2\sqrt{EA}+I.$$

Consequently $S\leq I+2E+2\sqrt{E^2+ES}$, from which we can derive the desired inequality (8.4): $S\leq 8E+\frac{3}{2}I$. (8.5) can be derived similarly and the proof is omitted. q.e.d.

Suppose that $\int_D dd^c r\wedge\theta$ and $\int_D (1-r)\, dd^c|z|^2\wedge\theta$ are finite. Then, for $\varphi\in C^\infty(\bar{D})$

$$\int_D d\varphi\wedge d^c r\wedge\theta=\int_D dr\wedge d^c\varphi\wedge\theta=-\int_D d(1-r)\wedge d^c\varphi\wedge\theta=\int_D (1-r)dd^c\varphi\wedge\theta.$$

Since

$$S(\varphi)=\int_D \varphi\, dd^c r\wedge\theta+\int_D d\varphi\wedge d^c r\wedge\theta,$$

we have

$$S(\varphi)=\int_D \varphi\, dd^c r\wedge\theta+\int_D (1-r)\, dd^c\varphi\wedge\theta,\quad \varphi\in C^\infty(\bar{D}). \tag{8.6}$$

Note that the right hand side still makes sense even when r and θ are not smooth. This suggests the way of defining the surface integral $S(\varphi)$ for general r and θ.

THEOREM 8.3. *Suppose that D is a bounded domain defined as $D=\{r<1\}$ by a non-negative continuous psh function r on a neighbourhood of $\bar{D}$ and that θ is a closed positive current on D of bidegree $(n-1, n-1)$ satisfying*

$$\int_D dd^c r \wedge \theta < \infty, \quad \int_D (1-r)\, dd^c |z|^2 \wedge \theta < \infty. \tag{8.7}$$

Then there exists a unique positive measure s on ∂D such that

$$\int_{\partial D} \varphi(z)\, ds(z) = \int_D \varphi\, dd^c r \wedge \theta + \int_D (1-r)\, dd^c \varphi \wedge \theta, \quad \varphi \in C^\infty(\bar{D}). \tag{8.8}$$

Further the inequalities (8.4) and (8.5) persist to hold if we interpret $S(\varphi^2)$ as the integral of φ^2 by the measure s on ∂D.

Proof. For $\varphi \in C^\infty(\bar{D})$, we define $S(\varphi)$ by the right hand side of (8.8). By (8.7), $S(\varphi)$ is finite and linear in φ. So if we can show the implication

$$\varphi \in C^\infty(\bar{D}), \quad \varphi \geq 0 \quad \text{on} \quad \partial D \quad \Rightarrow \quad S(\varphi) \geq 0. \tag{8.9}$$

Then $S(\varphi)$ is uniquely determined by the restriction of φ to the boundary ∂D and S can be extended to a unique positive linear functional on $C(\partial D)$ proving the first assertion of Theorem 8.3.

To prove (8.9), we let $r_m = \alpha_{1/m} * r$, $\theta_m = \alpha_{1/m} * \theta$ by a mollifier α_ε and consider the sets

$$D_{m,\delta} = \{r_m < 1-\delta\}, D_\delta = \{r < 1-\delta\}, \quad \delta > 0, \quad m = 1, 2, \ldots.$$

By Sard's theorem, we can find a sequence $\delta_j \downarrow 0$ such that $D_{m,j} = D_{m,\delta_j}$ has a regular boundary and Stokes' theorem is applicable, $m=1,2,\ldots$, $j=1,2,\ldots$. We may also assume that the sets $D_{m,j}$ and $D_j = D_{\delta_j}$ are continuous with respect to the measures $dd^c r \wedge \theta$ and $dd^c |z|^2 \wedge \theta$.

We now have from (8.6) that, for $\varphi \in C^\infty(\bar{D})$.

$$\int_{\partial D_{m,j}} \varphi\, d^c r_m \wedge \theta_m = \int_{D_{m,j}} \varphi\, dd^c r_m \wedge \theta_m + \int_{D_{m,j}} (1-\delta_j - r_m)\, dd^c \varphi \wedge \theta_m.$$

Since r_m decreases to r and $dd^c r_m \wedge \theta_m$ (resp. $dd^c \varphi \wedge \theta_m$) vaguely converges to $dd^c r \wedge \theta$ (resp. $dd^c \varphi \wedge \theta$) as $m \to \infty$, we see that the right hand side of the above equation is convergent, as $m \to \infty$, to

$$\int_{D_j} \varphi\, dd^c r \wedge \theta + \int_{D_j} (1-\delta_j - r)\, dd^c \varphi \wedge \theta.$$

Hence

$$\lim_{j\to\infty} \lim_{m\to\infty} \int_{\partial D_{m,j}} \varphi\, d^c r_m \wedge \theta_m = S(\varphi), \quad \varphi \in C^\infty(\bar{D}). \tag{8.10}$$

(8.9) follows immediately from (8.10). We have already seen that

$$\lim_{j\to\infty} \lim_{m\to\infty} \int_{D_{m,j}} \varphi\, dd^c r_m \wedge \theta_m = I(\varphi).$$

An analogous relation holds for $\mathscr{E}^\theta(\varphi, \varphi)$. Therefore we get the second assertion of Theorem 8.3 from Lemma 8.2. q.e.d.

As an application of Theorem 8.3, we have the following property of Hartogs' type, which is contained in Bedford-Taylor [3; Lemma 2.5] however:

THEOREM 8.4. *Suppose that D is a bounded domain defined as $D=\{r<1\}$ by a non-negative continuous psh function r belonging to $\mathscr{P}_+(D)$. Suppose a closed positive current θ on D of bidegree $(n-1, n-1)$ vanishes on $\{1-\delta<r<1\}$ for some $\delta>0$, then θ vanishes identically on D.*

Proof. Take δ_0 such that $0<\delta_0<\delta$ and use formula (8.8) for the subdomain $D_{\delta_0}=\{r<1-\delta_0\}=(r/(1-\delta_0)<1\}$. First choose a function $\varphi \in C^\infty(\bar{D}_{\delta_0})$ such that $\varphi=1$ on ∂D_{δ_0} and $\varphi=0$ on $\{r<1-\delta\}$ to get $s=0$ on ∂D_{δ_0}. Next choose φ identically equal to 1 on $\bar{D}_{\delta_0}$ to conclude that

$$\int_{D_{\delta_0}} dd^c r \wedge \theta = 0 \quad \text{and consequently } \theta = 0 \text{ on } D_{\delta_0}$$

q.e.d.

This theorem particularly means that the positive current of the type (8.2) is not closed.

§9. Appendix: superharmonic functions for general Dirichlet forms

Let X be a locally compact separable metric space and m be a positive Radon measure on X with $\text{supp}\,[m]=X$. Consider a dense subalgebra $\mathscr{D}$ of $C_0(X)$ satisfying the following two properties:

($\mathscr{D}$.1) For any compact K and open G with $K\subset G\subset\subset X$, $\mathscr{D}$ contains a non-negative function taking value 1 on K and 0 on $X-G$.

($\mathscr{D}$.2) For any $\varepsilon>0$, there exists a real function $\beta_\varepsilon(t)$ satisfying that $\beta_\varepsilon=t$ on $[0,1]$, $-\varepsilon\leq\beta_\varepsilon\leq 1+\varepsilon$ everywhere and $0\leq\beta_\varepsilon(t')-\beta_\varepsilon(t)\leq t'-t$ for $t<t'$, and that $\beta_\varepsilon(\varphi)\in\mathscr{D}$ whenever $\varphi\in\mathscr{D}$.

For instance, if X is an Euclidean domain D, then $C_0^\infty(D)$ and $C_0^\infty(\bar{D})$ have those properties.

Let $\mathscr{E}$ with domain $\mathscr{F}$ be a Dirichlet form on $L^2(X;m)$ possessing $\mathscr{D}$ as its core: $\mathscr{D}$ is dense in $\mathscr{F}$. Thus $\mathscr{E}$ on $\mathscr{D}$ has the Markovian property:

$$\mathscr{E}(\beta_\varepsilon(\varphi),\beta_\varepsilon(\varphi))\leq\mathscr{E}(\varphi,\varphi),\quad \varphi\in\mathscr{D},\quad \text{for } \beta_\varepsilon \text{ of } (\mathscr{D}.2).$$

We assume further a specific local property: $u,v\in\mathscr{D}$, $u=\text{constant}$ on a neighbourhood of $\text{supp}\,[\varphi]\Rightarrow\varepsilon(u,v)=0$. The associated diffusion process $\mathbf{M}=(X_t,P_x,\zeta)$ on X then admits no killing inside X ([9; Theorem 4.5.3]):

$$P_x(X_{\zeta-}\in X;\zeta<\infty)=0,\quad x\in X,$$

ζ being the killing time. In what follows, the terms "quasi-continuous" and "q.e." are used in relation to this Dirichlet form $\mathscr{E}$. For a function u on X, we write as $u\in\mathscr{F}_{\text{loc}}$ if for any open set $G\subset\subset X$ there exists $w\in\mathscr{F}$ such that $u=w$ on G.

We now recall a few facts from §4.4 of [9]. For a Borel set $E\subset X$, we let

$$\mathscr{F}_E=\{\varphi\in\mathscr{F}:\tilde{\varphi}=0 \text{ q.e. on} X-E\}. \tag{9.1}$$

$\tilde{\varphi}$ being a quasi-continuous version of φ. Using the hitting time

$$\sigma_E=\inf\{t>0:X_t\in E\},$$

we define, for $x\in X$ and $\alpha>0$.

$$R_\alpha^E f(x)=E_x\left(\int_0^{\sigma_E}e^{-\alpha s}f(X_s)\,ds\right),\quad H_\alpha^E\varphi(x)=E_x(e^{-\alpha\sigma_E}\varphi(X_{\sigma_E})). \tag{9.2}$$

For non-negative $f \in L^2(X;m)$, $R_\alpha^E f$ is a quasi-continuous function in $\mathscr{F}_{X-E}$ and satisfies for any $\varphi \in \mathscr{F}$

$$\mathscr{E}_\alpha(\varphi, R_\alpha^E f) = (\varphi - H_\alpha^E \tilde{\varphi}, f). \tag{9.3}$$

Here (,) is the inner product of $L^2(X;m)$ and $\mathscr{E}_\alpha(\varphi, \psi) = \mathscr{E}(\varphi, \psi) + \alpha(\varphi, \psi)$. When $X - E \subset\subset X$, (9.3) holds for any $\varphi \in \mathscr{F}_{\text{loc}}$.

For an open set $G \subset X$, denote by $\mathbf{M}_G$ and $\mathscr{E}_G$ the parts on G of the diffusion $\mathbf{M}$ and the Dirichlet form $\mathscr{E}$ respectively. $\mathbf{M}_G$ is obtained from $\mathbf{M}$ by shortening the life time from ζ to $\zeta \wedge \tau_G$ $(\tau_G = \inf\{t>0\, X_t \notin G\})$ and $\mathscr{E}_G$ is the restriction of $\mathscr{E}$ to the space $\mathscr{F}_G$. According to Theorem 4.4.2 of [9]. $\mathscr{E}_G$ is a regular Dirichlet form on $L^2(G;m)$ and possesses $\mathbf{M}_G$ as its associated process. A subset of G (resp. a function on G) is polar (resp. quasi-continuous) with respect to $\mathscr{E}_G$ if and only if it is so with respect to $\mathscr{E}$.

PROPOSITION 9.1. *For an open set G, $\mathscr{D}_G = \{\varphi \in \mathscr{D}$: $\text{supp}[\varphi] \subset G\}$ is a core of the Dirichlet space $(\mathscr{F}_G, \mathscr{E}_G)$ on $L^2(G;m)$.*

This is a consequence of the spectral synthesis [9; Theorem 3.3.4 and Problem 3.3.4]. For instance, if $X = \mathbf{R}^d$, $\mathscr{F} = H^1(\mathbf{R}^d)$, then this proposition implies $\mathscr{F}_G = H_0^1(G)$.

A function $w \in \mathscr{F}_{\text{loc}}$ is said to be *$\mathscr{E}$-superharmonic* on an open set $G \subset X$ if $\mathscr{E}(w, \varphi) \geq 0$ for any non-negative $\varphi \in \mathscr{D}_G$.

LEMMA 9.2. *The next conditions are equivalent for $w \in \mathscr{F}_{\text{loc}}$:*

(i) *w is $\mathscr{E}$-superharmonic (on X).*

(ii) *$\mathscr{E}(w, \varphi) \geq 0$ for any non-negative $\varphi \in \mathscr{F} \cap C_0(X)$.*

(iii) *There exists a positive Radon measure μ charging no set of zero capacity such that*

$$\mathscr{E}(w, \varphi) = \int_X \tilde{\varphi}(x)\mu(dx), \quad \varphi \in \mathscr{F}_G, \tag{9.4}$$

holds for any open set $G \subset\subset X$.

Proof. (i)⇒(ii): Take $\varphi \in C_0(X) \cap \mathscr{F}$ with $0 \leq \varphi \leq 1$ and choose η satisfying the condition of ($\mathscr{D}$.1) for $K = \text{supp}[\varphi]$ and open G with $K \subset G \subset\subset X$. Choose $\varphi_n \in \mathscr{D}$ which converges to φ in $\mathscr{F}$. Then, as in the proof of Theorem 2.1.2 of [9], we see that $\tilde{\varphi}_n=\beta_{1/n}(\varphi_n)\cdot\eta$ is weakly convergent in $\mathscr{F}$ to φ. (ii) now follows from $\mathscr{E}(w,\tilde{\varphi}_n)\geqslant-(1/n)\,\mathscr{E}(w,\eta)$.

(ii)$\Rightarrow$(iii): (ii) implies the existence of a unique positive Radon measure μ with the property (9.4) holding for $\varphi\in\mathscr{F}\cap C_0(X)$. Let $G\subset\subset D$ and take $\xi\in\mathscr{D}$ with $\xi=1$ on G. Then, for any $\varphi\in\mathscr{F}\cap C_0(X)$ with $\mathrm{supp}\,[\varphi]\subset G$,

$$\int|\varphi|\,d\mu=\mathscr{E}(\xi w,|\varphi|)\leqslant\sqrt{\mathscr{E}(\xi w,\xi w)}\,\sqrt{\mathscr{E}(\varphi,\varphi)},$$

which means that $\mu|_G$ is of finite energy integral with respect to $\mathscr{E}_G$. Hence $\mu|_G$ charges no set of $\mathscr{E}_G$-capacity zero and the validity of (9.4) for the $\mathscr{E}_G$-quasi-continuous version $\tilde{\varphi}$ of $\varphi\in\mathscr{F}_G$ follows just as in the proof of Theorem 3.2.2 of [9].

(iii)$\Rightarrow$(i): trivial. q.e.d.

THEOREM 9.3. *Consider an open set $G_0\subset X$ and let $w\in\mathscr{F}_{\mathrm{loc}}$ be quasi-continuous on X and $\mathscr{E}$-superharmonic on G_0. Then*

$$E_x(w(X_{\tau_G\wedge T}))\leqslant w(x)\quad \text{q.e. } x\in X \tag{9.5}$$

for any open $G\subset\subset G_0$ and $T>0$.

Proof. First consider the case that $G_0=X$. Let μ be the positive Radon measure on X associated with w by the preceding lemma. Let A be the positive continuous additive functional of M corresponding to the smooth measure μ. Then it holds that

$$P_x(w(X_t)-w(X_0)=M_t^{[w]}-A_t,\,t<\zeta)=1,\quad \text{q.e. } x,$$

where $M^{[w]}$ is the local martingale additive functional associated with w. The optional sampling theorem for the martingale then yields (9.5) for any open $G\subset\subset X$ and $T>0$. See the proof of Lemma 1 of [11] for more details.

For a general open set G_0, it suffices to replace $\mathscr{E}$ and $\mathbf{M}$ by their parts $\mathscr{E}_{G_0}$ and $\mathbf{M}_{G_0}$ respectively and observe that

$$P_x(\tau_G\wedge T<\zeta\wedge\tau_{G_0})=1,\quad \text{q.e. } x\in G_0$$

for open $G\subset\subset G_0$ and $T>0$. q.e.d.

Next we consider the Poincaré type inequality for $\mathscr{E}$:

$$\|\varphi\|^2_{L^2(X;m)} \leq C\cdot\mathscr{E}(\varphi,\varphi), \quad \varphi\in\mathscr{D}, \tag{9.6}$$

which means the transience of $\mathscr{E}$ and more than that the existence of a bounded linear operator G from $L^2(X;,m)$ into $\mathscr{F}$ such that

$$\mathscr{E}(\psi, Gf) = (\psi, f), \quad f\in L^2(X;m), \quad \psi\in\mathscr{F}. \tag{9.7}$$

THEOREM 9.4. *Assume that* $\mathscr{E}$ *has the property* (9.6).

(i) $E_x(\tau_G)<\infty$ q.e. $x\in X$ *for any open* $G\subset\subset X$. *If* $m(X)<\infty$ *in addition, then* $E_x(\zeta)<\infty$ q.e. $x\in X$.

(ii) *Let* K *be compact. If* $w\in\mathscr{F}_{\text{loc}}\cap L^\infty_{\text{loc}}(X;m)$ *be quasi-continuous on* X *and* $\mathscr{E}$-*superharmonic on* $G_0=X-K$, *then*

$$E_x(w(X_{\tau_G\wedge\sigma_K}))\leq w(x) \quad \text{q.e. } x\in X. \tag{9.8}$$

for any open G *such that* $K\subset G\subset\subset X$.

Proof. (i) For non-negative Borel $f\in L^2(X;m)$, $Rf(x)=E_x(\int_0^\zeta f(X_s)\,ds)$ is a quasi-continuous version of Gf and consequently finite q.e. In particular $E_x(\tau_G)\leq RI_G(x)<\infty$ q.e. for open $G\subset\subset X$. The second statement is clear from $E_x(\zeta)=R1(x)$.

(ii) We take an open H with $K\subset H\subset\subset G$. Then we have

$$E_x(w(X_{\tau_{G-\bar{H}}}))\leq w(x) \quad \text{q.e. } x\in X.$$

by letting $T\uparrow\infty$ in (9.5). w is, being quasi-continuous, continuous along the sample path X_t ([9; §4.3]). Hence we get (9.8) by making $H\downarrow K$. q.e.d.

We mention an additional remark on (9.6). It means that $\mathscr{E}$ and $\mathscr{E}_1$ define the equivalent norms and accordingly the potential theory and its probabilistic interpretation in [9] can be formulated in terms of $\mathscr{E}$ instead of $\mathscr{E}_1$. In particular, the associated capacity is defined for an open set E by

$$\operatorname{Cap}(E) = \inf\{\mathscr{E}(\varphi,\varphi)\colon \varphi\in\mathscr{F}, \varphi\geq 1\ m\text{-a.e. on } E\} \tag{9.9}$$

and is extended to any set as an outer capacity. For a compact set $K\subset X$, we then have

$$\text{Cap}(K) = \inf\{\mathscr{E}(\varphi, \varphi): \varphi \in \mathscr{F} \cap C_0(X), \varphi \geq 1 \text{ on } K\} \tag{9.10}$$

and further the function $e_K(x) = P_x(\sigma_K < \infty)$, $x \in X$, belongs to $\mathscr{F}$ and

$$\text{Cap}(K) = \mathscr{E}(e_K, e_K). \tag{9.11}$$

(cf. [9; Theorem 3.3.1, Problem 3.3.2 and Theorem 4.3.5]).

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ON QUASI-SUPPORTS OF SMOOTH MEASURES AND CLOSABILITY OF PRE-DIRICHLET FORMS

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1. Introduction

Consider a singular perturbation of the Laplacian Δ on the d-dimensional Euclidean space R^d:

$$\hat{L} = -\Delta + L_\Gamma .$$

Here L_Γ is a linear operator "living on" a closed subset $\Gamma \subset R^d$ which might be of zero Lebesgue measure and as irregular as a fractal set. The problem is how and when we can give $\hat{L}$ a proper sense. One way to formulate this is to introduce a perturbed bilinear form

$$\hat{\mathcal{E}}(f,g) = \boldsymbol{D}(f,g) + \mathcal{E}_\Gamma(f|_\Gamma, g|_\Gamma), \qquad f,g \in C_0^\infty(R^d),$$

where $\boldsymbol{D}$ is the Dirichlet integral and $\mathcal{E}_\Gamma$ is a closable pre-Dirichlet form on $L^2(\Gamma;\mu)$ for some positive Radon measure μ on Γ such that $C_0^\infty(R^d)|_\Gamma \subset \mathcal{D}[\mathcal{E}_\Gamma]$. If $\hat{\mathcal{E}}$ is proven to be closable on $L^2(R^d)$, the L^2-space based on the Lebesgue measure dx, then the associated self-adjoint operator on $L^2(R^d)$ may be thought of as a realization of $\hat{L}$.

Some sufficient conditions for the closability of the perturbed pre-Dirichlet form $\hat{\mathcal{E}}$ are known (M. Fukushima [2; §2.1], J.F. Brasche and W. Karwowski [1]). It is plausible that $\hat{\mathcal{E}}$ ought to be closable on $L^2(R^d)$ under the sole potential theoretic assumption that μ charges no set of zero (Newtonian) capacity. A purpose of the present paper is to affirm this in a more general context as will be stated in §2 and proven in §4.

The proof in §4 involves the notion of the quasi-support of a measure and its characterization crucially. The quasi-notions have appeared in potential theory in diverse contexts. Another aim of the present paper is to show in §3 the existence of the quasi-support along with its useful characterizations in terms of classes of quasi-continuous functions.

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Using the characterization, the quasi-support of a smooth measure is seen to be identical with the probabilistic notion of the support of the additive functional associated with the measure. The latter notion has been adopted recently by M. Fukushima, K. Sato and S. Taniguchi[3] to describe the closable part of a pre-Dirichlet form and to give a necessary and sufficient criterion for the closability. In §5, we restate some basic results of [3] in terms of the quasi-support together with alternative proof using a characterization of §3 instead of the usage of additive functionals. In particular, a characterization due to M. Röckner and N. Wielens[5] for the closability is recovered. The arguments in §4 can be regarded as a reduction of those in §5 to a simpler specific situation.

2. Statements on closability of perturbed pre-Dirichlet forms

In what follows, we fix a locally compact separable metric space X. $C_0(X)$ denotes the family of continuous functions on X with compact support. Suppose that a pair $(\mathcal{E}, \mathcal{C})$ satisfies the following conditions:

(1) $\mathcal{C}$ is a dense subalgebra of $C_0(X)$ such that, for any compact set K and relatively compact open set $G \supset K$, there exists $u \in \mathcal{C}$ with $u=1$ on K, $u=0$ on $X-G$ and $0 \leq u \leq 1$ on X. Furthermore, for any $\varepsilon>0$, there exists a real function $\varphi_\varepsilon(t)$ with $\varphi_\varepsilon(t)=1$, $t \in [0, 1]$, $-\varepsilon \leq \varphi_\varepsilon(t) \leq 1+\varepsilon$, $t \in R$ and $0 \leq \varphi_\varepsilon(t')-\varphi_\varepsilon(t) \leq t'-t$ for $t<t'$ such that $\varphi_\varepsilon(\mathcal{C}) \subset \mathcal{C}$.

(2) $\mathcal{E}$ is a non-negative definite symmetric bilinear form on $\mathcal{C}$ such that, for each $\varepsilon>0$ and for some function φ_ε satisfying the condition in (1), $\mathcal{E}(\varphi_\varepsilon(u), \varphi_\varepsilon(u)) \leq \mathcal{E}(u, u)$, $u \in \mathcal{C}$.

Then we call $\mathcal{E}$ or the pair $(\mathcal{E}, \mathcal{C})$ a *pre-Dirichlet form* over X.

Denote by $\mathcal{M}$ the family of positive Radon measure on X and let

$$\mathcal{M}' = \{m \in \mathcal{M} : \operatorname{supp} m = X\}.$$

where supp m denotes the topological support of m. A pre-Dirichlet form $(\mathcal{E}, \mathcal{C})$ is called *closable* on $L^2(X; m)$ for $m \in \mathcal{M}'$ if $\mathcal{E}(u_n, u_n) \to 0$ whenever $u_n \in \mathcal{C}$, $\{u_n\}$ is $\mathcal{E}$-Cauchy and $u_n \to 0$ in $L^2(X; m)$. If this is the case, the closure $(\mathcal{E}, \mathcal{F})$ of $(\mathcal{E}, \mathcal{C})$ on $L^2(X; m)$ is a regular Dirichlet form on $L^2(X; m)$. Conversely, given a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X; m)$, the restriction of $\mathcal{E}$ to any subfamily $\mathcal{C} \subset \mathcal{F}$ satisfying condition (1) is a pre-Dirichlet form closable on $L^2(X; m)$. In this case, $\mathcal{E}$ satisfies a stronger condition than (2) in the sense that the statement "for some function φ_ε" in (2) can be strengthened to "for any function φ_ε".

If $(\mathcal{E}, \mathcal{C})$ is a pre-Dirichlet form closable on $L^2(X; m)$ for some $m \in \mathcal{M}'$, we have the associated notion of capacity which can be evaluated for compact set K as

$$\operatorname{Cap}(K) = \inf \{\mathcal{E}_1(u, u) : u \in \mathcal{C}, u \geq 1 \text{ on } K\},$$

where $\mathcal{E}_1(u, v)$ is the sum of $\mathcal{E}(u, v)$ and the inner product $(u, v)_m$ in $L^2(X; m)$. The set of zero capacity is called *$\mathcal{E}_1$-polar* to indicate its relevence to $\mathcal{E}$ and m.

Theorem 2.1. *Let $(\mathcal{E}, \mathcal{C})$ and $(\hat{\mathcal{E}}, \mathcal{C})$ be pre-Dirichlet forms closable on $L^2(X; m)$ and $L^2(X; \hat{m})$ respectively for some $m, \hat{m} \in \mathcal{M}'$. We assume that*

$$\hat{\mathcal{E}}(u, u) \geq \mathcal{E}(u, u), \qquad u \in \mathcal{C},$$

and that $\hat{m}$ charges no $\mathcal{E}_1$-polar set. Then $(\hat{\mathcal{E}}, \mathcal{C})$ is closagle on $L^2(X; m)$.

Theorem 2.1 will be proven in §4 by using quasi-notions studied in the next section. We now show an immediate consequence of Theorem 2.1.

Theorem 2.2. (superposition of closable forms). *Let $(\mathcal{E}, \mathcal{C})$ be a pre-Dirichlet form closable on $L^2(X; m)$ for some $m \in \mathcal{M}'$. Consider a collection $\{\Gamma_\theta; \theta \in \Theta\}$ of closed subsets of X and suppose that, for each $\theta \in \Theta$, there exist a positive Radon measure μ_θ on X charging no $\mathcal{E}_1$-polar set with supp $\mu_\theta = \Gamma_\theta$, and a pre-Dirichlet form $\mathcal{E}_\theta$ over Γ_θ closable on $L^2(\Gamma_\theta, \mu_\theta)$ with $\mathcal{C}|_{\Gamma_\theta} \subset \mathcal{D}[\mathcal{E}_\theta]$. Further let $(\Theta, \mathcal{A}, \nu)$ be an auxiliary σ-finite measure space such that $\mathcal{E}_\theta(f|_{\Gamma_\theta}, f|_{\Gamma_\theta})$ and $\mu_\theta(K)$ are, as functions of $\theta \in \Theta$, ν-integrable for every $f \in \mathcal{C}$ and every compact $K \subset X$. Then the form $\hat{\mathcal{E}}$ defined by*

$$\hat{\mathcal{E}}(f, g) = \mathcal{E}(f, g) + \int_\Theta \mathcal{E}_\theta(f|_{\Gamma_\theta}, g|_{\Gamma_\theta})\, \nu(d\theta), \qquad f, g \in \mathcal{C},$$

is a pre-Dirichlet form closable on $L^2(X; m)$.

Proof. In view of the remark made after the definition of the pre-Dirichlet form, we can see that $(\hat{\mathcal{E}}, \mathcal{C})$ is a pre-Dirichlet form. If we let

$$\hat{m} = m + \int_\Theta \mu_\theta(\cdot)\nu(d\theta),$$

the $\hat{m}$ is a positive Radon measure on X charging no $\mathcal{E}_1$-polar set. Furthermore in the same manner as in the proof of [2; Theorem 2.1.3], we can show that $\hat{\mathcal{E}}$ is also closable on $L^2(X; \hat{m})$. Hence $\hat{\mathcal{E}}$ is closable on $L^2(X; m)$ by Theorem 2.1.

3. Quasi-supports of smooth measures and their characterizations

Let m be in $\mathcal{M}'$ and $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(X; m)$. Then we have the associated capacity Cap, and we use the terms "quasi-continuous", "polar" and "q.e." in relation to Cap. We add "$\mathcal{E}_1$-" in front of these terms when it is necessary to emphasize their relevence to $\mathcal{E}$ and m. Without loss of generality, we assume that each element of $\mathcal{F}$ is quasi-continuous. Two elements of $\mathcal{F}$ represent an equivalence class of $\mathcal{F} \subset L^2(X; m)$ iff they coincide q.e. An increasing sequence $\{F_n\}$ of closed sets with $\lim_{n\to\infty}$ Cap $(X - F_n) = 0$ is said to be a *nest*.

A set $E \subset X$ is called *quasi-open* if there exists a nest $\{F_n\}$ such that $E \cap F_n$ is open in F_n in the relative topology for each n. The complement of a quasi-open set is called *quasi-closed*.

Lemma 3.1. (*i*) *A set $F \in X$ is quasi-closed if and only if there exists a quasi-continuous function u with $F = u^{-1}(\{0\})$ q.e.*

(*ii*) *If F is quasi-closed, then, for any relatively compact open set G, there exists a non-negative quasi-continuous function u in $\mathcal{F}$ such that $u=0$ q.e. on F and $u>0$ on $G-F$.*

Proof. The "if" part of (i) is evident. We give probabilistic constructions for the rest of the proof. Consider a Hunt process $\boldsymbol{M}=(X_t, P_x)$ on X associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X; m)$. For any Borel set B, denote by e_B the 1-order hitting probability of B:

$$e_B(x) = E_x(e^{-\sigma_B}), \qquad x \in X,$$

where $\sigma_B = \inf\{t>0: X_t \in B\}$. When Cap $(B)<\infty$, e_B is a quasi-coninuous version of the (1-)equilibrium potential of B ([2; Th.4.3.5]). But we can see that e_B is quasi-continuous for any Borel B. In fact, for any relatively compact $E \subset X$, $w = e_B \wedge e_E$ is a 1-excessive function dominated by $e_E \in \mathcal{F}$. Hence $w \in \mathcal{F}$ (by [2; Lemma 3.3.2]) and w (and consequently e_B) is quasi-continuous (by [2; Th.4.3.2]).

Now, for any quasi-closed set F, the function

$$u(x) = 1 - e_F(x), \qquad x \in X,$$

has the required property in (i) because $F - F^r$ is polar (by [2; Th.4.2.3]) and $e_F(x)<1$ q.e. $x \in X-F$ (by [2; (4.3.5)]). The properties stated in (ii) is satisfied by

$$u(x) = (e_G(x) - e_{F \cap \bar{G}}(x)) \cdot v(x)$$

where v is a (quasi-continuous) bounded function in $\mathcal{F}$ such that $v>0$ on G and $v=0$ q.e. on $X-G$ (eg. $v(x) = E_x(\int_0^{\sigma_{X-G}} e^{-t} f(X_t)dt)$ for bounded $f>0$, $f \in L^2(X; m)$).

Corollary 3.2. *Any m-negligible quasi-open set is polar.*

Proof. Let E be an m-negligible quasi-open set. By Lemma 3.1(i), $E=\{u \neq 0\}$ for some quasi-continuous u. Then $u=0$ m-a.e. and consequently q.e., namely, E is polar.

A measure $\mu \in \mathcal{M}$ is said to be *of finite energy integral* if $\mathcal{F} \subset L^1(X; \mu)$ and

$$\int |v(x)| \mu(dx) \leq C\sqrt{\mathcal{E}_1(v, v)}, \qquad v \in \mathcal{F}, \tag{3.1}$$

for some positive constant C. The family of all measures of finite energy integrals is denoted by S_0. A positive Borel measure μ is said to be *smooth* if μ charges no polar set and there exists an incresaing sequence $\{F_n\}$ of closed sets such that

$$(3.2) \qquad \mu\left(X-\bigcup_{n=1}^{\infty} F_n\right)=0, \lim_{n\to\infty} \mathrm{Cap}\,(K-F_n)=0 \quad \text{for any compaat set } K$$

and $\mu(F_n)<\infty$ for each n. S will denote the totality of smooth measures. S contains the class $\mathcal{M}_0$ defined by

$$\mathcal{M}_0=\{\mu\in\mathcal{M}: \quad \mu \text{ charges no polar set}\}.$$

It is known ([2; Th.3.2.3]) that $\mu\in S$ iff there exists an increasing sequence $\{F_n\}$ of closed sets satisfying (3.2) and $I_{F_n}\cdot\mu\in S_0$ for each n.

For set $A, B\subset X$, we write

$$A\subset B \quad q.e. \qquad (resp.\ A=B\ q.e.)$$

if the set $A-B$ (resp. the symmetric differnece $A\ominus B$) is polar. For $\mu\in S$, a set $\tilde{F}\subset X$ is said to be a *quasi-support* of μ if

(a) $\tilde{F}$ is quasi-closed and $\mu(X-\tilde{F})=0$

(b) if $\check{F}$ is another set with property (a), then $\tilde{F}\subset\check{F}$ *q.e.*

The quasi-support $\tilde{F}$ of $\mu\in S$ is unique up to a polar set. Let $F=\mathrm{supp}\,\mu$ be the toplogical support of μ. Since any closed set is quasi-closed, we have $\tilde{F}\subset F$ *q.e.*, and by deleting a polar set from $\tilde{F}$ if necessary, we can always assume that $\tilde{F}\subset F$.

Theorem 3.3.

(*i*) *Any $\mu\in S$ admits a quasi-support.*

(*ii*) *For $\mu\in S$ and quasi-closed $F\subset X$, the following conditions are equivalent:*

(1) *F is a quasi-support of μ.*

(2) *$u=0$ $\mu-a.e.$ on X if and only if $u=0$ q.e. on F for any $u\in\mathcal{F}$.*

(3) *Condition* (2) *holds for any quasi-continuous function u.*

Proof. We first prove (ii).

(2)⇒(1): For $\mu\in S$ and quasi-closed F, we set

$$(3.3) \qquad \mathcal{N}_\mu=\left\{u\in\mathcal{F}: \int |u|\,d\mu=0\right\}$$

$$(3.4) \qquad \mathcal{F}_{F^c}=\{u\in\mathcal{F}: u=0 \quad q.e. \quad \text{on } F\}$$

and assume that $\mathcal{N}_\mu=\mathcal{F}_{F^c}$. For any relatively compact set G, take a function u of Lemma 3.1 (ii). Then $u\in\mathcal{F}_{F^c}$ and hence $u\in\mathcal{N}_\mu$, which means $\mu(G-F)=0$. Consequently we get $\mu(X-F)=0$. Consider other quasi-closed set F_1 with

$\mu(X-F_1)=0$. Take again a function u_1 of Lemma 3.1 (ii) for G and F_1. Then $u_1\in\mathcal{N}_\mu$ and hence $u_1\in\mathcal{F}_{F^c}$, which means $F\cap G\subset F_1\cap G$ *q.e.* Accordingly $F\subset F_1$ *q.e.* proving that F is a quasi-supo support of μ.

(1)$\Rightarrow$(3): Suppose (1) is satisfied. Then the "if" part of condition (2) is celarly satisfied for any Borel function u. If u is quasi-continuous and $u=0$ $\mu-a.e.$, then the set $\check{F}=\{u\neq 0\}$ has the property (a) and hence $F\subset\check{F}$ *q.e.* and $u=0$ *q.e.* on F.

The implication (3)$\Rightarrow$(2) is trivial.

(i) can be proved as follows. For any $\mu\in S$, the space $\mathcal{N}_\mu$ defined by (3.3) is a closed subspace of the separable Hilbert space $(\mathcal{F},\mathcal{E}_1)$ because $I_{F_n}\cdot\mu\in S_0$ for some increasing closed sets F_n satisfying (3.2) and $\mathcal{F}$ is continuously embedded into $L^1(X;I_{F_n}\cdot\mu)$ for each n by (3.1). Choose a countable dense subcollection $\{u_k\}$ of $\mathcal{N}_\mu$ and let

$$(3.5)\qquad F=v^{-1}(\{0\})\quad\text{for}\quad v(x)=\sum_{k=1}^{\infty}2^{-k}\frac{|u_k(x)|}{\|u_k\|_{\mathcal{E}_1}}$$

Since $v\in\mathcal{N}_\mu$, F is quasi-closed by Lemma 3.1(i) and further $\mu(X-F)=0$. Hence we arrive at the equality $\mathcal{N}_\mu=\mathcal{F}_{F^c}$ for $\mathcal{F}_{F^c}$ defined by (3.4) for this F. We can then conclude that F is a quasi-support of μ from (ii).

Corollary 3.4. *The underlying measure m has the full quasi-support X.*

Finally we state an important probabilistic consequence of Theorem 3.3 although we shall not use it in this paper.

Corollary 3.5. *For $\mu\in S$, the support of the associated positive continuous additive functional (PCAF) of $\boldsymbol{M}$ is a quasi-support of μ.*

Proof. Denote by A a PCAF of $\boldsymbol{M}$ associated with the smooth measure $\mu\in S$ (cf. [2; Chap. 5]). The support F_A of A is defined by

$$F_A=\{x\in X-N\colon P_x(A_t>0\text{ for any }t>0)=1\},$$

where N is an ecxeptional (polar) set for A. Then

$$F_A=\{x\in X-N\colon e_{F_A}(x)=1\}$$

and consequently F_A is quasi-closed since e_{F_A} is quasi-continuous as was seen in the proof of Lemma 3.1. Furthermore we can check the property (3) of Theorem 3.3 (ii) for μ and F_A in the same way as in the last paragraph of the proof of [2; Th.5.5.1].

4. Proof of Theorem 2.1.

We prove Theorem 2.1 by a series of lemmas. Suppose that $\mathcal{E}$, $\hat{\mathcal{E}}$, m and

$\hat{m}$ satisfy the conditions of Theorem 2.1. Let $(\mathcal{E}, \mathcal{F})$ and $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$ be the closures of $(\mathcal{E}, C)$ and $(\hat{\mathcal{E}}, C)$ on $L^2(X; m)$ and $L^2(X, \hat{m})$ respectively. $(u, v)_m$ (resp. $(u, v)_{\hat{m}}$) denotes the inner product in $L^2(X; m)$ (resp. $L_2(X; \hat{m})$). Recall that $\mathcal{E}_1(u, v)$ (resp. $\hat{\mathcal{E}}_1(u, v)$) stands for $\mathcal{E}_1(u, v)+(u, v)_m$ (resp. $\hat{\mathcal{E}}_1(u, v)+(u, v)_{\hat{m}}$). Note further that the conditions of Theorem 2.1 are never destroyed if we replace $\hat{m}$ by $m+\hat{m}$. Hence we can assume without loss of generality that

$$\hat{m} \geq m . \tag{4.1}$$

Lemma 4.1. *We let*

$$\hat{\mathcal{E}}^m(u, v) = \hat{\mathcal{E}}(u, v)+(u, v)_m , \qquad u, v \in \hat{\mathcal{F}} . \tag{4.2}$$

Then $(\hat{\mathcal{E}}^m, \hat{\mathcal{F}})$ is a Dirichlet form on $L^2(X; \hat{m})$ possessing C as a core. Moreover this is transient, namely, there exists a strictly positive $\hat{m}$-integrable function $\hat{g}$ such that

$$\int_X |v(x)| \hat{g}(x)\hat{m}(dx) \leq \sqrt{\hat{\mathcal{E}}^m(v, v)} , \qquad v \in \hat{\mathcal{F}} . \tag{4.3}$$

Proof. The first assertion is evident because $\hat{\mathcal{E}}_1^m(u, u)=\hat{\mathcal{E}}^m(u, u)+(u, u)_{\hat{m}}$ is equivalent to $\hat{\mathcal{E}}_1(u, u)$ for $u \in \hat{\mathcal{F}}$. Since $\hat{m} \in \mathcal{M}_0$ with respect to the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X; m)$ by assumption, there exist increasing closed sets F_n such that (3.2) holds for $\hat{m}$ and (3.1) holds for $I_{F_n} \cdot \hat{m}$ and some positive constant C_n for each n. Hence we can find a function $\hat{g}$ with the required properties.

Lemma 4.2. *The measure m has the full quasi-support X with respect to the Dirichlet form of Lemma* 4.1.

Proof. Let $\check{S}$ be a quasi-support of m with respect to the Dirichlet form $(\hat{\mathcal{E}}^m, \hat{\mathcal{F}})$ on $L^2(X; \hat{m})$ of Lemma 4.1. $E=X-\check{S}$ is then $\hat{\mathcal{E}}_1^m$-quasi-open and $m(E)=0$. Due to the domination of $\hat{\mathcal{E}}_1^m$ over $\mathcal{E}_1$, E is also $\mathcal{E}_1$-quasi-open and consequently $\mathcal{E}_1$-polar by Corollary 3.2. Then E is $\hat{m}$-negligible by the assumption and $\hat{\mathcal{E}}_1^m$-polar by Corollary 3.2 again.

The next lemma particularly implies Theorem 2.1. Denote by $(\hat{\mathcal{E}}^m, \mathcal{G})$ the extended Dirichlet space of the transient Dirichlet form of Lemma 4.1. $\mathcal{G}$ is the completion of $\hat{\mathcal{F}}$ with respect to the metric $\hat{\mathcal{E}}^m$. We may assume that each element of $\mathcal{G}$ is $\hat{\mathcal{E}}_1^m$-quasi-continuous.

Lemma 4.3. *$(\hat{\mathcal{E}}, \mathcal{G})$ is a Dirichlet form on $L^2(X; m)$ possessing C as its core.*

Proof. If $u \in \mathcal{G}$ and $u=0$ m—*a.e.* on X, then $u=0$ $\hat{\mathcal{E}}_1^m$—*q.e.* on X by Lemma 4.2 and Theorem 3.3, and hence $\hat{\mathcal{E}}(u, u)=0$. Therefore $(\hat{\mathcal{E}}, \mathcal{G})$ can be regarded as a symmetric form on $L^2(X; m)$. The rest of the proof is clear from Lemma 4.1.

5. Closable part and closability criterion in terms of quasi support

In this section, we restate some basic results of [3] in terms of the quasi-support and we see how an analytical characterization of §3 simplifies the arguments. For $m\in\mathcal{M}'$, let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(X; m)$ possessing as its core a set C satisfying condition (1) in §2. $(\mathcal{E}, \mathcal{F})$ is assumed to be either transient or irreducible. For simplicity of presentation, we only describe the case that $(\mathcal{E}, \mathcal{F})$ is transient. The irreducible case can be treated in the same way however by considering the transient Dirichlet form $\mathcal{E}^{\mu}(\cdot, \cdot)=\mathcal{E}(\cdot, \cdot)+(\cdot, \cdot)_{L^2(X;\mu)}$ ([3], [4]).

Take any non-trivial $\mu\in\mathcal{M}_0$. Denote by F and $\widetilde{F}$ the (topological) support and the quasi-support of μ respectively. As was noticed in §3, we may asume that $\widetilde{F}\subset F$. For a moment, we do not assume that $F=X$. Consider the extended Dirichlet space $(\mathcal{F}_e, \mathcal{E})$ of $(\mathcal{F}, \mathcal{E})$. $\mathcal{F}_e$ is a Hilbert space with inner product $\mathcal{E}$ and each element of $\mathcal{F}_e$ can be assumed to be $\mathcal{E}_1$-quasi-continuous. Let

$$\mathcal{F}_{e, X-\widetilde{F}} = \{u\in\mathcal{F}_e: u = 0 \ q.e. \text{ on } \widetilde{F}\}.$$

This is a closed subspace of $(\mathcal{F}_e, \mathcal{E})$. Denote by $P_{\widetilde{F}}$ the orthogonal projection on the orthogonal complement of $\mathcal{F}_{e, X-\widetilde{F}}$.

Note that, if $v_1, v_2\in\mathcal{F}_e$ and $v_1=v_2$ $\mu-a.e.$ on F, then $v_1=v_2$ $q.e.$ on $\widetilde{F}$ by virtue of Theorem 3.3, and consequently $P_{\widetilde{F}}v_1=P_{\widetilde{F}}v_2$. Therefore the following definition makes sense:

$$\mathcal{F}^{\mu} = \{u\in L^2(F; \mu): u = v \quad \mu-a.e. \text{ on } F \text{ for some } v\in\mathcal{F}_e\}$$
$$\mathcal{E}^{\mu}(u, u) = \mathcal{E}(P_{\widetilde{F}}v, P_{\widetilde{F}}v) \text{ for } v \text{ as in the above braces.}$$

Lemma 5.1. *$(\mathcal{F}^{\mu}, \mathcal{E}^{\mu})$ is a Dirichlet form on $L^2(F; \mu)$ possessing $C|_F$ as a core. Here $C|_F$ denotes the restrictions to F of elements of C.*

Proof. It can be readily seen that, for $u\in\mathcal{F}^{\mu}$,

$$\mathcal{E}^{\mu}(u, u) = \inf\{\mathcal{E}(v, v): v\in\mathcal{F}_e \quad v = u \quad \mu-a.e. \text{ on } F\}.$$

Denote by Tu the unit contraation $0\vee u\wedge 1$ of $u\in\mathcal{F}^{\mu}$. Then $Tu\in\mathcal{F}^{\mu}$ and

$$\begin{aligned}\mathcal{E}^{\mu}(Tu, Tu) &= \inf\{\mathcal{E}(v, v): v\in\mathcal{F}_e, v=Tu \ \mu-a.e. \text{ on } F\}\\ &\le \inf\{\mathcal{E}(Tv, Tv): v\in\mathcal{F}_e, v=u \ \mu-a.e. \text{ on } F\}\\ &\le \inf\{\mathcal{E}(v, v): v\in\mathcal{F}_e, v=u \ \mu-a.e. \text{ on } F\}\\ &= \mathcal{E}(u, u),\end{aligned}$$

proving that the unit contraction operates. The closedness of $(\mathcal{F}^{\mu}, \mathcal{E}^{\mu})$ on $L^2(F; \mu)$ is easily verified. We refer to [3] for the last statement about the core.

A pre-Dirichlet form $(\mathcal{A}, C)$ is called the *closable part* of a pre-Dirichlet form $(\mathcal{E}, C)$ with respect to $\mu\in\mathcal{M}'$ if $(\mathcal{A}, C)$ is closable on $L^2(X; \mu)$, $\mathcal{A}(u, u)\le\mathcal{E}(u, u)$, $u\in C$, and $(\mathcal{A}, C)$ is the maximum among those. Lemma 5.1

leads us to the next assertion (cf. [3; Lemma 4.3]).

Theorem 5.2. *For $\mu \in \mathcal{M}' \cap \mathcal{M}_0$, $(\mathcal{E}^{\mu}, C)$ is the closable part of $(\mathcal{E}, C)$ with respect to μ.*

Theorem 5.3. *For $\mu \in \mathcal{M}' \cap \mathcal{M}_0$, $(\mathcal{E}, C)$ is closable on $L^2(X; \mu)$ if and only if μ has the full quasi-support X.*

Proof. By Theorem 5.2, we have the following series of equivalent conditions:

$$\begin{aligned} &(\mathcal{E}, C) \text{ is closable on } L^2(X; \mu) \\ &\quad \Leftrightarrow P_{\tilde{F}} f = f \quad \text{for any } f \in \mathcal{F}_e \\ &\quad \Leftrightarrow f = 0 \ q.e. \text{ on } \tilde{F} \text{ iff } f=0 \ q.e. \text{ on } X \text{ for any } f \in \mathcal{F}_e \end{aligned}$$

Since $\tilde{F}$ is a quasi-support of μ, we see by Theorem 3.3 that the last condition is equivalent to

$$\text{“}f=0 \ \mu\text{—}a.e. \text{ on } X \quad \text{iff } f=0 \ q.e. \text{ on } X \text{ for any } f \in \mathcal{F}_e\text{”},$$

which is in turn equivalent to "X is a quasi-support of μ" by the same theorem.

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Dirichlet forms, diffusion processes and spectral dimensions for nested fractals

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1. Introduction

The (regular local)Dirichlet form for the Sierpinski gasket in the Euclidean k-space R^k has been introduced in Fukushima-Shima[5] as a basis to formulate the spectral analysis for the gasket. The associated self-adjoint operator coincides with the Laplacian introduced previously by Kigami[6]. The associated diffusion recovers the Brownian motion constructed by Kusuoka[9] and Barlow- Parkins[1].

The Dirichlet form in [5] has two special properties ; the first is that the Dirichlet norm is simply obtained as the limit of an increasing sequence of finite sums evaluated on the successive pre-gaskets (see §2), and the second is that the Dirichlet space is continuously embedded into the space of continuous functions. These properties are never shared by the ordinary Sobolev space H^1 on R^k except for the case that $k = 1$. They seem to be due to the following geometrical property of the gasket called the finite ramifiedness:if we try to connect any distinct two points of the gasket by a continuous curve on it,the curve should cross at least one of the finite number of specific points.

Indeed Dirichlet forms with these properties are nicely extended to general classes of finitely ramified fractals by Kusuoka[10] (for Lindstrom's nested fractals) and Kigami[7] (for post critically finite (PCF) self similar sets), yielding a quickest way to construct diffusion processes on the respective fractal sets. Accordingly the diffusion associated with the Dirichlet form of [10] recovers the Brownian motion on the nested fractal already constructed by Lindstrom[11].

In this paper, we consider the Dirichlet form of [10] and observe a simple scaling property it exhibits when the size of the underlying finite nested fractal is expanded. We shall present two straightforward applications of this scaling property. The first is to prove the point recurrence of the Brownian motion on the infinite nested fractal. The second is to identify the spectral dimension (which is strictly smaller than 2)with the exponent of polynomial growth of eigenvalues for the finite nested fractal. This identification was established in [11] by a different method. But we shall simultaneously derive similar tail behaviors of the integrated density of states (IDS) for the infinite nested fractal.

Recently Kumagaya[8] employs a method similar to ours to identify the spectral dimension of Kigami's PCF self similar set. On the other hand, Shima[16] uses the Dirichlet-Neumann bracketing method in terms of the Dirichlet form in getting the Lifschitz tail behavior(which also involves the spectral dimension in an exponential decay rate) of the

IDS for the infinite nested fractal under the presence of random Poisson obstacles or random Poisson noise potentials. The Lifschitz tail of the IDS for the Sierpinski gasket in R^2 with Poisson obstacles has been derived by Katarzyna Pietuska-Paluba[13]using a different method.

Thus the knowledge of the Dirichlet form and its scaling property is enough to extract the notion of the spectral dimension from the behaviors of eigenvalues for the fairly general finitely ramified fractal set. However more subtle spectral properties were studied so far only for the Sierpinski gasket. For instance, the IDS for the Sierpinski gasket has been shown in [5] to be purely discontinuous.[1] It is interesting to know if such wild spectral phenomena are common among the finitely ramified fractal sets.

2. The Dirichlet form for the nested fractal

In this section, we describe those notions and relations in Lindstrom[11] and Kusuoka[10] which we shall use later on.

For $\alpha > 1$, a mapping Ψ from R^k to R^k is said to be an *α-similitude* if $\Psi x = \alpha^{-1}Ux + \beta, x \in R^k$, for some unitary map U and $\beta \in R^k$. Given a collection $\Psi = \{\Psi_1, \Psi_2, \dots, \Psi_N\}$ of α-similitudes,we let $\Psi(A) = \bigcup_{i=1}^N \Psi_i(A), A \subset R^k$. There exists then a unique compact set $E \subset R^k$ such that $\Psi(E) = E$. The pair (Ψ, E) is called a *self similar fractal.*

For $A \subset R^k$ and integer $n \geq 1$, we let

$$A_{i_1 \dots i_n} = \Psi_{i_1} \dots \Psi_{i_n}(A), \qquad 1 \leq i_1, \dots, i_n \leq N$$

$$A^{(n)} = \Psi^{(n)}(A) = \bigcup_{1 \leq i_1, \dots, i_n \leq N} A_{i_1 \dots i_n}, \qquad A^{(0)} = A$$

. We denote by F the set of all essential fixed points of Ψ([11]). $\sharp F \leq N$. Lindstrom[11] calls a self similar fractal (Ψ, E) a *nested fractal* if three axioms (axioms of connectivity, symmetry and nesting) and the open set condition are fulfilled and $\sharp F \geq 2$. We refer the readers to [11] for details but we note that the nesting axiom requires

$$E_{i_1 \dots i_n} \cap E_{j_1 \dots j_n} = F_{i_1 \dots i_n} \cap F_{j_1 \dots j_n} \qquad (i_1, \dots, i_n) \neq (j_1, \dots, j_n),$$

which expresses the finite ramifiedness mentioned in §1.

Given a nested fractal (Ψ, E), the Hausdorff dimension of E is known to be equal to $\frac{\log N}{\log \alpha}$. The normalized Hausdorff measure on E is denoted by $\mu : \mu(E) = 1$. The sequence $\{F^{(n)}\}$ of finite sets is increasing and $\cup_{n=0}^{\infty} F^{(n)}$ is denoted by $F^{(\infty)}$, which may be called the pre-nested fractal since $\overline{F^{(\infty)}} = E$.

A typical example of nested fractals is the Sierpinski gasket in R^k. We now explane how to introduce a natural Dirichlet form on the Siepinski gasket in R^2. Let $F = \{p_1, p_2, p_3\}$ be the vertices of the regular triangle of side length 1 in R^2. Let Ψ_i be the 2-similitude on R^2 without rotation ($U = I$) making the point p_i fixed ($i = 1, 2, 3$). The self-similar fractal$E$ determined by $\Psi = (\Psi_1, \Psi_2, \Psi_3)$ is the finite Sierpinski gasket,which is easily seen to be a nested fractal.

[1]See also [14] in this connection

For any real valued function f on the pre-gasket $F^{(\infty)}$, we set
(2.1)
$$\mathcal{E}^{(n)}(f,f)=c\left(\frac{5}{3}\right)^n\sum_{1\le i_1,\dots,i_n\le 3}\sum_{\xi,\eta\in F}(f(\Psi_{i_1}\dots\Psi_{i_n}\xi)-f(\Psi_{i_1}\dots\Psi_{i_n}\eta))^2,\quad n=0,1,2,\dots$$

c being a positive constant. It is easy to see that $\mathcal{E}^{(n)}(f,f)$ is increasing in n. In fact $\mathcal{E}^{(0)}(f,f)\le\mathcal{E}^{(1)}(f,f)$ reduces to the elementary absolute inequality

$$(2.2)\quad (A_1-A_2)^2+(A_2-A_3)^2+(A_3-A_1)$$
$$\le\frac{5}{3}\left\{(a_1-a_2)^2+(a_2-a_3)^2+(a_3-a_1)^2\right\}$$
$$+\frac{5}{3}\left\{(A_1-a_2)^2+(A_1-a_3)^2+(A_2-a_1)^2+(A_2-a_3)^2+(A_3-a_1)^2+(A_3-a_2)^2\right\}$$

holding for any real numbers $A_i, a_i, i=1,2,3$. This inequality is honestly inherited to the next step inequality $\mathcal{E}^{(1)}(f,f)\le\mathcal{E}^{(2)}(f,f)$ and so on.

Hence it seems natural to introduce the space

$$(2.3)\quad \mathcal{F}=\left\{f: functionon F^{(\infty)}, \sup_n\mathcal{E}^{(n)}(f,f)<\infty\right\}$$
$$\mathcal{E}(f,g)=\lim_{n\to\infty}\mathcal{E}^{(n)}(f,g),\qquad f,g\in\mathcal{F}.$$

It is actually proven in [5] that any function in $\mathcal{F}$ is extended to be a continuous function on the gasket $\mathcal{E}=\overline{F^{(\infty)}}$ and that $\mathcal{E}$ on $\mathcal{F}$ is a local regular Dirichlet form on $L^2(E;\mu)$. It is clear that behind this analytical approach is the sequence of random walks on $F^{(n)}, n=0,1,2,\dots$,moving to the nearest neighbours with equal probability. The renormalizing factor $\frac{5}{3}$ should have to do with these random walks. In fact,the equality is attained in (2.2) when each a_i is the harmonic average of A_1, A_2, A_3 with respect to the random walk on $F^{(1)}$.

As was proved in Kusuoka[10], the above mentioned situation for the Sierpinski gasket is totally unchanged for the general nested fractal if one takes, as the random walks on $F^{(n)}$'s, those governed by Lindstrom's invariant probability[11].

In what follows, we work with a fixed nested fractal (Ψ, E) in R^k. By a random walk on F, we mean a Markov chain on F whose transition probability from $x\in F$ to $y\in F, x\ne y$, depends only on the distance $|x-y|$ and decreases strictly if the value $|x-y|$ increases. Thus we let

$$\{|x-y|: x,y\in F, x\ne y\}=\{\ell_1,\dots,\ell_r\},\quad 0<\ell_1<\cdots<\ell_r$$

$$m_s=\sharp\{y\in F: |x-y|=\ell_s\}\quad\text{for a fixed } x\in F, 1\le s\le r$$

$$\mathcal{P}=\{\mathbf{P}=(p_1,\dots,p_r): p_1>p_2>\cdots>p_r>0\}.$$

m_s is independent of $x\in F$ because of the axiom of symmetry.

Each element $\mathbf{P} \in \mathcal{P}$ not only decides a random walk on F, but also determines a Markov chain on $F^{(1)}$ with transition probability

$$p(x,y,\mathbf{P}) = \begin{cases} \frac{\rho(x,y)}{\rho(x)} p_s & \rho(x,y) \geq 1, |x-y| = \alpha^{-1}\ell_s \\ 0 & \text{otherwise,} \end{cases}$$

where $\rho(x) = \sharp\{i : x \in F_i\}, \rho(x,y) = \sharp\{i : x,y \in F_i\}$. This Markov chain on $F^{(1)}$ in turn induces a Markov chain on F by letting the sample path start at $x \in F$ and observing the first time the path hits $F \setminus \{x\}$ and so on. The invariant probability is an element of $\mathcal{P}$ such that the induced Markov chain on F in the above sense coincides in law with the original walk on F. Lindstrom[11] proved its existence and Barlow[2] recently shows its uniqueness for a certain class of nested fractals including Lindstrom's snow flake..

Let us denote the invariant probability by $\mathbf{P}_0 = (p_1, \ldots, p_r)$. Let $(X(n), P_x)$ be the Markov chain on $F^{(1)}$ with trasition probability $p(x,y,\mathbf{P}_0), x,y \in F^{(1)}$ and let

$$c = P_x(X_{\tilde{\sigma}} = x),\ x \in F, \text{ where } \tilde{\sigma} = \{n > 0 : X(n) \in F\}.$$

By virtue of the axiom of symmetry,c is independent of $x \in F$ and evidently $0 < c < 1$. In view of the above description of the invariance property of $\mathbf{P}_0$, we may expect the quanitity c to play an intrinsically important role.

For $\xi, \eta \in F, \xi \neq \eta$, we let $\pi_{\xi\eta} = p_s$ if $|\xi - \eta| = \ell_s$. Then $\sum_{\eta \in F} \pi_{\xi\eta} = 1$ and $\pi_{\xi\eta} = \pi_{\eta\xi}$. We denote by $\mathcal{D}$ the set of all real valued functions on $F^{(\infty)} = \cup_{n=0}^{\infty} F^{(n)}$. For $f, g \in \mathcal{D}$, we define $\mathcal{E}^{(n)}(f,g)$ by

$$\text{(2.4)} \quad \mathcal{E}^{(n)}(f,g) = \frac{1}{2}(1-c)^{-n} \sum_{1 \leq k_1, \ldots, k_n \leq N} \sum_{\substack{\xi,\eta \in F \\ \xi \neq \eta}} (f(\Psi_{k_1} \cdots \Psi_{k_n}\xi) - f(\Psi_{k_1} \cdots \Psi_{k_n}\eta)) (g(\Psi_{k_1} \cdots \Psi_{k_n}\xi) - g(\Psi_{k_1} \cdots \Psi_{k_n}\eta)) \pi_{\xi\eta}$$

In the case of the Sierpinski gasket in R^2, $c = \frac{2}{5}, \pi_{\xi\eta} = \frac{1}{2}, \xi \neq \eta$, and hence (2.4) reduces to (2.1).

THEOREM 2.1(KUSUOKA[10]).

(1) *For any $f \in \mathcal{D}, \mathcal{E}^{(n)}(f,f)$ defined by (2.4) is non-decreasing in n. Hence $(\mathcal{F}, \mathcal{E})$ is well defined by (2.3).*

(2) *Any function of $\mathcal{F}$ can be uniquely extended to a continuous function on $E = \overline{F^{(\infty)}}$.*

(3) *$(\mathcal{F}, \mathcal{E})$ is a regular local Dirichlet form on $L^2(E;\mu)$.*[2]

Since $\pi_{\xi\eta}/p_r \geq 1, \xi, \eta \in F$, we have for any function f on F

$$\text{(2.5)} \qquad \max\{|f(\xi) - f(\eta)| : \xi, \eta \in F\} \leq \frac{1}{\sqrt{p_r}} \left\{ \frac{1}{2} \sum_{\xi,\eta \in F} (f(\xi) - f(\eta))^2 \pi_{\xi\eta} \right\}^{1/2}$$

This elementary esimate leads us to the next theorem. Actually Theorem 2.1(2) is a corollary of the much stronger assertion Theorem 2.2(1).

[2] μ can be replaced by any everywhere dense positive Radon measure on E.

THEOREM 2.2(KUSUOKA[10]).

(1) *Let B be the set of those functions f on $F^{(\infty)}$ with $\sup_n \mathcal{E}^{(n)}(f,f) \leq 1$. Then*

$$\lim_{\varepsilon\to 0}\sup\left\{|f(x)-f(y)| : f\in B, x,y\in F^{(\infty)}, |x-y|<\varepsilon\right\}=0.$$

(2)

$$\sup\{|f(x)-f(y)| : x,y\in E\} \leq 4N\left(\frac{N}{p_r}\right)^{1/2}\frac{1-c}{c}\mathcal{E}_1(f,f)^{1/2},\ f\in\mathcal{F}.$$

(3)

$$\sup\{|f(x)| : x\in E\} \leq \sqrt{2}\left(\frac{N}{p_r}\right)^{1/2}\frac{1-c}{c}\mathcal{E}_1(f,f)^{1/2},\ f\in\mathcal{F},$$

where $\mathcal{E}_\alpha(f,g)=\mathcal{E}(f,g)+\alpha(f,g)_{L^2(\mu)}$, $\alpha>0$.

We can easily draw several conclusions from the above two theorems.

THEOREM 2.3.

(1) *For each $\alpha>0$, the Hilbert space $(\mathcal{F},\mathcal{E}_\alpha)$ admits a reproducing kernel $g_\alpha(x,y)$: for each $y\in E$ there exists $g_\alpha(\cdot,y)\in\mathcal{F}$ such that*

$$\mathcal{E}_\alpha(g_\alpha(\cdot,y),v)=v(y),\quad v\in\mathcal{F}. \tag{2.6}$$

(2) *$g_\alpha(x,y)$ is positive continuous symmetric on $E\times E$.*

(3) *Each one point set has a positive capacity:*

$$Cap(\{y\})=\frac{1}{g_1(y,y)},\qquad y\in E.$$

(4) *The associated diffusion on E is point recurrent:*

$$P_x\left(\sigma_{\{y\}}<\infty\right)=1 \text{ for any } x,y\in E,$$

$\sigma_{\{y\}}$ being the first hitting time for $\{y\}$.

PROOF: (1) follows from Theorem 2.2(3) which implies that the map sending $f\in\mathcal{F}$ to $f(y)\in R$ is bounded.
(2). Symmetry is obvious from (2.6). From (2.6), we also have

$$\mathcal{E}_\alpha(g_\alpha(\cdot,y),g_\alpha(\cdot,y))=g_\alpha(y,y)$$

which is positive because otherwise $v(y)$ vanishes for any $v\in\mathcal{F}$. Further, by Theorem 2.2(3)

$$\sup_{y\in E}\sqrt{g_\alpha(y,y)} \le \sup_{y\in E}\frac{\sup_{x\in E} g_\alpha(x,y)}{\sqrt{g_\alpha(y,y)}} \le \sup_{f\in\mathcal{F}}\frac{\sup_{x\in E}|f(x)|}{\sqrt{\mathcal{E}_\alpha(f,f)}} < \infty.$$

Therefore the family of functions $\{g_\alpha(\cdot,y); y\in E\}$ is equi uniformly continuous on E by Theorem 2.2(1) and we can get the continuity of $g_\alpha(x,y)$ in x,y from

$$|g_\alpha(x,y)-g_\alpha(x\prime,y\prime)| \le |g_\alpha(x,y)-g_\alpha(x\prime,y)| + |g_\alpha(y,x\prime)-g_\alpha(y\prime,x\prime)|.$$

(3).If we let $p_1^y(x)=\frac{g_1(x,y)}{g_1(y,y)}$, then $p_1^y\in\mathcal{F}, p_1^y(y)=1$ and $\mathcal{E}_1(p_1^y,v)\ge 0$ for any $v\in\mathcal{F}$ with $v(y)\ge 0$. Therefore p_1^y is the 1-equilibrium potential of the one point set {y} and

$$Cap(\{y\}) = \mathcal{E}_1(p_1^y,p_1^y) = \frac{1}{g_1(y,y)}.$$

(4). $(\mathcal{F},\mathcal{E})$ is irreducible because otherwise $\mathcal{F}$ must contain a discontinuous function([4]) contradicting to Theorem 2.1(2). Since $1\in\mathcal{F}$ and $\mathcal{E}(1,1)=0, (\mathcal{F},\mathcal{E})$ is recurrent([4]). Becuase of (3), the associated diffusion is point recurrent. In particular,

$$g_\alpha(x,y) = E_x\left(e^{-\alpha\sigma_{\{y\}}}\right) g_\alpha(y,y) > 0, \quad x,y\in E.$$

We denote by Δ the self-adjoint operator on $L^2(E;\mu)$ associated with $(\mathcal{F},\mathcal{E})$:

$$\mathcal{D}(\Delta)\subset\mathcal{F} \qquad \mathcal{E}(f,g) = -(\Delta f,g), f\in\mathcal{D}, g\in\mathcal{F}.$$

By virtue of Theorem 2.3 (2), $-\Delta$ is of compact resolvent and Mercer's theorem leads to the absolutely uniformly convergent series expansion :

$$g_\alpha(x,y) = \sum_{i=1}^{\infty}\frac{1}{\alpha+\lambda_i}\varphi_i(x)\varphi_i(y), \tag{2.7}$$

where

$$0\le\lambda_1\le\lambda_2\le\cdots\le\lambda_n\le\ldots$$

are eigenvalues of $-\Delta$ and $\{\varphi_n\}$ are the corresponding normalized eigenfunctions. Notice that κ is an eigenvalue of $-\Delta$ with an eigenfunction f if

$$f\in\mathcal{F} \quad \text{and} \quad \mathcal{E}(f,g) = \kappa(f,g)_{L^2(E;\mu)} \quad \text{for any} \quad g\in\mathcal{F}. \tag{2.8}$$

3. Dirichlet forms on expanded nested fractals

As in the preceding section, we fix a nested fractal (Ψ, E) in R^k. Without loss of generality, we assume that $0 \in F$ and $\Psi_1 x = \alpha^{-1}x, x \in R^k$. We let

$$E^{\langle \ell \rangle} = \alpha^\ell E, \quad \ell = 0, 1, \ldots, \quad E^{\langle \infty \rangle} = \bigcup_{\ell=0}^{\infty} E^{\langle \ell \rangle}. \tag{3.1}$$

We call $E^{\langle \ell \rangle}$ (resp.$E^{\langle 0 \rangle} = E$)the *infinite* (resp.*unit*) nested fractal. $E^{\langle \ell \rangle}$ is the union of N^ℓ -number of sets congruent to E :

$$E^{\langle \ell \rangle} = \bigcup_{1 \le i_1, \ldots, i_\ell \le N} E^{\langle \ell \rangle}_{i_1 \ldots i_\ell} \quad \text{and} \quad E^{\langle \ell \rangle}_{i_1 \ldots i_\ell} = \Phi_{i_1 \ldots i_\ell} E \quad \text{with} \quad \Phi_{i_1 \ldots i_\ell} = \alpha^\ell \Psi_{i_1} \ldots \Psi_{i_\ell}.$$

We define the mapping σ_ℓ by

$$(\sigma_\ell f)(x) = f(\alpha^\ell x) \qquad x \in E, \tag{3.2}$$

which maps a function f on $E^{\langle \ell \rangle}$ to a function $\sigma_\ell f$ on the unit nested fractal E.

Recall that we have a Hausdorff measure μ on E with $\mu(E) = 1$ and a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E; \mu)$ introduced in Theorem 2.1. We extend μ to $E^{\langle \infty \rangle}$ by defining its value on B to be $\mu\left(\Phi^{-1}_{i_1 \ldots i_\ell} B)\right)$ if $B \subset E^{\langle \ell \rangle}_{i_1 \ldots i_\ell}$ (which does not depend on the choice of ℓ). We also define a Dirichlet form $(\mathcal{F}_{E^{\langle \ell \rangle}}, \mathcal{E}_{E^{\langle \ell \rangle}})$ on $L^2(E^{\langle \ell \rangle}; \mu)$ by

$$\begin{aligned} (3.3) \quad & \mathcal{F}_{E^{\langle \ell \rangle}} = \sigma_\ell^{-1} \mathcal{F} \\ & \mathcal{E}_{E^{\langle \ell \rangle}}(f, g) = \sum_{1 \le i_1, \ldots, i_\ell \le N} \mathcal{E}\left(f\left(\Phi_{i_1 \ldots i_\ell} \cdot\right), g\left(\Phi_{i_1 \ldots i_\ell} \cdot\right)\right), \quad f, g \in \mathcal{F}^{\langle \ell \rangle}, \ell = 1, 2, \ldots. \end{aligned}$$

LEMMA 3.1(SCALING PROPERTIES).

(1) *For a function F on $E^{\langle \ell \rangle}$,*

$$\int_{E^{\langle \ell \rangle}} f \, d\mu \quad = \quad N^\ell \int_E (\sigma_\ell f) \, d\mu.$$

(2) *For $f \in \mathcal{F}_{E^{\langle \ell \rangle}}$,*

$$\mathcal{E}_{E^{\langle \ell \rangle}}(f, f) \quad = \quad (1 - c)^\ell \, \mathcal{E}(\sigma_\ell f, \sigma_\ell f).$$

PROOF: (1) In the case that $f = I_B$ for $B \subset E^{\langle \ell \rangle}_{i_1 \ldots i_\ell}$, the right hand side equals

$$N^\ell \mu\left(\alpha^{-\ell} B\right) = N^\ell \mu\left(\alpha^{-\ell} \Phi^{-1}_{i_1 \ldots i_\ell} B\right) = \mu\left(\Phi^{-1}_{i_1 \ldots i_\ell} B\right)$$

which coincides with the left hand side.

(2) By (3.3) and (2.4),

$$\mathcal{E}_{E^{(\ell)}}(f,f) = \sum_{1\le i_1,\dots,i_\ell\le N} \mathcal{E}\left\{\sigma_\ell f(\Psi_{i_1\dots i_\ell}\cdot),\sigma_\ell f(\Psi_{i_1\dots i_\ell}\cdot)\right\}$$

$$= \lim_{n\to\infty}\frac{1}{2}(1-c)^{-n}\sum_{1\le i_1,\dots,i_{\ell+n}\le N}\sum_{\substack{\xi,\eta\in F\\ \xi\ne\eta}}\left\{\sigma_\ell f\left(\Psi_{i_1}\dots\Psi_{i_{\ell+n}}\xi\right)-\sigma_\ell f\left(\Psi_{i_1}\dots\Psi_{i_{\ell+n}}\eta\right)\right\}^2\pi_{\xi\eta}$$

$$= (1-c)^\ell\mathcal{E}\left(\sigma_\ell f,\sigma_\ell f\right).$$

At the end of §2,we introduced the self-adjoint operator Δ on $L^2(E;\mu)$ associated with $(\mathcal{F},\mathcal{E})$. Analogously we consider the self-adjoint operator $\Delta^{(\ell)}$ on $L^2(E^{(\ell)};\mu)$ associated with $(\mathcal{F}_{E^{(\ell)}},\mathcal{E}_{E^{(\ell)}})$. From the above lemma and (2.8), we get

Corollary 3.2. *κ is an eigenvalue of $-\Delta$ iff $\left(\frac{1-c}{N}\right)^\ell\kappa$ is an eigenvalue of $-\Delta^{(\ell)}$.*

It is clear from definition (3.3) that

$$\mathcal{E}_{E^{(\ell)}}\left(f|_{E^{(\ell)}},f|_{E^{(\ell)}}\right)\le\mathcal{E}_{E^{(m)}}(f,f),\qquad \ell<m,\qquad f\in\mathcal{F}_{E^{(m)}},$$

the equality holding when f vanishes on $E^{(m)}-\overline{E^{(\ell)}}$. We define the space $\mathcal{F}_{E^{(\infty)}}$ of functions f on the infinite nested fractal $E^{(\infty)}$ by

(3.4)

$$\mathcal{F}_{E^{(\infty)}}=\left\{f: f|_{E^{(\ell)}}\in\mathcal{F}_{E^{(\ell)}}\text{ for each }\ell\text{ and }\lim_{\ell\to\infty}\mathcal{E}_{E^{(\ell)}}\left(f|_{E^{(\ell)}},f|_{E^{(\ell)}}\right)<\infty\right\}\cap L^2(E^{(\infty)};\mu).$$

We also let

$$\mathcal{E}_{E^{(\infty)}}(f,g)=\lim_{\ell\to\infty}\mathcal{E}_{E^{(\ell)}}\left(f|_{E^{(\ell)}},g|_{E^{(\ell)}}\right),\qquad f,g\in\mathcal{F}_{E^{(\infty)}}. \tag{3.5}$$

Denote by $C(E^{(\infty)})$ (resp.$C_0(E^{(\infty)})$) the space of continuous functions(resp.continuous functions with compact suport) on $E^{(\infty)}$. We have $\mathcal{F}_{E^{(\infty)}}\subset C(E^{(\infty)})$. If $f\in C_0(E^{(\infty)})$, supp$f\subset E^{(m)}$ and $f\in\mathcal{F}_{E^{(m)}}$, then $f\in\mathcal{F}_{E^{(\infty)}}$ and

$$\mathcal{E}_{E^{(\infty)}}(f,f)=\mathcal{E}_{E^{(m)}}(f,f). \tag{3.6}$$

Lemma 3.3. *There exist functions $\phi_\ell\in\mathcal{F}_{E^{(\infty)}}\cup C_0\left(E^{(\infty)}\right)$ such that*

$$\phi_\ell=1 \text{ on } E^{(\ell-1)},\ \phi_\ell=0\text{ on }E^{(\infty)}-E^{(\ell)},\ 0\le\phi_\ell\le1\text{ and }\lim_{\ell\to\infty}\mathcal{E}_{E^{(\infty)}}(\phi_\ell,\phi_\ell)=0.$$

Proof: Take the function ψ on E with the properties that $\psi\in\mathcal{F},\psi=1$ on $F_1(=\Psi_1F),\psi=0$ on $F^{(1)}\setminus F_1$ and ψ is $\mathcal{E}$-harmonic on $E\setminus F^{(1)}$. Then $\psi=1$ on $E^{(1)}=\psi_1E$ and $0\le\psi\le1$ on E. It suffices to let ϕ_ℓ be $\sigma_\ell^{-1}\psi$ on $E^{(\ell)}$ and 0 on $E^{(\infty)}\setminus E^{(\ell)}$. We get from Lemma 3.1 and (3.6)

$$\mathcal{E}_{E^{(\infty)}}(\phi_\ell,\phi_\ell)=\mathcal{E}_{E^{(\ell)}}(\phi_\ell,\phi_\ell)=(1-c)^\ell\mathcal{E}(\psi,\psi),$$

which tends to zero as $\ell\to\infty$.

THEOREM 3.4. *$(\mathcal{F}_{E^{\langle\infty\rangle}}, \mathcal{E}_{E^{\langle\infty\rangle}})$ is a regular local Dirichlet form on $L^2\left(E^{\langle\infty\rangle};\mu\right)$. For each $\alpha > 0, \mathcal{E}_{E^{\langle\infty\rangle},\alpha}$ admits a positive symmetric continuous reproducing kernel $g_\alpha(x,y)$ on $E^{\langle\infty\rangle} \times E^{\langle\infty\rangle}$ and*

$$Cap(\{y\}) = \frac{1}{g_\alpha(y,y)}(>0), \qquad y \in E^{\langle\infty\rangle}.$$

The associated diffusion on $E^{\langle\infty\rangle}$ is point recurent.

PROOF: The preceding lemma implies that $(\mathcal{F}_{E^{\langle\infty\rangle}}, \mathcal{E}_{E^{\langle\infty\rangle}})$ is non-transient([4]). Other assertions except for the regularity can be proven similarly as the proof of theorems of §2. To prove the regularity of $(\mathcal{F}_{E^{\langle\infty\rangle}}, \mathcal{E}_{E^{\langle\infty\rangle}})$, take any bounded function $f \in \mathcal{F}_{E^{\langle\infty\rangle}}$ and set $f_\ell = f \cdot \phi_\ell, \ell = 1, 2, \ldots,$ for ϕ_ℓ of Lemma 3.3. We write for $g \in \mathcal{F}_{E^{\langle\infty\rangle}}$

$$\mathcal{E}_{E^{\langle\infty\rangle}\setminus E^{\langle\ell\rangle}}(g,g) = \mathcal{E}_{E^{\langle\infty\rangle}}(g,g) - \mathcal{E}_{E^{\langle\ell\rangle}}\left(g|_{E^{\langle\ell\rangle}}, g|_{E^{\langle\ell\rangle}}\right).$$

We have then

$$\begin{aligned}\mathcal{E}_{E^{\langle\infty\rangle}}(f - f_\ell, f - f_\ell) &= \mathcal{E}_{E^{\langle\infty\rangle}\setminus E^{\langle\ell-1\rangle}}(f(1-\phi_\ell), f(1-\phi_\ell)) \\ &\leq \|f\|_\infty^2 \mathcal{E}_{E^{\langle\infty\rangle}}(\phi_\ell, \phi_\ell) + \mathcal{E}_{E^{\langle\infty\rangle}\setminus E^{\langle\ell-1\rangle}}(f,f) \ \rightarrow \ \infty, \ \ell \rightarrow \infty.\end{aligned}$$

Since $f_\ell \in \mathcal{F}_{E^{\langle\infty\rangle}} \cap C_0\left(E^{\langle\infty\rangle}\right), \mathcal{F}_{E^{\langle\infty\rangle}} \cap C_0\left(E^{\langle\infty\rangle}\right)$ is $\mathcal{E}_{E^{\langle\infty\rangle},1}$-dense in $\mathcal{F}_{E^{\langle\infty\rangle}}$.

4. Asymptotics of the eigenvalue distribution and the integrated density of states

For Δ and $\Delta^{\langle\ell\rangle}$ considered in §3, we let

$$\rho(\lambda) = \sharp\{\text{eigenvalues of } -\Delta \leq \lambda\}$$

$$k_\ell(\lambda) = \sharp\left\{\text{eigenvalues of} - \Delta^{\langle\ell\rangle} \leq \lambda\right\}.$$

Together with $k_\ell(\lambda)$, we consider

$$k_\ell^0(\lambda) = \sharp\left\{\text{eigenvalues of} - \Delta_0^{\langle\ell\rangle} \leq \lambda\right\},$$

where $\Delta_0^{\langle\ell\rangle}$ is the self-adjoint operator on $L^2(E^{\langle\ell\rangle};\mu)$ corresponding to the Dirichlet space $\left(\mathcal{F}^0_{E^{\langle\ell\rangle}}, \mathcal{E}_{E^{\langle\ell\rangle}}\right)$ with

$$\mathcal{F}^0_{E^{\langle\ell\rangle}} = \left\{f \in \mathcal{F}_{E^{\langle\ell\rangle}} : f(p) = 0 \quad p \in \alpha^\ell F\right\}.$$

We have then the inequality $k_\ell^0(\lambda) \leq k_\ell(\lambda)$. Further we see that $\frac{k_\ell(\lambda)}{N^\ell}$ is non-increasing in ℓ. To see this, consider the space

$$\tilde{\mathcal{F}}_{E^{\langle\ell+1\rangle}} = \left\{f : \text{function on } E^{\langle\ell+1\rangle}, \ f(\alpha\Psi_k \cdot) \in \mathcal{F}_{E^{\langle\ell\rangle}}, \quad 1 \leq k \leq N\right\}$$

and the self-adjoint operator $\tilde{\Delta}^{\langle\ell+1\rangle}$ on $L^2(E^{\langle\ell+1\rangle},\mu)$ associated with $\left(\tilde{\mathcal{F}}_{E^{\langle\ell+1\rangle}}, \mathcal{E}_{E^{\langle\ell+1\rangle}}\right)$. Then

$$\sharp\left\{\text{eigenvalues of } -\tilde{\Delta}^{\langle\ell+1\rangle} \leq \lambda\right\} = N\, k_\ell(\lambda).$$

But the left hand side is not smaller than $k_{\ell+1}(\lambda)$ because $\tilde{\mathcal{F}}_{E^{(\ell+1)}} \supset \mathcal{F}_{E^{(\ell+1)}}$. In the same way, we can see that $\frac{k_\ell^0(\lambda)}{N^\ell}$ is non-decreasing in ℓ. Therefore there exists a non-trivial right continuous non-decreasing function $\mathcal{N}(\lambda), \lambda \geq 0$, such that

$$\lim_{\ell\to\infty} \frac{k_\ell(\lambda)}{N^\ell} = \mathcal{N}(\lambda) \tag{4.1}$$

at each continuity point λ of $\mathcal{N}(\lambda)$. $\mathcal{N}(\lambda)$ is called the *integrated density of states* .

On the other hand, we have from Corollary 3.2 that

$$k_\ell(\lambda) = \rho\left(\left(\frac{N}{1-c}\right)^\ell \lambda\right). \tag{4.2}$$

Fix a $\lambda_0 > 0$ such that $\mathcal{N}(\lambda_0) > 0$. Let

$$d_s = 2\,\frac{\log N}{\log N - \log(1-c)}. \tag{4.3}$$

For $x \in \left[\left(\frac{N}{1-c}\right)^{\ell-1}\lambda_0, \left(\frac{N}{1-c}\right)^{\ell}\lambda_0\right]$, we have $N^{\ell-1}\lambda_0^{d_s/2} \leq x^{d_s/2} \leq N^\ell \lambda_0^{d_s/2}$ and , by (4.2)

$$\frac{k_{\ell-1}(\lambda_0)}{N^{\ell-1}}\frac{\lambda_0^{-d_s/2}}{N} \leq \frac{\rho(x)}{x^{d_s/2}} \leq \frac{k_\ell(\lambda_0)}{N^\ell} N \lambda_0^{-d_s/2}. \tag{4.4}$$

THEOREM 4.1.

(1)

$$0 < \varliminf_{x\to\infty} \frac{\rho(x)}{x^{d_s/2}} \leq \varlimsup_{x\to\infty} \frac{\rho(x)}{x^{d_s/2}} < \infty.$$

(2) $\mathcal{N}\left(\left(\frac{N}{1-c}\right)\lambda\right) = N\cdot\mathcal{N}(\lambda)$ *for any* λ.

$$0 < \varliminf_{x\to\infty} \frac{\mathcal{N}(x)}{x^{d_s/2}} \leq \varlimsup_{x\to\infty} \frac{\mathcal{N}(x)}{x^{d_s/2}} < \infty,$$

$$0 < \varliminf_{x\downarrow 0} \frac{\mathcal{N}(x)}{x^{d_s/2}} \leq \varlimsup_{x\downarrow 0} \frac{\mathcal{N}(x)}{x^{d_s/2}} < \infty.$$

PROOF: By letting $x \to \infty$ in (4.4), we get (1) with lower bound $\mathcal{N}(\lambda_0)\frac{\lambda_0^{-d_s/2}}{N}$ and upper bound $\mathcal{N}(\lambda_0)N\lambda_0^{-d_s/2}$. (4.1) and(4.2) lead us to the above scaling property of $\mathcal{N}$ which in turn implies the above asymptotics of $\mathcal{N}$ at 0 and ∞.

In the case of the Sirpinski gasket,ρ and $\mathcal{N}$ are so wild that no equality holds in the middle of each of three inequalities in the above theorem([5]).

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On a Spectral Analysis for the Sierpinski Gasket

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Abstract. A complete description of the eigenvalues of the Laplacian on the finite Sierpinski gasket is presented. We then demonstrate highly oscillatory behaviours of the distribution function of the eigenvalues, the integrated density of states (for the infinite gasket) and the spectrum of the Laplacian on the infinite gasket. The method has two ingredients: the decimation method in calculating eigenvalues due to Rammal and Toulouse and a simple description of the Dirichlet form associated with the Laplacian.

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Key words. Sierpinski gasket, Laplacian, Dirichlet form, eigenvalue, spectral dimension, integrated density of states, decimation method.

1. Introduction

In this paper, we are mainly concerned with eigenvalue problems for the Sierpinski gasket in $R^{N-1}(N \geqslant 3)$. See Section 2 for the notions about the Sierpinski gasket. Our purposes are threefold. In Section 5 we shall give a complete description of the distribution of eigenvalues of the Laplacian on the finite Sierpinski gasket. In particular we shall see that the number $\rho(\lambda)$ of the eigenvalues not greater than λ increases with rate $\lambda^{d_s/2}$ ($d_s = 2(\log N/\log(N+2))$ is the so called spectral dimension of the gasket) as $\lambda \to \infty$, but $\rho(\lambda)$ does not vary regularly and the ratio $\rho(\lambda)/\lambda^{d_s/2}$ is oscillating and non-convergent as $\lambda \to \infty$.

In Section 6, we shall determine the integrated density of states $\mathfrak{N}(\lambda)$ of the Laplacian on the infinite gasket, which is defined as the normalized limit of $\rho(\lambda)$ as the size of the finite gasket being expanded to infinity. It turns out that $\mathfrak{N}(\lambda)$ increases only with jumps and enjoys a certain scaling property. The set of jumps points of $\mathfrak{N}(\lambda)$ on a typical interval shall be seen after a rescaling to be accumulating to the Julia set of the quadratic transformation $\Phi(x) = x(N+2-x)$. Further we shall see that $\mathfrak{N}(\lambda)$ behaves exactly in the same manner as $\rho(\lambda)$ not only when $\lambda \to \infty$ but also when $\lambda \downarrow 0$. This legitimates the naming of the spectral dimension for d_s by Rammal and Toulouse [10].

We may well expect that the properties of the integrated density of states $\mathfrak{N}(\lambda)$ reflect the nature of the spectrum of the Laplacian defined as a self-adjoint operator on an L^2-space on the infinite gasket. Indeed we shall prove in Section 7 that each jump point of $\mathfrak{N}(\lambda)$ is a point spectre of the Laplacian (in the preceding sense) with infinite multiplicity.

The above mentioned high singularity of the integrated density of states $\mathfrak{N}(\lambda)$ was discovered by Rammal [11] who worked however only with the discrete Laplacians (certain difference operators) on the pre-gaskets approximating the true gasket, and in that case, $\mathfrak{N}(\lambda)$ was finitely supported. Recently, Kigami [5] introduced the Laplacian on the finite Sierpinski gasket as a limit of the discrete Laplacians on pre-gaskets and constructed explicitly the solutions of the associated Dirichlet problem and Poisson equation on the gasket. It is the Friedrichs extension of Kigami's Laplacian that we now employ in formulating the present spectral analysis.

We would like to capture the whole eigenvalues of the Laplacian on the finite gasket as appropriate limits of those eigenvalues of the approximating discrete Laplacians. Our method in doing so has two ingredients. The first is the decimation method due to Rammal and Toulouse [10] which relates the eigenvalues of successive discrete Laplacians by the map $\Phi(x)$. One of the present authors has used the method to decide the eigenvalues and their multiplicities of the discrete Laplacian completely [12]. The second ingredient is a direct description of the closed symmetric form (actually a Dirichlet form) corresponding to the Friedrichs extension of Kigami's Laplacian. The Dirichlet norm will be quite simply defined as an increasing limit of renormalized Dirichlet norms on the pre-gaskets.

The organization of the present paper is as follows. We collect in Section 2 those preliminary notions and relations from Kigami [5] which we shall use subsequently. In Section 3, we shall take from [12] the list of the eigenvalues of the discrete Laplacian on the pre-gasket together with the decimation diagram. Section 4 will be devoted to the abovementioned description of the Dirichlet form on the finite gasket and that on the infinite gasket as well. After these preparations, we proceed to the already explained spectral analysis on the gasket in Sections 5, 6 and 7.

As will be seen in Section 4, our Dirichlet space on the gasket is continuously embedded into the space of continuous functions, and accordingly, it has a reproducing kernel and further each point of the gasket has a positive capacity. These properties are never shared by the classical Sobolev space $H_0^1(D)$ on an Euclidean domain D except for the case that D is a one-dimensional interval.

Since our Dirichlet form is local and regular, it admits an associated diffusion process on the gasket – the so called Brownian motion. The Brownian motion on the infinite gasket has been already constructed by Kusuoka [7] and Barlow–Perkins [1] as a limit of random walks on pre-gaskets. Besides, [1] gave an upper and lower bound of the transition probability density, which in particular implies through a

Tauberian theorem that the ratio $\rho(\lambda)/\lambda^{d_s/2}$ remains as $\lambda \to \infty$ bounded above and bounded away from zero. Theorem 5.2 of the present paper confirms a conjecture in [1] about the oscillatory nature of this ratio.

The present approach to the Dirichlet form on the finite Sierpinski gasket is being extended by Kigami [6] to a more general class of fractals called *post critically finite self-similar sets* and by Kusuoka to the class of nested fractals introduced by Lindstrøm [9]. Kusuoka [8] also gives a different description of the Dirichlet form and identifies the dimensions of the associated spaces of martingales. Both classes include the gasket and the snowflake but exclude the Sierpinski carpet.

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2. Preliminaries

In this section, we collect those preliminary notions and relations from [5] which we shall use in the subsequent sections.

For $p_i \in R^{N-1}$, $i = 1, 2, \dots, N$, such that $\overrightarrow{p_1 p_j}$, $j = 2, \dots, N$, are independent, we let

$$|p_1 p_2 \cdots p_N| = \left\{ p : \overrightarrow{p_1 p} = \sum_{i=2}^{N} \lambda_i \overrightarrow{p_1 p_i} \text{ where } \lambda_i \geqslant 0 \text{ and } \sum_{i=2}^{N} \lambda_i \leqslant 1 \right\}$$

and call it an $(N-1)$ dimensional simplex. For $M = |p_1 p_2 \cdots p_N|$, the set of vertices, the set of $N(N-1)/2$ midpoints of edges and the set of new N simplices are defined respectively by

$$\begin{aligned} V(M) &= \{p_1, p_2, \dots, p_N\}, \\ Son(M) &= \{\tfrac{1}{2}(p_i p_j) : i > j\}, \\ Dau(M) &= \{|\tfrac{1}{2}(p_1 p_i)\tfrac{1}{2}(p_2 p_i) \cdots \tfrac{1}{2}(p_N p_i)| : i = 1, 2, \dots, N\}. \end{aligned}$$

Denote by K_0 a simplex with the length of each edge equal to 1. For $m \geqslant 0$, the set F_m of simplices is defined inductively by

$$\begin{aligned} F_0 &= \{K_0\}, \\ F_n &= \cup_{M \in F_{n-1}} Dau(M) \quad \text{for } n \geqslant 1. \end{aligned}$$

We then let

$$\begin{aligned} V_m &= \cup_{M \in F_m} V(M), \\ V_* &= \cup_{m \geqslant 0} V_m, \\ K &= \bar{V}_*. \end{aligned}$$

K is the *(finite Sierpinsiki) gasket*. We call V_m the (m-th step) pre-gasket.

Some more notations. For $p \in V_m$, the set of m-neighbours is defined by

$$V_{m,p} = \bigcup_{\substack{M \ni p \\ M \in F_m}} V(M) - \{p\}.$$

V_0 is called the boundary of K and denotes by ∂K. We let

$$V_m^o = V_m - V_0,$$
$$V_*^o = V_* - V_0,$$
$$K^o = K - \partial K.$$

For $p \in V_*$, $i(p)$ denotes the first step when p appears: $i(p) = \min\{n : p \in V_m\}$. For $p \in V_*^o$, there exists a unique $M_p \in F_{i(p)-1}$ with $p \in Son(M_p)$. M_p is said to be the *mother simplex* of p and $Son(M_p) - \{p\}$ the *brothers* of p. The brothers of $p \in V_*^o$ are the unions of

$$B_p = Son(M_p) \cap V_{i(p),p},$$
$$C_p = Son(M_p) - (\{p\} \cup B_p).$$

It can be easily seen that

$$\left\{\begin{aligned} &\#(V_m) = \frac{N(N^m+1)}{2}, \\ &\#(V_{m,p}) = \begin{cases} 2(N-1) & \text{for } p \in V_m^o, \\ N-1 & \text{for } p \in \partial K, \end{cases} \\ &\#(B_p) = 2N-4, \quad \#(C_p) = \frac{(N-2)(N-3)}{2} \quad \text{for } p \in V_*^o. \end{aligned}\right\} \tag{2.1}$$

For a real function f on V_m,

$$H_{m,p}f = \sum_{q \in V_{m,p}} f(q) - 2(N-1)f(p) \quad \text{for } p \in V_m^o, \tag{2.2}$$

is called the m-harmonic difference of f at p.

Denote by $C(K)$ the totality of real continuous functions on K. $f \in C(K)$ is called m-harmonic if $H_{n,p}f = 0$ for any $n > m$ and $p \in V_* - V_m$. It is known [5; Theorem 3.3] that, given $\rho : V_m \to R$, there exists a unique m-harmonic function f with $f|_{V_m} = \rho$. For $p \in V_m$, we denote by ψ_p^m the m-harmonic function corresponding to $\rho(q) = \delta_{qp}$, $q \in V_m$. Then, for any $f : V_* \to R$, the function $P_m f$ defined by

$$P_m f = \sum_{p \in V_m} f(p)\psi_p^m \tag{2.3}$$

is a unique m-harmonic function such that $P_m f|_{V_m} = f|_{V_m}$.

We let $\psi_p = \psi_p^{i(p)}$, $H_p^* = H_{i(p),p}$ for $p \in V_*$. The m-harmonic function $P_m f$ of (2.3) admits a more useful expansion in terms of ψ_p's ([5; Lemma 4.3]);

$$P_m f = \sum_{p \in V_m} \alpha_p(f)\psi_p \left(= \sum_{k=0}^{m} \sum_{i(p)=k} \alpha_p(f)\psi_p^k \right), \tag{2.4}$$

where the coefficients are given by

$$\alpha_p(f) = \begin{cases} f(p) & \text{if } p \in \partial K, \\ a_N H_p^* f + b_N \sum_{q \in B_p} H_q^* f + c_N \sum_{q \in C_p} H_q^* f & \text{if } p \in V_*^o, \end{cases} \tag{2.5}$$

where $a_N = -(N+6)/\{2N(N+2)\}$, $b_N = -3/\{2N(N+2)\}$, $c_N = -1/\{N(N+2)\}$. If $f \in C(K)$, then $\Sigma_{p \in V_m} \alpha_p(f)\psi_p$ converges to f as $m \to \infty$ uniformly on K in view of the maximum principle holding for m-harmonic functions.

The following relations taken from [5] will be used in Section 4. For a function f on V_m and for $p \in \partial K$, we set

$$D_{m,p} f = \sum_{q \in V_{m,p}} f(q) - (N-1) f(p). \tag{2.6}$$

LEMMA 2.1. (i) *For* $p \in V_m^o$, $q \in V_k$, $0 \leqslant k \leqslant m$,

$$\left(\frac{N+2}{N}\right)^{m-k} H_{m,p}\psi_q^k = \begin{cases} -2(N-1) & \textit{if } p = q, \\ 1 & \textit{if } p \in V_{k,q}, \\ 0 & \textit{otherwise}. \end{cases}$$

(ii) *For* $q \in V_*^o$, $q' \in V_*$,

$$H_q^* \psi_{q'} = \begin{cases} -2(N-1) & \textit{if } q' = q, \\ 1 & \textit{if } q' \in B_q, \\ 0 & \textit{otherwise}. \end{cases}$$

(iii) *For* $p, q \in \partial K$

$$\left(\frac{N+2}{N}\right)^{m} D_{m,p}\psi_q = \begin{cases} -(N-1) & \textit{if } p = q \\ 1 & \textit{if } p \neq q. \end{cases}$$

In accordance with [5], let us introduce the discrete Laplacian Δ_m on the pre-gasket V_m by

$$\Delta_m f(p) = (N+2)^m H_{m,p} f \quad \text{if} \quad p \in V_m^o, \tag{2.7}$$

for $f: V_m \to R$.

Kigami's Laplacian Δ is then defined by

$$\mathscr{D} = \{f \in C(K)\text{: there exists a function}$$
$$g \in C(K) \text{ and } \lim_{m\to\infty} \Delta_m f(p) = g(p) \text{ uniformly in } p \in V_*^o\} \tag{2.8}$$
$$\Delta f = g \quad f \in \mathscr{D} \tag{2.9}$$

where g is the function appearing in (2.8).

We shall also consider the spaces $C_0(K) = \{f \in C(K): f = 0 \text{ on } \partial K\}$ and $\mathscr{D}_0 = \mathscr{D} \cap C_0(K)$.

Finally we take a lemma from [5] concerning an estimate of a Green operator – an inverse operator of Δ.

LEMMA 2.2. *For $v \in C(K)^*$, we let*

$$f_m(v) = -\frac{N}{2} \sum_{p \in V_m^o} \langle \omega_p, v \rangle \left(\frac{N}{N+2}\right)^{i(p)} \psi_p$$

where

$$\omega_p = a_N \psi_p + b_N \sum_{q \in B_p} \psi_q + c_N \sum_{q \in C_p} \psi_q.$$

Then

$$\|f_m(v) - f_{m-1}(v)\|_\infty \leqslant \left(\frac{N}{N+2}\right)^m \|v\|_{C(K)^*}. \tag{2.10}$$

In particular, f_m converges uniformly to a function in $C_0(K)$ which we denote by $G^0 v$. $G^0 v$ then satisfies

$$\Delta_m(G^0 v)(p) = -\tfrac{1}{2} N^{m+1} \langle \psi_p^m, v \rangle, \quad p \in V_m^o. \tag{2.11}$$

COROLLARY. (i) $\|G^0 v\|_\infty \leqslant (N/2)\|v\|_{C(K)^*}$, $v \in C(K)^*$,
(ii) *If $v_l, v \in C(K)^*$ and v_l converges weakly v as $l \to \infty$, then $G^0 v_l$ converges to $G^0 v$ uniformly.*

Proof. (i) is immediate from (2.10). (ii) is obtained by writing

$$G^0 v_l - G^0 v = f_k(v_l - v) + \sum_{m=k+1}^{\infty} \{f_m(v_l - v) - f_{m-1}(v_l - v)\}$$

and applying (2.10) to the second term of the right hand side. ■

In what follows, we denote by μ the normalized Hausdorff measure on the gasket K. μ is by definition the weak limit as $m \to \infty$ of discrete measures

$$\mu_m = \frac{2}{N^{m+1}} \sum_{p \in V_m} \delta_p. \tag{2.12}$$

3. Eigenvalues of $-H_m^0$

In this section, we deal with the pre-Sierpinski gasket $V_m \subset R^{N-1}, m = 0, 1, 2, \ldots, N \geqslant 3$. Denote by $\ell(V_m)$ the totality of real functions on V_m and let

$$\ell_0(V_m) = \{f \in \ell(V_m): f = 0 \text{ on } V_0\}.$$

We are concerned with the eigenvalue problem for the linear operator H_m^0 on $\ell_0(V_m)$ defined for $f \in \ell_0(V_m)$ by

$$H_m^0 f(p) = \begin{cases} H_{m,p} f & \text{if } p \in V_m^o, \\ 0 & \text{if } p \in V_0. \end{cases} \tag{3.1}$$

We take from Shima [12] the following results about the eigenvalues and eigenfunctions of $-H_m^0$, $m = 1, 2, \ldots$.

PROPOSITION 3.1. (i) *$-H_1^0$ possesses the eigenvalues* 2, $N + 2$ *and* $2N$ *with multiplicities* 1, $N - 1$ *and* $(N/2)(N - 3)$ *respectively.*
(ii) *$2N$ is an eigenvalue of $-H_m^0$ with multiplicity* $(N/2)(N^m - 2N^{m-1} - 1), m = 1, 2, \ldots$.

The decimation method consists in the next two propositions. Denote by $\mathscr{A}_m$ the collection of the eigenvalues of $-H_m^0$. Further define the quadratic function Φ by

$$\Phi(x) = x(N + 2 - x), \quad x \in R.$$

PROPOSITION 3.2. (i) $\mathscr{A}_{m+1} \backslash \{N + 2, 2N\} = \Phi^{-1}(\mathscr{A}_m) \backslash \{2\}, m = 1, 2, \ldots$.
(ii) *If $\lambda_m \in \mathscr{A}_m$, $\lambda_{m+1} \in \mathscr{A}_{m+1}$ and $\lambda_m = \Phi(\lambda_{m+1})$, then λ_m and λ_{m+1} have a common multiplicity.*

PROPOSITION 3.3. *Let λ_m and λ_{m+1} be as in Proposition* 3.2(ii). *The restriction to V_m of any eigenfunction ($\in \ell_0(V_m)$) belonging to λ_{m+1} is an eigenfunction belonging to λ_m. Conversely any eigenfunction ($\in \ell_0(V_m)$) belonging to λ_m can be uniquely extended to an eigenfunction belonging to λ_{m+1}.*

We denote by $\phi_\pm(x)$ as the real valued inverse functions of $\Phi(x)$. That is

$$\phi_\pm(x) = \frac{N+2}{2}\left(1 \pm \sqrt{1 - \frac{4}{(N+2)^2}x}\right), \quad x \in \left(-\infty, \frac{(N+2)^2}{4}\right].$$

$(\Phi)^{(n)}$ denotes the n-th iteration of Φ, $n \geqslant 1$. We let $(\Phi)^{(0)}(x) = x$. If $w = (\Phi)^{(n)}(x)$, w is called a *successor* of x of order n with respect to Φ, and x is called a *predecessor* of w of order n with respect to Φ. Then $\mathscr{P}_w^{(n)}$ denotes the collection of predecessors of w of order n with respect to Φ.

Proposition 3.1 and Proposition 3.2 leads us to the following diagram and Theorem.

OPERATOR	*EIGENVALUE*
$-H_1^0$	2
	↙ ↘
$-H_2^0$	$\phi_-(2)$ $\qquad$ $\phi_+(2)$
	↙ ↘ $\qquad$ ↙ ↘
$-H_3^0$	$\phi_-(\phi_-(2))$ $\quad$ $\phi_+(\phi_-(2))$ $\quad$ $\phi_-(\phi_+(2))$ $\quad$ $\phi_+(\phi_+(2))$
⋮	⋮
$-H_m^0$	$\mathscr{P}_2^{(m-1)}$

OPERATOR	*EIGENVALUE*
$-H_1^0$	$N+2$
	↙ ↘
$-H_2^0$	$\phi_-(N+2)$ $\quad$ $\phi_+(N+2)$ $\quad$ $N+2$
	↙ ↘ $\quad$ ↙ ↘ $\quad$ ↙ ↘
$-H_3^0$	$\phi_-(\phi_-(N+2))$ $\phi_+(\phi_-(N+2))$ $\phi_-(\phi_+(N+2))$ $\phi_+(\phi_+(N+2))$ $\phi_-(N+2)$ $\phi_+(N+2)$ $N+2$
⋮	⋮
$-H_m^0$	$\cup_{1\leqslant j\leqslant m}\mathscr{P}_{N+2}^{(m-j)}$

OPERATOR	*EIGENVALUE*
$-H_1^0$	$2N$
	↓
$-H_2^0$	$\phi_+(2N)=N$ $\quad$ $2N$
	↙ $\quad$ ↘ $\quad$ ↓
$-H_3^0$	$\phi_-(N)$ $\quad$ $\phi_+(N)$ $\quad$ N $\quad$ $2N$
⋮	⋮
$-H_m^0$	$\cup_{1\leqslant j\leqslant m-1}\mathscr{P}_N^{(m-j-1)}\cup\{2N\}$

THEOREM 3.1. *Let* $\alpha_m = (N/2)(N^m - 2N^{m-1} - 1)$ *and* $\beta_m = (N/2)(N^{m-1} - 2N^{m-2} + 1)$. *Then the eigenvalues of* $-H_m^0$ $(m \geqslant 1)$ *are sorted as follows*:

eigenvalue	...	*multiplicity*	
$2N$	...	α_m	
predecessors of 2 of order $m-1$	...	1	
predecessors of N *of order* $m-j-1$	...	α_j	$1\leqslant j\leqslant m-1$
predecessors of $N+2$ *of order* $m-j$	...	β_j	$1\leqslant j\leqslant m.$

Later we need the following lemmas taken from [12].

LEMMA 3.1. *Let* $V_0 = \{p_1, p_2, \ldots, p_N\}$ *and*

$$g_k(p) = \begin{cases} 1 & \text{if } p = \tfrac{1}{2}(p_1 p_j),\quad j = 2, 3, \overset{k}{\vee}\!\ldots, N, \\ -1 & \text{if } p = \tfrac{1}{2}(p_j p_k),\quad j = 2, 3, \overset{k}{\vee}\!\ldots, N, \\ 0 & \text{otherwise.} \end{cases}$$

Then $g_2, \ldots, g_N$ *are linearly independent eigenfunctions of* $-H_1^0$ *belonging to the eigenvalue* $N + 2$.

LEMMA 3.2. *Let* $f \in \ell_0(V_m)$. *If* $(-H_m^0 f)|_{Son(M)} = 2Nf|_{Son(M)}$ *for each* $M \in F_{m-1}$, *then automatically*

$$-H_m^0 f = 2Nf.$$

LEMMA 3.3. *Let* $f \in \ell_0(V_{m+1})$ *be the eigenfunction of the eigenvalue* λ_{m+1} $(\neq 2N, N + 2)$ *of* $-H_{m+1}^0$ *and if* $|\lambda_{m+1} - (N + 2)| \geqslant |\lambda_{m+1} - 2|$, *then we have*

$$\sum_{i(p)=m+1} f^2(p) \leqslant \frac{4(N-1)}{(\lambda_{m+1} - 2)^2} \sum_{p \in V_m} f^2(p).$$

4. Dirichlet Form and the Laplacian

PROPOSITION 4.1. *For* $f: V_* \to R$, *we let*

$$\mathscr{E}^{(m)}(f, f) = \frac{2}{N}\left(\frac{N+2}{N}\right)^m \sum_{M \in F_m} \sum_{p,q \in V(M)} (f(p) - f(q))^2. \tag{4.1}$$

Then

$$\mathscr{E}^{(m)}(f, f) = \frac{2}{N} \sum_{k=0}^{m} \left(\frac{N+2}{N}\right)^k (f, f)_k, \tag{4.2}$$

where

$$\left\{\begin{aligned} (f, f)_k &= \sum_{i(p)=k} \left\{2\alpha_p(f)^2 + \frac{1}{2} \sum_{q \in B_p} (\alpha_p(f) - \alpha_q(f))^2\right\}, \quad \text{for } k \geqslant 1, \\ (f, f)_0 &= \frac{1}{2} \sum_{p,q \in \partial K} (f(p) - f(q))^2. \end{aligned}\right\} \tag{4.3}$$

In particular, $\mathscr{E}^{(m)}(f, f)$ *is non-decreasing in m and, if f is (the restriction to* V_**) of an m-harmonic function, then* $\mathscr{E}^{(m)}(f, f) = \mathscr{E}^{(m+1)}(f, f) = \ldots.$

Proof. For $f, g: V_* \to R$, *we have*

$$\sum_{M \in F_m} \sum_{p,q \in V(M)} (f(p) - f(q))(g(p) - g(q)) = \sum_{p \in V_m} \sum_{q \in V_{m,p}} (f(p) - f(q))g(p),$$

and consequently

$$\mathscr{E}^{(m)}(f,g) = -\frac{2}{N}\left(\frac{N+2}{N}\right)^m \left\{\sum_{p \in V_m^\circ} H_{m,p} fg(p) + \sum_{p \in \partial K} D_{m,p} fg(p)\right\}, \tag{4.4}$$

where $\mathscr{E}^{(m)}(f,g)$ is defined by (4.1) with $(f(p) - f(q))^2$ being replaced by $(f(p) - f(q))(g(p) - g(q))$.

(4.4) particularly implies that $\mathscr{E}^{(m)}(h,g)$ vanishes whenever h is 0-harmonic and g vanishes on ∂K. For $f: V_* \to \mathrm{R}$, we have $P_m f = f$ on V_m and hence

$$\begin{aligned}\mathscr{E}^{(m)}(f,f) &= \mathscr{E}^{(m)}(P_m f, P_m f)\\ &= \mathscr{E}^{(m)}((P_m - P_0)f, (P_m - P_0)f) + \mathscr{E}^{(m)}(P_0 f, P_0 f).\end{aligned}$$

Let us compute the last two terms. From $(P_m - P_0)f = \Sigma_{q \in V_m^\circ} \alpha_q(f)\psi_q$, we get

$$\begin{aligned}&\mathscr{E}^{(m)}((P_m - P_0)f, (P_m - P_0)f)\\ &\quad = -\frac{2}{N}\left(\frac{N+2}{N}\right)^m \sum_{p \in V_m^\circ} \left(\sum_{q \in V_m^\circ} \alpha_q(f) H_{m,p}\psi_q\right)\left(\sum_{q' \in V_m^\circ} \alpha_{q'}(f)\psi_{q'}(p)\right)\\ &\quad = -\frac{2}{N}\left(\frac{N+2}{N}\right)^m \sum_{k=1}^m \sum_{q' \in V_m^\circ} \sum_{i(q)=k} \alpha_q(f)\alpha_{q'}(f)\left(\sum_{p \in V_m^\circ} H_{m,p}\psi_q \psi_{q'}(p)\right).\end{aligned}$$

The expression in the last braces equals

$$\begin{aligned}\sum_{p \in V_m^\circ} H_{m,p}\psi_q^k \psi_{q'}(p) &= \left(\frac{N}{N+2}\right)^{m-k} H_{k,q}\psi_{q'}\\ &= \left(\frac{N}{N+2}\right)^{m-k} H_q^* \psi_{q'},\end{aligned}$$

by Lemma 2.1(i). Using Lemma 2.1(ii), we arrive at

$$\begin{aligned}&\mathscr{E}^{(m)}((P_m - P_0)f, (P_m - P_0)f)\\ &\quad = -\frac{2}{N}\sum_{k=1}^m \left(\frac{N+2}{N}\right)^k \sum_{i(q)=k} \left\{-2(N-1)\alpha_q(f)^2 + \sum_{q' \in B_q} \alpha_q(f)\alpha_{q'}(f)\right\},\end{aligned}$$

which is equal to $2/N\, \Sigma_{k=1}^m (N + 2/N)^k (f,f)_k$ on account of (2.1). On the other hand, we get from $P_0 f = \Sigma_{q \in \partial K} f(q)\psi_q$ and Lemma 2.1(iii) that

$$\begin{aligned}\mathscr{E}^{(m)}(P_0 f, P_0 f) &= -\frac{2}{N}\left(\frac{N+2}{N}\right)^m \sum_{p\in\partial K}\left(\sum_{q\in\partial K} f(q) D_{m,p}\psi_q\right) f(p)\\ &= \frac{2}{N}\sum_{p\in\partial K}\sum_{q\in\partial K} f(p)(f(p)-f(q))\\ &= \frac{2}{N}(f,f)_0.\end{aligned}$$

■

REMARK. The fact that $\mathscr{E}^{(m)}(f,f)$ is non-decreasing in m can be checked directly at least when $N = 3$.

We now let for $f: V_* \to R$

$$\begin{cases}\mathscr{E}(f,f) = \lim_{m\to\infty} \mathscr{E}^{(m)}(f,f),\\ \mathscr{F} \quad = \{f: V_* \to \mathrm{R},\ \mathscr{E}(f,f) < \infty\}.\end{cases} \tag{4.5}$$

PROPOSITION 4.2. *Any $f \in \mathscr{F}$ is uniquely extended to be an element of $C(K)$ and*

$$\|f - P_0 f\|_\infty \leqslant \frac{N\sqrt{N+2}}{2}\sqrt{\mathscr{E}(f - P_0 f, f - P_0 f)}.$$

Proof. By the preceding proof,

$$\mathscr{E}(f - P_0 f, f - P_0 f) = \mathscr{E}(f,f) - \frac{2}{N}(f,f)_0$$

and by virtue of (4.2)

$$\mathscr{E}(f - P_0 f, f - P_0 f) \geqslant \frac{4}{N}\left(\frac{N+2}{N}\right)^k \left(\max_{i(p)=k} |\alpha_p(f)|\right)^2$$

for each $k \geqslant 1$. Therefore, we have

$$\sqrt{\mathscr{E}(f - P_0 f, f - P_0 f)} \geqslant \frac{2}{N\sqrt{N+2}} \sum_{k=1}^{\infty} \max_{k=1} |\alpha_p(f)|.$$

Since $0 \leqslant \Sigma_{i(p)=k}\psi_p(x) \leqslant 1$ by maximum principle for the k-harmonic function, we see that the function

$$\tilde{f}(x) = \sum_{k=1}^{\infty} \sum_{i(p)=k} \alpha_p(f)\psi(x). \quad x \in K,$$

is uniformly convergent, $\tilde{f}|_{V_*} = f - P_0 f$ and, moreover,

$$\|\tilde{f}\|_\infty \leqslant \frac{N\sqrt{N+2}}{2}\sqrt{\mathscr{E}(f - P_0 f, f - P_0 f)}. \qquad \blacksquare$$

By the preceding proposition, we have the inclusion $\mathscr{F} \subset C(K) \subset L^2(K;\mu)$. Denote by $(\,,)_\mu$ the inner product of $L^2(K;\mu)$. We let $\mathscr{E}_\alpha(f,g) = \mathscr{E}(f,g) + \alpha(f,g)_\mu$ for $\alpha > 0$ and $f, g \in \mathscr{F}$.

PROPOSITION 4.3.

$$\|f\|_\infty \leqslant C\sqrt{\mathscr{E}_1(f,f)}, \quad f \in \mathscr{F}, \tag{4.6}$$

for some constant $C > 0$.

Proof. We first show the inequality (4.6) for 0-harmonic functions. Suppose f is 0-harmonic, then $f = \Sigma_{q\in\partial K} f(q)\psi_q$. By setting $\Pi_{pq} = \int_K \psi_p(x)\psi_q(x)\mu(\mathrm{d}x)\,(>0)$, $p, q \in \partial K$, we have from Proposition 4.1

$$\begin{aligned}\mathscr{E}_1(f,f) &= \frac{1}{N}\sum_{p,q\in\partial K}(f(p) - f(q))^2 + 2\sum_{p,q\in\partial K} f(p)f(q)\Pi_{pq}\\ &\geqslant C_1 \sum_{q\in\partial K} f(q)^2 \geqslant C_1\left(\max_{q\in\partial K}|f(q)|\right)^2\\ &\geqslant C_1\|f\|_\infty^2\end{aligned}$$

for some constant $C_1 > 0$, proving (4.6) for the 0-harmonic function f.

For a general $f \in \mathscr{F}$, we then get from Proposition 4.2,

$$\begin{aligned}\|f\|_\infty &\leqslant \|f - P_0 f\|_\infty + \|P_0 f\|_\infty\\ &\leqslant \frac{N\sqrt{N+2}}{2}\sqrt{\mathscr{E}(f,f)} + \frac{1}{\sqrt{C_1}}\sqrt{\mathscr{E}(f,f) + (P_0 f, P_0 f)_\mu},\end{aligned}$$

which leads us to (4.6) because

$$\begin{aligned}(P_0 f, P_0 f)_\mu &\leqslant 2(f,f)_\mu + 2(f - P_0 f, f - P_0 f)_\mu\\ &\leqslant 2(f,f)_\mu + 2\|f - P_0 f\|_\infty^2\\ &\leqslant 2(f,f)_\mu + \frac{N^2(N+2)}{2}\mathscr{E}(f,f). \qquad \blacksquare\end{aligned}$$

By the same reasoning as in the above proof, we also have

PROPOSITION 4.4.

$$\|f\|_\infty \leqslant \frac{N\sqrt{N+2}+2\sqrt{N}}{2}\sqrt{\mathscr{E}(f,f)} \tag{4.7}$$

for any $f \in \mathscr{F}$ vanishing at some point of ∂K.

THEOREM 4.1. (i) *$(\mathscr{E},\mathscr{F})$ is a local regular Dirichlet form on $L^2(K;\mu)$. $\mathscr{E}_\alpha$ on $\mathscr{F}$ admits a reproducing kernel $g_\alpha(x,y)$, $x,y \in K$, for each $\alpha > 0$:*

$$g_\alpha(\cdot,y) \in \mathscr{F},\ \mathscr{E}_\alpha(g_\alpha(\cdot,y),v) = v(y),\ v \in \mathscr{F}.$$

(ii) *Let $\mathscr{F}_0 = \{f \in \mathscr{F} : f = 0 \text{ on } \partial K\}$. Then $(\mathscr{E},\mathscr{F}_0)$ is a local regular Dirichlet form on $L^2(K^\circ;\mu)$. $\mathscr{E}$ on $\mathscr{F}_0$ admits a reproducing kernel $g^0(x,y)$, $x,y \in K^\circ$:*

$$g^0(\cdot,y) \in \mathscr{F}_0,\ \mathscr{E}(g^0(\cdot,y),v) = v(y),\ v \in \mathscr{F}_0.$$

Proof. We first show that the space $\mathscr{F}$ is complete with metric $\mathscr{E}_1$. Consider an $\mathscr{E}_1$-Cauchy sequence $f_n \in \mathscr{F}$. Then f_n converges to an $f \in C(K)$ uniformly on K by virtue of Proposition 4.3. Since $(f_n,f_n)_k \to (f,f)_k$, $n \to \infty$, for each k, we have from Proposition 4.1

$$\frac{2}{N}\sum_{k=1}^{\infty}\left(\frac{N+2}{N}\right)^k (f_m-f,f_m-f)_k = \frac{2}{N}\sum_{k=1}^{\infty}\left(\frac{N+2}{N}\right)^k \lim_{n\to\infty}(f_m-f_n,f_m-f_n)_k$$
$$\leqslant \lim_{n\to\infty}\mathscr{E}(f_m-f_n,f_m-f_n),$$

proving that $f \in \mathscr{F}$ and $f_n \to f$ in $\mathscr{E}_1$.

Next we see from Proposition 4.1 that, for any $f \in C(K)$, $\mathscr{E}(P_m f, P_m f) = \mathscr{E}^{(m)}(f,f)$ and hence $P_m f \in \mathscr{F}$. Since $P_m f$ converges to f uniformly, we conclude that $\mathscr{F}$ is dense in $C(K)$. Further the expression (4.1) of the approximating form $\mathscr{E}^{(m)}$ implies that every unit contraction operates on $\mathscr{E}$: if $u \in \mathscr{F}$, then $v = (0 \vee u) \wedge 1$ and $\mathscr{E}(v,v) \leqslant \mathscr{E}(u,u)$. It also implies that $\mathscr{E}$ is local: if $supp[u] \cap supp[v] = \varnothing$ for $u,v \in \mathscr{F}$, then $\mathscr{E}(u,v) = 0$.

We have proven the first half of (i). The first half of (ii) is now clear. The second assertions of (i) and (ii) follow from Proposition 4.3 and Proposition 4.2 respectively. ■

At the end of Section 2, we have introduced the discrete Laplacian Δ_m. Kigami's Laplacian Δ and Kigami's Green operator G^0. We now study their relationship to $\mathscr{E}^{(m)}$, $\mathscr{E}$ and $\mathscr{F}_0$.

LEMMA 4.1. (i) *For functions* f, g *on* V_m *vanishing on* ∂K,

$$\mathscr{E}^{(m)}(f,g) = -\int_{V_m^o} (\Delta_m f)(x) g(x) \mu_m(\mathrm{d}x)$$

where μ_m *is defined by (2.12).*
(ii) $\mathscr{D}_0 \subset \mathscr{F}_0$ *and*

$$\mathscr{E}(f,g) = -\int_K \Delta f g \,\mathrm{d}\mu, \quad f \in \mathscr{D}_0, \quad g \in \mathscr{F}_0,$$

where $\mathscr{D}_0$ *is defined in the paragraph after (2.9).*
(iii) *For any* $f \in C(K)$, $G^0 f \in \mathscr{D}_0$ *and* $-\Delta(G^0 f) = f$, *where* $G^0 f$ *denotes* $G^0 \nu$ *for the measure* $\nu = f\mu$.
(iv) *For any* $\nu \in C(K)^*$, $G^0 \nu \in \mathscr{F}_0$ *and*

$$\mathscr{E}(G^0 \nu, g) = \int_K g \,\mathrm{d}\nu, \quad g \in \mathscr{F}_0.$$

Proof. (i) is immediate from (4.4). (ii) follows from (i) in view of the definition of Δ and μ. (iii) is a consequence of (2.11). To see (iv), we note that $\nu \in C(K)^*$ is a weak limit of $f_m \mu$ for some $f_m \in C(K)$, $m = 1, 2, \ldots$. By virtue of Corollary to Lemma 2.2, $G^0 f_m$ converges uniformly to $G^0 \nu$ as $m \to \infty$. By (ii) and (iii)

$$\mathscr{E}(G^0 f_m - G^0 f_n, G^0 f_m - G^0 f_n) = (f_m - f_n, G^0 f_m - G^0 f_n)_\mu$$

which implies that $\{G^0 f_m\}$ is $\mathscr{E}$-Cauchy and hence $G^0 \nu \in \mathscr{F}_0$. The final equation is now evident. ■

THEOREM 4.2. (i) *The self-adjoint operator on* $L^2(K;\mu)$ *associated with the Dirichlet form* $(\mathscr{F}_0, \mathscr{E})$ *is the Friedrichs extension of Kigami's Laplacian* Δ *with domain* $\mathscr{D}_0$.
(ii) *The reproducing kernel* $g^0(x,y)$, $x, y \in K$, *in Theorem 4.1(ii) (defined to be zero when* x *or* $y \in \partial K$) *is identical with* $G^0 \delta_y(x)$. *In particular it is symmetric and continuous in* $x, y \in K$ *and further strictly positive for* $x, y \in K^o$.
(iii) *For* $\nu \in C(K)^*$,

$$G^0 \nu(x) = \int_K g^0(x,y) \nu(\mathrm{d}y), \quad x \in K.$$

Proof. (i) On account of Lemma 4.1(ii), it suffices to show that $\mathscr{D}_0$ is a core of $(\mathscr{E}, \mathscr{F}_0)$, namely $\mathscr{D}_0$ is $\mathscr{E}$-dense in $\mathscr{F}_0$. But this follows from Lemma 4.1(iii) because $\{G^0 f : f \in C(K)\}$ $(\subset \mathscr{D}_0)$ is seen to be $\mathscr{E}$-dense in $\mathscr{F}_0$. (ii) The first half follows from Lemma 4.1(iv). Symmetry is clear and the strict positivity is immediate from the definition of $G^0 \delta_y$, $y \in K^o$. To see the joint continuity, suppose $y_n \in K$ converges to

$y \in K$. Then $G^0\delta_{y_n}$ converges to $G^0\delta_y$ uniformly on K by virtue of Corollary to Lemma 2.2.

(iii)
$$\begin{aligned} G^0\nu(y_1) &= \mathscr{E}(g^0(\cdot, y_1), G^0\nu) \\ &= \int_K g^0(y_2, y_1)\nu(dy_2) \\ &= \int_K g^0(y_1, y_2)\nu(dy_2) \end{aligned}$$

by Lemma 4.1(iv). ∎

Hereafter Δ will stand for the self-adjoint operator on $L^2(K;\mu)$ associated with $(\mathscr{E}, \mathscr{F}_0)$: $f \in \mathscr{D}(\Delta)$, $\Delta f = g \in L^2(K;\mu)$ iff

$$f \in \mathscr{F}_0, \quad \mathscr{E}(f,h) = -(g,h)_\mu, \quad h \in \mathscr{F}_0. \tag{4.8}$$

We call Δ the Laplacian on the gasket K with Dirichlet boundary condition.

Theorem 4.2 means that the inverse G^0 of $-\Delta$ has a symmetric continuous kernel $g^0(x,y)$, $x, y \in K$, which is strictly positive definite. Therefore, the spectrum of $-\Delta$ consists only of positive eigenvalues with finite multiplicity accumulating only at $+\infty$. By Mercer's theorem, $g^0(x,y)$ can be expanded as a uniformly convergent series

$$g^0(x,y) = \sum_{i=1}^{\infty} \frac{\phi_i(x)\phi_i(y)}{\kappa_i}, \quad x, y \in K, \tag{4.9}$$

where $0 < \kappa_1 \leqslant \kappa_2 \leqslant \dots$ are eigenvalues of $-\Delta$ and $\phi_1, \phi_2, \dots$ are the associated eigenfunctions constituting an ONS of $L^2(K;\mu)$. κ_i's will be decided completely in the next section.

For the sake of Sections 5 and 6, we now expand the size of the gasket K and introduce the related notions. Place the finite Sierpinski gasket K in R^{N-1} so that one of the vertices of K coincides with the origin O. We then let

$$\begin{aligned} &K^{(n)} = 2^n K \quad n = 0, 1, 2, \dots \\ &K^{(\infty)} = \bigcup_{n=0}^{\infty} K^{(n)}. \end{aligned} \tag{4.10}$$

We call $K^{(\infty)}$ the infinite Sierpinski gasket. Accordingly, the m-step pre-gaskets approximating $K^{(n)}$ and $K^{(\infty)}$ are defined respectively by

$$\begin{aligned} &V_m^{(n)} = 2^n V_{m+n} \quad n = 0, 1, 2, \dots \\ &V_m^{(\infty)} = \bigcup_{n=0}^{\infty} V_m^{(n)}, \end{aligned} \tag{4.11}$$

for $m = 0, 1, 2, \dots$. Finally we let

$$V_*^{(n)} = \bigcup_{m=0}^{\infty} V_m^{(n)}$$
$$V_*^{(\infty)} = \bigcup_{n=0}^{\infty} V_*^{(n)}. \tag{4.12}$$

In particular $V_m = V_m^{(0)}$, $K = K^{(0)}$.

Denote by μ the Hausdorff measure on $K^{(\infty)}$ normalized to be $\mu(K) = 1$. μ is the vague limit of the uniform measure μ_m on $V_m^{(\infty)}$ which is defined by (2.12) with V_m being replaced by $V_m^{(\infty)}$.

For each $n = 0, 1, 2, \ldots, \infty$, and for $f: V_*^{(n)} \to \mathrm{R}$, define $\mathscr{E}_{K^{(n)}}^{(m)}(f, f)$ by (4.1) with V_m being replaced by $V_m^{(n)}$ and let $\mathscr{E}_{K^{(n)}}(f, f)$ be the (non-decreasing) limit of $\mathscr{E}_{K^{(n)}}^{(m)}(f, f)$ as $m \to \infty$. Denote by $C(K^{(n)})$ the space of continuous functions on $K^{(n)}$. If $\mathscr{E}_{K^{(n)}}(f, f)$ is finite, then, as before, f can be and will be identified with an element of $C(K^{(n)})$. We also use the notation $C_0(K^{(n)}) = \{f \in C(K^{(n)}): f = 0 \text{ on } \partial K^{(n)} \ (= V_0^{(n)})\}$.

Let us set

$$\begin{cases} \mathscr{F}^{(n)} = \{f: V_*^{(n)} \to R,\ \mathscr{E}_{K^{(n)}}(f, f) < \infty\} & n = 0, 1, 2, \ldots \\ \mathscr{F}_0^{(n)} = \{f \in \mathscr{F}^{(n)}: f = 0 \text{ on } \partial K^{(n)}\} & n = 0, 1, 2, \ldots \end{cases} \tag{4.13}$$

$$\begin{cases} \mathscr{F}^{(\infty)} = \{f: V_*^{(\infty)} \to R,\ \mathscr{E}_{K^{(\infty)}}: f(f, f) < \infty\} \cap \mathrm{L}^2(K^{(\infty)}; \mu) \\ \mathscr{F}_0^{(\infty)} = \{f \in \mathscr{F}^{(\infty)}: f(O) = 0\}. \end{cases} \tag{4.14}$$

For $n = 0, 1, 2, \ldots$ and for a function f on $K^{(n)}$, we put

$$(\sigma_n f)(x) = f(2^n x), \quad x \in K. \tag{4.15}$$

σ_n is then a surjection from $\mathscr{F}^{(n)}$ (resp. $\mathscr{F}_0^{(n)}$) to $\mathscr{F}$ (resp. $\mathscr{F}_0$).

LEMMA 4.2. (*Scaling properties*)

(i)
$$\int_{K^{(n)}} f(x)\mu(dx) = N^n \int_K (\sigma_n f)(x)\mu(\mathrm{d}x)$$

(ii)
$$\mathrm{E}_{K^{(n)}}(f, f) = \left(\frac{N}{N+2}\right)^n \mathscr{E}(\sigma_n f, \sigma_n f).$$

Proof. (i) For continuous f, the left hand side is the limit as $m \to \infty$ of

$$\frac{2}{N^{m+1}} \sum_{p \in V_m^{(n)}} f(p) = N^n \frac{2}{N^{m+n+1}} \sum_{p \in V_{m+n}} (\sigma_n f)(p).$$

(ii) The left hand side is the limit as $m \to \infty$ of

$$\mathscr{E}_{K^{(n)}}^{(m)}(f, f) = \left(\frac{N}{N+2}\right)^n \mathscr{E}^{(m+n)}(\sigma_n f, \sigma_n f).$$ ■

We denote by $\Delta^{(n)}$ the Laplacian on $K^{(n)}$ with Dirichlet boundary condition. In other words, $\Delta^{(n)}$ is a self-adjoint operator on $L^2(K^{(n)};\mu)$ characterized as follows: $f\in\mathscr{D}(\Delta^{(n)})$ and $\Delta^{(n)}f=g\in L^2(K^{(n)};\mu)$ iff

$$f\in\mathscr{F}_0^{(n)},\quad \mathscr{E}_{K^{(n)}}(f,h)=-(g,h)_{L^2(K^{(n)};\mu)},\quad h\in\mathscr{F}_0^{(n)}. \tag{4.16}$$

PROPOSITION 4.5. *κ is an eigenvalue of $-\Delta$ $(=-\Delta^{(0)})$ iff $[1/(N+2)^n]\kappa$ is an eigenvalue of $-\Delta^{(n)}$.*

Proof. If $\tilde{\kappa}$ is an eigenvalue of $-\Delta^{(n)}$ with an eigenfunction f, then $f\in\mathscr{F}_0^{(n)}$ and $\mathscr{E}_{K^{(n)}}(f,h)=\tilde{\kappa}(f,h)_{L^2(K^{(n)};\mu)}$, $h\in\mathscr{F}_0^{(n)}$, by (4.16). Lemma 4.2 leads us to $\mathscr{E}(\sigma_n f,\sigma_n h)=(N+2)^n\tilde{\kappa}(\sigma_n f,\sigma_n h)_{L^2(K;\mu)}$, which means that $(N+2)^n\tilde{\kappa}$ is an eigenvalue of $-\Delta$ with eigenfunction $\sigma_n f$. ∎

Finally we are concerned with the spaces $\mathscr{F}^{(\infty)}$ and $\mathscr{F}_0^{(\infty)}$ defined by (4.14). We set

$$C_0(K^{(\infty)})=\{f\in C(K^{(\infty)})\colon \text{the support of } f \text{ is compact}\}$$
$$C_{00}(K^{(\infty)})=\{f\in C_0(K^{(\infty)})\colon f(O)=0\}.$$

$C_{00}(K^{(\infty)})$ can be identified with $C_0(K_0^{(\infty)})$ where $K_0^{(\infty)}=K^{(\infty)}-\{O\}$.

LEMMA 4.3. *There exists $\phi_n\in\mathscr{F}^{(\infty)}\cap C_0(K^{(\infty)})$ such that $0\leqslant\phi_n\leqslant 1$, $\phi_n=1$ on $K^{(n-1)}$, $\phi_n=0$ on $K^{(\infty)}\backslash K^{(n)}$ and $\lim_{n\to\infty}\mathscr{E}_{K^{(\infty)}}(\phi_n,\phi_n)=0$.*

Proof. Let ψ be a 1-harmonic function on the unit gasket such that

$$\psi(p)=\begin{cases}1 & p=O \text{ or } |p|=2^{-1}\\ 0 & p\in V_1^{(0)}\quad |p|\neq 2^{-1}.\end{cases}$$

ψ is then non-negative and identically 1 on the set $\{x\in K\colon |x|\leqslant\frac{1}{2}\}$. $\mathscr{E}(\psi,\psi)$ is finite by Proposition 4.1. We set $\phi_n=\sigma_n^{-1}\psi$ and regard ϕ_n as an element of $C_0(K^{(\infty)})$ by setting $\phi_n(x)=0$, $x\in K^{(\infty)}\backslash K^{(n)}$. On account of Lemma 4.2,

$$\begin{aligned}\mathscr{E}_{K^{(\infty)}}(\phi_n,\phi_n)&=\mathscr{E}_{K^{(n)}}(\phi_n,\phi_n)\\ &=\left(\frac{N}{N+2}\right)^n\mathscr{E}(\psi,\psi)\to 0,\quad (n\to\infty).\end{aligned}$$ ∎

The discrete Laplacian $\Delta_m^{(\infty)}$ on the infinite pregasket $V_m^{(\infty)}$ is defined by

$$\Delta_m^{(\infty)}f(p)=(N+2)^m\left\{\sum_{q\in V_{m,p}^{(\infty)}}f(q)-2(N-1)f(p)\right\},\quad p\in V_m^{(\infty)}\backslash\{O\},$$

for $f\colon V_m^{(\infty)}\to R$, where $V_{m,p}^{(\infty)}=\cup_{n=0}^{\infty}2^nV_{m+n,p/2^n}$. In accordance with (2.8), we introduce

$$\mathscr{D}_0^{(n)} = \{f \in C_0(K^{(n)}): \lim_{m\to\infty} \Delta_m^{(\infty)} f(p) = g(p) \text{ uniformly in } p \in V_*^{(n)}\backslash\{O\} \text{ for some } g \in C(K^{(n)})\}.$$

Each function $f \in \mathscr{D}_0^{(n)}$ is extended to $K^{(\infty)}$ by setting $f(p) = 0$, $p \in K^{(\infty)}\backslash K^{(n)}$. We then let

$$\mathscr{D}_0^{(\infty)} = \bigcup_{n=1}^{\infty} \mathscr{D}_0^{(n)},$$

$$\Delta^{(\infty)} f(p) = \begin{cases} g(p) & p \in K^{(n)}, \\ 0 & p \in K^{(\infty)}\backslash K^{(n)}, \end{cases}$$

for $f \in \mathscr{D}_0^{(n)}$ and g appearing in the above braces.

THEOREM 4.3. (i) *$(\mathscr{E}_{K^{(\infty)}}, \mathscr{F}^{(\infty)})$ is a local regular Dirichlet form on $L^2(K^{(\infty)}; \mu)$. $\mathscr{E}_{K^{(\infty)},\alpha}$ on $\mathscr{F}^{(\infty)}$ admits a reproducing kernel $g_\alpha^{(\infty)}(x, y)$, $x, y \in K^{(\infty)}$, for each $\alpha > 0$.*
(ii) *$(\mathscr{E}_{K^{(\infty)}}, \mathscr{F}_0^{(\infty)})$ is a local regular Dirichlet form on $L^2(K_0^{(\infty)}; \mu)$. $\mathscr{E}_{K^{(\infty)}}$ on $\mathscr{F}_0^{(\infty)}$ admits a reproducing kernel $g^{(\infty),0}(x, y)$, $x, y \in K_0^{(\infty)}$.*
(iii) *The self-adjoint operator on $L^2(K_0^{(\infty)}; \mu)$ associated with $(\mathscr{E}_{K^{(\infty)}}, \mathscr{F}_0^{(\infty)})$ is the Friedrichs extension of $\Delta^{(\infty)}$ with domain $\mathscr{D}_0^{(\infty)}$.*

Proof. (i) Since $\mathscr{E}_{K^{(\infty)}}(f, f)$ is an increasing limit of $\mathscr{E}_{K^{(n)}}(f, f)$ as $n \to \infty$, the first assertion except for the regularity follows from Theorem 4.1. To prove the regularity, take any bounded $f \in \mathscr{F}^{(\infty)}$ and let $f_n = f \cdot \phi_n$ for ϕ_n of Lemma 4.3. Denoting $\mathscr{E}_{K^{(l)}}(g, g) - \mathscr{E}_{K^{(n)}}(g, g)$ by $\mathscr{E}_{K^{(l)}\backslash K^{(n)}}(g, g)$ for $n \leqslant l \leqslant \infty$, we then have

$$\begin{aligned} \mathscr{E}_{K^{(\infty)}}(f - f_n, f - f_n) &= \mathscr{E}_{K^{(\infty)}\backslash K^{(n-1)}}(f(1 - \phi_n), f(1 - \phi_n)) \\ &\leqslant 2\|f\|_\infty^2 \mathscr{E}_{K^{(\infty)}}(\phi_n, \phi_n) + 2\mathscr{E}_{K^{(\infty)}\backslash K^{(n-1)}}(f, f), \end{aligned}$$

which converges to zero as $n \to \infty$. Since $f_n \in \mathscr{F}^{(\infty)} \cap C_0(K^{(\infty)})$, we have seen that $\mathscr{F}^{(\infty)} \cap C_0(K^{(\infty)})$ is $\mathscr{E}_{K^{(\infty)},1}$-dense in $\mathscr{F}^{(\infty)}$. We can also see in the same way as in the proof of Theorem 4.1 that $\mathscr{F}^{(\infty)} \cap C_0(K^{(\infty)})$ is uniformly dense in $C_0(K^{(\infty)})$.

It follows from Proposition 4.3 and Lemma 4.2 that, for any $f \in \mathscr{F}^{(\infty)}$,

$$\sup_{x \in K^{(n)}} |f(x)| \leqslant C_n \sqrt{\mathscr{E}_{K^{(\infty)},1}(f, f)}, \quad n = 1, 2, \ldots \tag{4.17}$$

where C_n is a positive constant depending only on N and n. The second assertion of (i) is immediate from this.

(ii) The first assertion follows from that of (i). It follows from Proposition 4.4 and Lemma 4.2 that, for any $f \in \mathscr{F}_0^{(\infty)}$,

$$\sup_{x \in K^{(n)}} |f(x)| \leqslant \frac{N\sqrt{N+2} + 2\sqrt{N}}{2} \left(\frac{N+2}{N}\right)^{n/2} \sqrt{\mathscr{E}_{K^{(\infty)}}(f, f)}, \quad n = 1, 2, \ldots. \tag{4.18}$$

The second assertion of (ii) is a consequence of this.

(iii) As in Lemma 4.1(ii), we have that $\mathscr{D}_0^{(\infty)} \subset \mathscr{F}_0^{(\infty)}$ and, for $f \in \mathscr{D}_0^{(\infty)}$, $g \in \mathscr{F}_0^{(\infty)} \cap C_0(K_0^{(\infty)})$,

$$\mathscr{E}_{K^{(\infty)}}(f, g) = -\int_{K^{(\infty)}} \Delta^{(\infty)} f \cdot g \, d\mu.$$

This extends to $g \in \mathscr{F}_0^{(\infty)}$ because of the regularity of $(\mathscr{E}_{K^{(\infty)}}, \mathscr{F}_0^{(\infty)})$. Therefore it suffices to prove that $\mathscr{D}_0^{(\infty)}$ is dense in $\mathscr{F}_0^{(\infty)}$.

By virtue of (4.18), there exists for each $f \in C_0(K^{(\infty)})$ a function $G^{(\infty),0} f \in \mathscr{F}_0^{(\infty)}$ such that

$$\mathscr{E}_{K^{(\infty)}}(G^{(\infty),0} f, v) = (f, v)_\mu, \quad v \in \mathscr{F}_0^{(\infty)}.$$

$\{G^{(\infty),0} f : f \in C_0(K^{(\infty)})\}$ is dense in $\mathscr{F}_0^{(\infty)}$. On the other hand, if we let for G^0 of Lemma 4.1(iii)

$$G^{(n),0} f = (N+2)^n \sigma_n^{-1} G^0 \sigma_n f \quad f \in C(K^{(n)}),$$

then $G^{(n),0} f \in \mathscr{D}_0^{(n)}$ and

$$\mathscr{E}_{K^{(\infty)}}(G^{(n),0} f, v) = (f, v)_\mu, \quad v \in \mathscr{F}_0^{(n)},$$

by Lemma 4.1(iii). From the above two equations, we can easily see that $G^{(n),0} f$ is $\mathscr{E}_{K^{(\infty)}}$-convergent to $G^{(\infty),0} f$ as $n \to \infty$. Since $G^{(n),0} f \in \mathscr{D}_0^{(\infty)}$, we conclude that $\mathscr{D}_0^{(\infty)}$ is dense in $\mathscr{F}_0^{(\infty)}$. ■

The next theorem will not be used later but it states specific potential theoretic and probabilistic features of the Dirichlet form $(\mathscr{E}_{K^{(\infty)}}, \mathscr{F}^{(\infty)})$.

THEOREM 4.4. (i) *Denote by Cap the 1-capacity on $K^{(\infty)}$ associated with $(\mathscr{E}_{K^{(\infty)}}, \mathscr{F}^{(\infty)})$. Then each point set has a positive capacity given by*

$$Cap(\{x\}) = \frac{1}{g_1^{(\infty)}(x, x)}, \quad x \in K^{(\infty)}. \tag{4.19}$$

(ii) *The μ-symmetric diffusion process on $K^{(\infty)}$ associated with $(\mathscr{E}_{K^{(\infty)}}, \mathscr{F}^{(\infty)})$ is point recurrent: starting at any point, the sample paths hit any one point set almost surely.*

Proof. (i) Since $g_1^{(\infty)}(x, x) > 0$, the function $h = g_1^{(\infty)}(x, \cdot)/g_1^{(\infty)}(x, x)$ is an element of $\mathscr{F}^{(\infty)}$ such that $h = 1$ on $\{x\}$, $\mathscr{E}_1(h, v) = v(x)/g_1^{(\infty)}(x, x) \geq 0$ for any $v \in \mathscr{F}^{(\infty)}$ with $v \geq 0$ on $\{x\}$. This characterizes h to be the 1-equilibrium potential for $\{x\}$ and $Cap(\{x\}) = \mathscr{E}_1(h, h) = 1/g_1^{(\infty)}(x, x)$.

(ii) $(\mathscr{E}_{K^{(\infty)}}, \mathscr{F}^{(\infty)})$ is irreducible because otherwise $\mathscr{F}^{(\infty)}$ should contain a discontinuous function. Lemma 4.3 then implies the recurrence of the Dirichlet form $(\mathscr{E}_{K^{(\infty)}}, \mathscr{F}^{(\infty)})$, which combined with (i) leads us to the point recurrence of the associated diffusion (cf. [4]). ■

5. Eigenvalues of $-\Delta$

In Section 4, we have defined by (4.8) the Laplacian Δ on the finite Sierpinski gasket K with Dirichlet boundary condition. As was pointed out there, the spectrum of $-\Delta$ consists only of positive eigenvalues with finite multiplicity accumulating only at ∞. In this section, we determined those eigenvalues as renormalized limits of eigenvalues of $-H_m^0$ specified in Section 3.

First we introduce some notations.

$$\mathfrak{S} = \{-,+\}^{\mathbf{N}}, \quad \mathfrak{S}_n = \{-,+\}^n, \quad n = 1,2,\ldots.$$

For $s = (\varepsilon_1, \varepsilon_2, \ldots) \in \mathfrak{S}$, we let $s_n = (\varepsilon_1, \ldots, \varepsilon_n) \in \mathfrak{S}_n$, $n = 1,2,\ldots$. We further let for $(\varepsilon_1, \ldots, \varepsilon_n) \in \mathfrak{S}_n$

$$\phi_{(\varepsilon_1,\ldots,\varepsilon_n)}(x) = \phi_{\varepsilon_n} \circ \phi_{\varepsilon_{n-1}} \circ \cdots \circ \phi_{\varepsilon_1}(x)$$

where $\phi_\pm$ is the inverse of $\Phi(x) = x(N+2-x)$:

$$\phi_\pm(x) = \frac{N+2}{2}\left(1 \pm \sqrt{1 - \frac{4}{(N+2)^2}x}\right), \quad x \in \left(0, \frac{(N+2)^2}{4}\right].$$

According to the decimation diagram of Section 3, all eigenvalues of $-H_m^0$, $m = 1,2,\ldots$, are obtained as follows: Take $i_0 \in \mathbf{N}$ and $s \in \mathfrak{S}$. Let λ_{i_0} be one of $\{2, N+2, 2N\}$ when $i_0 = 1$ and one of $\{N+2, 2N\}$ when $i_0 > 1$. Finally let $\lambda_{i_0+n} = \phi_{s_n}(\lambda_{i_0})$, $n = 0,1,2,\ldots$, where we define ϕ_{s_0} to be the identity ($\phi_{s_0}(x) = x$) by convention. In case that $\lambda_{i_0} = 2N$ however, we allow only those $s \in \mathfrak{S}$ with $s_1 = \{+\}$. For the series $\{\lambda_m\}_{m=i_0,i_0+1,\ldots}$ so obtained, λ_m is an eigenvalue of $-H_m^0$ and $\Phi(\lambda_{m+1}) = \lambda_m$. All eigenvalues of $-H_m^0$, $m = 1,2,\ldots$, arise in this way. Furthermore, according to Proposition 3.3, the above series $\{\lambda_m\}_{m=i_0,i_0+1,\ldots}$ admits a series of eigenfunctions: for any choice of the eigenfunction f_{i_0}, of $-H_{i_0}^0$ belonging to λ_{i_0}, there exist eigenfunctions f_m of $-H_m^0$ belonging to λ_m ($m = i_0, i_0+1, \ldots$) such that $f_m|_{V_l} = f_l$, $i_0 \leq l < m$.

In view of Theorem 4.2, the Laplacian Δ on the gasket is in a sense a limit of the discrete Laplacian $\Delta_m = (N+2)^m H_m^0$. We first study the convergence of the series $\{(N+2)^m \lambda_m\}$ of eigenvalues of $-\Delta_m$.

For any $s \in \mathfrak{S}$, we let $x_n = \phi_{s_n}(x)$, $\tilde{x}_n = (N+2)^n x_n$, $x \in (0, (N+2)^2/4]$, $n = 1,2,\ldots$. Since $\tilde{x}_n = \tilde{x}_{n+1}\{1 - (x_{n+1}/N+2)\}$, $\{\tilde{x}_n\}$ is increasing.

LEMMA 5.1. *Following conditions are equivalent:*

(i) $$\sum_{n=0}^{\infty} x_n < \infty$$

(ii) $$x_{n+1} = \phi_-(x_n) \textit{ from some } n \textit{ on.}$$

(iii) $\lim_{n\to\infty} \tilde{x}_n$ *is positive finite.*

Proof. Suppose (ii) holds, then $x_n \downarrow 0$ from some n on and, moreover, the convergence is geometrical because $x_{n+1}/x_n = 1/(N+2-x_{n+1})$. Hence (i) follows. Further

$$\frac{\tilde{x}_{n+1}}{\tilde{x}_n} = 1 + \frac{x_{n+1}}{N+2-x_{n+1}}$$

and

$$\sum_{n=0}^{\infty} \frac{x_{n+1}}{N+2-x_{n+1}} < \infty$$

imply the convergence of $\Pi_{n=1}^{\infty} \tilde{x}_{n+1}/\tilde{x}_n$ the validity of (iii). Conversely suppose (ii) does not hold, then x_n is not convergent to 0, and consequently neither (i) nor (ii) is valid. ∎

PROPOSITION 5.1. *Let* $\{\lambda_m\}_{m=i_0,i_0+1,\ldots}$ *and* $\{f_m\}_{m=i_0,i_0+1,\ldots}$ *be a series of eigenvalues of* $\{-H_m^0\}$ *and an associated series of eigenfunctions specified in the beginning of this section. If* $\kappa_m = (N+2)^m\lambda_m$ *converges as* $m\to\infty$ *to a finite limit* κ, *then* κ *is an eigenvalue of* $-\Delta$ *and* κ *admits an eigenfunction* $f\in\mathscr{F}_0$ *such that* $f|_{V_m} = f_m$, $m = i_0, i_0+1,\ldots$.

Proof. Let f be the function on V_* such that $f|_{V_m} = f_m$, $m = i_0, i_0+1,\ldots$. It holds then for any $g\in\mathscr{F}_0$

$$\begin{aligned}\mathscr{E}^{(m)}(f,g) &= -\frac{2}{N}\left(\frac{N+2}{N}\right)^m \sum_{p\in V_m^0} H_m^0(f|_{V_m})(p)g(p)\\ &= \kappa_m \int_K fg\,\mathrm{d}\mu_m.\end{aligned}$$

If we can show that $f\in\mathscr{F}_0$, then, by letting $m\to\infty$ in the above, we see

$$\mathscr{E}(f,g) = \kappa \int_K fg\,\mathrm{d}\mu,$$

which means that κ is an eigenvalue of $-\Delta$ and f is an eigenfunction of $-\Delta$ belonging to κ.

Let $a_m = \int_K f^2\,\mathrm{d}\mu_m$ and $b_m = (N-1/N)[\lambda_{m+1}(4-\lambda_{m+1})/(\lambda_{m+1}-2)^2]$, $m = i_0$, $i_0+1,\ldots$. Since κ_m is assumed to be convergent, we have $\Sigma|b_m| < \infty$ by Lemma 5.1. By Lemma 3.3, we can find m_0 ($\geqslant i_0$) such that for $m \geqslant m_0$

$$\begin{aligned}\frac{a_{m+1}}{a_m} - 1 &= \frac{\sum_{i(p)=m+1} f(p)^2 - (N-1)\sum_{p\in V_m} f(p)^2}{N\sum_{p\in V_m} f(p)^2}\\ &\leqslant b_m.\end{aligned}$$

a_m is bounded by the convergent product $a_{m_0}\Pi_{m=l_0}^{\infty}(1+b_m)$, so $\mathscr{E}^{(m)}(f,f)$ is bounded. Therefore we can conclude that $f\in\mathscr{F}_0$. ■

This proposition suggests the way to locate the eigenvalues of $-\Delta$. The following function plays an important role in doing so.

$$\psi(x) = \lim_{m\to\infty}(N+2)^{m+1}(\phi_-)^{(m)}(x), \quad -\infty < x \leqslant \frac{(N+2)^2}{4}. \tag{5.1}$$

Since ψ can be expressed as an infinite product as in the proof of Lemma 5.1 and $\phi_-(x)$ is strictly increasing continuous, we see that ψ is a continuous strictly increasing function. Obviously ψ satisfies

$$(N+2)\psi(\phi_-(x)) = \psi(x), \quad -\infty < x \leqslant \frac{(N+2)^2}{4}. \tag{5.2}$$

In Section 3, we have considered the set $\mathscr{P}_w^{(n)}$ of predecessors of $w\in R$ of order n with respect to $\Phi: x\in\mathscr{P}_w^{(n)}$ iff $(\Phi)^{(n)}(x) = w$. Denote by $v_w^{(n)}$ the uniform probability distribution of $\mathscr{P}_w^{(n)}: v_w^{(n)} = 2^{-n}\Sigma_{p\in\mathscr{P}_w^{(n)}}\delta_p$. By virtue of Theorem 3.1, the normalized counting measure of the eigenvalues of $-H_k^0$ (including the multiplicities) is then given by

$$\tilde{v}^{(k)} = \frac{1}{N_k}\left[2^{k-1}v_2^{(k-1)} + \sum_{1\leqslant j\leqslant k-1} 2^{k-j-1}\alpha_j v_N^{(k-j-1)} + \sum_{1\leqslant j\leqslant k} 2^{k-j}\beta_j v_{N+2}^{(k-j)} + \alpha_k\delta_{\{2N\}}\right], \tag{5.3}$$

where $N_k = (N/2)(N^k - 1)$. $\tilde{v}^{(k)}$ is a probability measure concentrated on $(0, 2N]$. Let $v^{(k)}$ be the image measure of $\tilde{v}^{(k)}$ by the map $\psi\circ\phi_+$:

$$v^{(k)} = \tilde{v}^{(k)}\cdot\phi_+^{-1}\circ\psi^{-1}. \tag{5.4}$$

$v^{(k)}$ is a probability measure concentrated on $\psi\circ\phi_+(0,2N] \subset [\psi(N), \psi(N+2)]$ (see Figure 1).

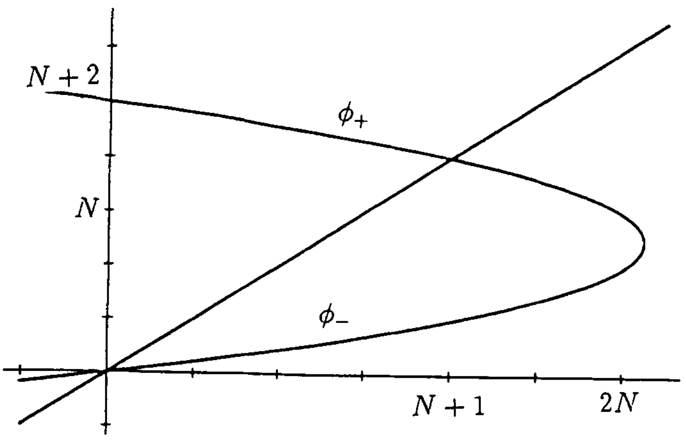

Fig. 1.

THEOREM 5.1. *Denote by S the spectrum of* $-\Delta$. *S consists only of the point spectra and*

$$S \ni \psi(2),\ \psi(N+2),$$
$$S \subset \{\psi(2)\} \cup \{\psi(N+2)\} \cup \bigcup_{k=1}^{\infty} I_k \quad \textit{(disjoint union)} \tag{5.5}$$

where $I_k = [(N+2)^k\psi(N), (N+2)^k\psi(N+2)]$.

$\psi(2)$ *and* $\psi(N+2)$ *have multiplicities* 1 *and* $N-1$ *respectively. The counting measure of* $I_k \cap S$ *taking the multiplicity of each eigenvalue into account is given by*

$$N_k \nu^{(k)}\left(\frac{\cdot}{(N+2)^k}\right) + \beta_{k+1}\delta_{\{(N+2)^k\psi(N+2)\}}(\cdot). \tag{5.6}$$

We call κ and f appearing in Proposition 5.1 a *raw eigenvalue* and a *raw eigenfunction* of $-\Delta$ respectively. By the *raw multiplicity* of the raw eigenvalue κ, we mean the multiplicity of the associated eigenvalue λ_{i_0} of $-H_{i_0}^0$. Since linearly independent eigenfunctions of $-H_{i_0}^0$ belonging to λ_{i_0} give rise to linearly independent f's, the raw multiplicity of κ is not greater than the true multiplicity of κ.

The proof of Theorem 5.1 is carried out by the following two lemmas. Denote by S' the collection of the raw eigenvalues of $-\Delta$, $S' \subset S$.

PROPOSITION 5.2. *Theorem 5.1 holds for the set* S' *of raw eigenvalues of* $-\Delta$ *equipped with raw multiplicities.*

PROPOSITION 5.3. *$S' = S$ and the raw multiplicity of each element of S' coincides with its (true) multiplicity.*

Proof of Proposition 5.2. Consider a series $\{\lambda_m\}_{m=i_0,i_0+1,\dots}$ of eigenvalues corresponding to a raw eigenvalue $\kappa \in S'$. Put

$$\tau = \min\{k \geqslant i_0 : \lambda_{k'+1} = \phi_-(\lambda_{k'}) \text{ for any } k' \geqslant \mathrm{k}\}.$$

Then τ is finite by Lemma 5.1 and

$$\kappa = (N+2)^{\tau-1}\psi(\lambda_\tau) \tag{5.7}$$

because

$$\kappa = \lim_{m\to\infty}(N+2)^m\lambda_m$$
$$= (N+2)^{\tau-1}\lim_{m\to\infty}(N+2)^{m-\tau+1}(\phi_-)^{(m-\tau)}(\lambda_\tau).$$

Denote by S'_k the set of raw eigenvalues with $\tau = k$ in (5.7), $k = 1, 2, \dots$. Then $S' = \cup_{k=1}^{\infty} S'_k$.

If $\tau = 1$, then λ_1 is either 2 or $N+2$ ($2N$ is excluded), and by (5.7)

$$S'_1 = \{\psi(2), \psi(N+2)\}.$$

Their raw multiplicities are 1 and $N-1$ respectively.

If $\tau = k > 1$, then either $\lambda_\tau = \phi_+(\lambda_{k-1})$ (when $k > i_0$) or $\lambda_\tau = \lambda_{i_0}$ (when $k = i_0$). In the former case, λ_{k-1} can take all possible eigenvalues of $-H^0_{k-1}$. In the latter case, λ_{i_0} can take only $N+2$. From (5.7), we see that $\kappa \in S'_k$ iff either $\kappa/[(N+2)^{k-1}] = \psi\circ\phi_+(\lambda_{k-1})$ for some λ_{k-1} or $\kappa = (N+2)^{k-1}\psi(N+2)$. Hence $S'_k \subset I_{k-1}$, $k = 2, 3, \dots$, and the counting measure of S'_k taking the individual raw multiplicity into account is given by (5.6) for $k-1$.

Since $\phi_-(N+2) < N$, we see from (5.2) that $\psi(N+2) < (N+2)\psi(N)$ and accordingly the right hand side of (5.5) is a disjoint union. ■

For the proof of Proposition 5.3, we need a lemma.

LEMMA 5.2. *Let $0 < \kappa'_1 < \kappa'_2 \leqslant \kappa'_3 \leqslant \cdots$ be the rearrangement of elements of S' each being repeated according to its raw multiplicity. Let $\{\lambda^{(i)}_m\}_{1\leqslant i\leqslant N_m}$ be the eigenvalues of $-H^0_m$ including multiplicities. Then*

$$\lim_{m\to\infty}\sum_{1\leqslant i\leqslant N_m}\frac{1}{\kappa^{(i)}_m} = \sum_{i=1}^{\infty}\frac{1}{\kappa'_i}$$

where $\kappa^{(i)}_m = (N+2)^m\lambda^{(i)}_m$.

Proof. Because (4.9) is valid and $S' \subset S$ and besides the raw multiplicity is not greater than the true one, we see that $\sum_{i=1}^{\infty} 1/\kappa_i'$ is finite. Further $\lim_{m\to\infty} \alpha_m/(N+2)^m = 0$. Therefore it suffices to show that

$$\sum_{\substack{1\leqslant i\leqslant N_m,\\ \lambda_m^{(i)}\neq 2N}} \frac{1}{\kappa_m^{(i)}} - \sum_{i=1}^{N^m} \frac{1}{\kappa_i'} \tag{5.8}$$

converges to zero as $m\to\infty$. By Proposition 5.2, $\{\kappa_1', \kappa_2', \ldots, \kappa_{N^m}'\}$ is an arrangement of elements of $\cup_{k=1}^{m} S_k'$ each being repeated according to its raw multiplicity. The first sum of (5.8) has also N^m terms, which can be rearranged according to the proof of Proposition 5.2 so that

$$\lim_{n\to\infty} (N+2)^{m+n}(\phi_-)^{(n)}(\lambda_m^{(i)}) = \kappa_i', \quad 1 \leqslant i \leqslant N^m.$$

After this arrangement, (5.8) equals

$$\begin{aligned}\sum_{k=2}^{m} \sum_{\kappa_i'\in I_{k-1}} &\left(\frac{1}{(N+2)^m\lambda_m^{(i)}} - \frac{1}{\kappa_i'}\right)\\ &+ \left\{\frac{1}{(N+2)^m(\phi_-)^{(m-1)}(2)} - \frac{1}{\psi(2)}\right\}\\ &+ \left\{\frac{N-1}{(N+2)^m(\phi_-)^{(m-1)}(N+2)} - \frac{N-1}{\psi(N+2)}\right\}.\end{aligned} \tag{5.9}$$

The last two terms converge to zero as $m\to\infty$.

If $\kappa_i'\in I_{k-1}$ $(k = 2,\ldots,m)$, then $\kappa_i' = (N+2)^{k-1}\psi(\lambda_k)$ for some $\lambda_k\in[N, N+2]$ and accordingly the corresponding $\lambda_m^{(i)}$ is of the form $\lambda_m^{(i)} = (\phi_-)^{(m-k)}(\lambda_k)$. Hence

$$\begin{aligned}0 &< \frac{1}{(N+2)^m\lambda_m^{(i)}} - \frac{1}{\kappa_i'}\\ &= \frac{1}{(N+2)^{k-1}}\left\{\frac{1}{(N+2)^{m-k+1}(\phi_-)^{(m-k)}(\lambda_k)} - \frac{1}{\psi(\lambda_k)}\right\}\end{aligned}$$

Since $1/\{(N+2)^{l+1}(\phi_-)^{(l)}(x)\}$ converges as $l\to\infty$ to $1/\psi(x)$ uniformly on $[N, N+2]$, the last expression is for any $\varepsilon > 0$ dominated by $\varepsilon/\{N(N-1)(N+2)^{k-1}\}$ whenever $m-k$ is greater than some number m_1. When $m-k \leqslant m_1$, the same expression is dominated by

$$\frac{1}{(N+2)^m M} \quad \text{for } M = \inf_{N\leqslant x\leqslant N+2} (\phi_-)^{(m_1)}(x).$$

The number of κ_i''s in I_{k-1} being $N_{k-1}+\beta_k$, we now see that the first sum of (5.9) is dominated by

$$\frac{\varepsilon}{N(N-1)}\sum_{k=2}^{m-m_1-1}\frac{N_{k-1}+\beta_k}{(N+2)^{k-1}}+\frac{1}{(N+2)^m M}\sum_{k=m-m_1}^{m}(N_{k-1}+\beta_k)$$
$$\leqslant\frac{\varepsilon}{2}+\left(\frac{N}{N+2}\right)^m\frac{1}{M},$$

which can be made smaller than ε for large enough m. The proof of Lemma 5.2 is complete. ∎

Proof of Proposition 5.3. By virtue of Lemma 4.1, we see that discrete Laplacian $\Delta_m = (N+2)^m H_m^0$ is associated with the symmetric form $\mathscr{E}^{(m)}(f,g)$ on the space $\ell_0(V_m)$ equipped with the inner product

$$(f,g)_{\mu_m} = \sum_{p\in V_m^o} f(p)g(p)\frac{2}{N^{m+1}}.$$

By Proposition 4.1, $\mathscr{E}^{(m)}(f,f) \geqslant c(f,f)_{\mu_m}$ for $f\in\ell_0(V_m)$ and for a constant $c>0$, and consequently the inverse operator $G_m^0 = (-\Delta_m)^{-1}$ exists: for $f\in\ell_0(V_m)$, $G_m^0 f\in\ell_0(V_m)$ and

$$\mathscr{E}^{(m)}(G_m^0 f, g) = (f,g)_{\mu_m}, \quad g\in\ell_0(V_m).$$

We denote $(N^{m+1}/2)G_m^0\delta p(q)$ by $g_{m,p}^0(q)$ or $g_m^0(p,q)$, $p,q\in V_m^o$.

Recall the reproducing kernel $g^0(x,y)$, $x,y\in K$, of the Dirichlet form $(\mathscr{E},\mathscr{F}_0)$ (Theorem 4.1(ii)). We then have

$$g^0(p,q) = g_m^0(p,q), \quad p,q\in V_m^o.$$

In fact, extending $g_{m,p}^0(\cdot)$ to an m-harmonic function on K, we see

$$g_{m,p}^0(q) = \mathscr{E}(g_q^0, g_{m,q}^0) = \mathscr{E}^{(m)}(g_q^0, g_{m,p}^0)$$
$$= \frac{N^{m+1}}{2}(\delta_p, g_q^0)_{\mu_m} = g_q^0(p) = g_p^0(q).$$

Therefore, we have the relation

$$\int_K g^0(p,p)\,\mathrm{d}\mu(p) = \lim_{m\to\infty}\int_K g^0(p,p)\,\mathrm{d}\mu_m(p)$$
$$= \lim_{m\to\infty}\int_{V_m^o} g_m^0(p,p)\,\mathrm{d}\mu_m(p).$$

On the other hand, (4.9) implies

$$\int_K g^0(p,p)\,\mathrm{d}\mu(p) = \sum_{i=1}^{\infty}\frac{1}{\kappa_i},$$

where $0<\kappa_1<\kappa_2\leqslant\cdots$ are eigenvalues of $-\Delta$ each being repeated according to its multiplicity. Similarly we get

$$\int_{V_m^o} g_m^0(p,p)\,\mathrm{d}\mu_m(p) = \sum_{1 \leqslant i \leqslant N_m} \frac{1}{\kappa_m^{(i)}}.$$

Lemma 5.2 now leads to $\Sigma_{i=1}^{\infty} 1/\kappa_i = \Sigma_{i=1}^{\infty} 1/\kappa_i'$, which shows $\kappa_i = \kappa_i'$, $i = 1, 2, \ldots$. ∎

Denote by $\rho(x)$ the number of eigenvalues of $-\Delta$ (taking the multiplicities into account) not exceeding x. By Theorem 5.1, $N^{k+1} \leqslant \rho(x) \leqslant N^{k+2}$ if $x \in ((N+2)^k \psi(N+2), (N+2)^{k+1}\psi(N+2)]$, $k = 0, 1, 2, \ldots$, and hence we get the bound

$$\psi(N+2)^{-d_s/2} \leqslant \frac{\rho(x)}{x^{d_s/2}} \leqslant N^2 \psi(N+2)^{-d_s/2}, \quad x > \psi(N+2). \tag{5.10}$$

But we can show that $\rho(x)$ varies highly irregularly:

THEOREM 5.2.

$$0 < \varliminf_{x\to\infty} \frac{\rho(x)}{x^{d_s/2}} < \varlimsup_{x\to\infty} \frac{\rho(x)}{x^{d_s/2}} < \infty.$$

$\rho(x)$ is not regularly varying.

Proof. By Theorem 5.1, we have

$$\rho((N+2)^k \psi(N+2)-) = N^k + \frac{N(N^k - 1)}{2}$$
$$\rho((N+2)^k \psi(N+2)) = N^{k+1}.$$

Consequently, for $x_k = (N+2)^k \psi(N+2)$,

$$\frac{\rho(x_k)}{x_k^{d_s/2}} = N\psi(N+2)^{-d_s/2}$$
$$\lim_{k\to\infty} \frac{\rho(x_k -)}{x_k^{d_s/2}} = \left(1 + \frac{N}{2}\right)\psi(N+2)^{-d_s/2},$$

from which follows the first assertion.

Suppose $\rho(x)$ varies regularly, then $\rho(x) = x^{\delta} L(x)$ for a finite δ and a slowly varying L. We get for any $0 < c < 1$

$$c^{\delta} = \lim_{k\to\infty} \frac{\rho(cx_k)}{\rho(x_k)}$$
$$\leqslant \lim_{k\to\infty} \frac{\rho(x_k-)}{\rho(x_k)} = \frac{1}{N} + \frac{1}{2}.$$

Letting $c \uparrow 1$, we arrive at a contradiction. ■

6. Integrated Density of States

In the preceding section, we have considered the normalized counting measure $\tilde{\nu}^{(k)}$ of the eigenvalues of $-H_k^0$ taking each individual multiplicity into account. See (5.3). The integrated density of states will be eventually described by the weak limit of $\tilde{\nu}^{(k)}$. Let

$$\tilde{\nu} = \frac{N-2}{N^2} \sum_{i=0}^{\infty} \left(\frac{2}{N}\right)^i \nu_N^{(i)} + \frac{N-2}{N^2} \sum_{i=0}^{\infty} \left(\frac{2}{N}\right)^i \nu_{N+2}^{(i)} + \frac{N-2}{N} \delta_{\{2N\}}. \tag{6.1}$$

$\tilde{\nu}$ is a probability measure on $(0, 2N]$.

LEMMA 6.1. *$\tilde{\nu}^{(k)}$ converges weakly to $\tilde{\nu}$ as $k \to \infty$.*

Proof. It is known by Brolin [2] that $\nu_w^{(n)}$ $(w = 2, N, N+2)$ converges as $n \to \infty$ to a certain probability measure ν^* on the Julia set $\mathcal{J}$ of the quadratic map Φ. $\mathcal{J}$ can be defined by

$$\mathcal{J} = \overline{\bigcup_{n=1}^{\infty} \{\alpha \in R : \Phi^{(n)}(\alpha) = \alpha,\ |(\Phi^{(n)})'(\alpha)| > 1\}} \tag{6.2}$$

and $\mathcal{J}$ is a closed perfect set contained in $[0, N+2]$.

Take a continuous function f on $[0, 2N]$ and put

$$a_w^{(n)} = \int_{[0,2N]} f(x)\nu_w^{(n)}(dx), \quad a^* = \int_{[0,2N]} f(x)\nu^*(dx).$$

From (5.3),

$$\int_{[0,2N]} f(x)\tilde{\nu}^{(k)}(dx) = \frac{2}{N(N^k-1)} [2^{k-1}a_2^{(k-1)} + \sum_{1\leqslant j\leqslant k-1} 2^{k-j-1}\alpha_j a_N^{(k-j-1)}$$
$$+ \sum_{1\leqslant j\leqslant k} 2^{k-j}\beta_j a_{N+2}^{(k-j)} + \alpha_k f(2N)]. \tag{6.3}$$

Since $a_w^{(n)} \to a^*$, $n \to \infty$, the first term of the right hand side converges as $k \to \infty$ to zero. The fourth term converges to $(N-2)/Nf(2N)$ as $k \to \infty$.

Take any $\varepsilon > 0$ and choose L such that $|a_N^{(n)} - a^*| < \varepsilon$ for $n \geqslant L$. Take any k with $k > L$. The second term of the right hand side of (6.3) is a sum of I_k and II_k, where

$$I_k = \frac{2^k}{N(N^k - 1)} \sum_{j=1}^{k-L} \frac{\alpha_j}{2^j} a_N^{(k-j-f)}$$

$$II_k = \frac{2^k}{N(N^k - 1)} \sum_{j=k-L+1}^{k-1} \frac{\alpha_j}{2^j} a_N^{(k-j-1)}.$$

Since

$$\lim_{k\to\infty} \frac{2^k}{N(N^k - 1)} \sum_{j=1}^{k-L} \frac{\alpha_j}{2^j} = \frac{1}{2}\left(\frac{2}{N}\right)^L$$

and

$$\lim_{k\to\infty} II_k = \lim_{k\to\infty} \frac{2^k}{N(N^k - 1)} \sum_{j=0}^{L-2} \frac{\alpha_{k-j-1}}{2^{k-j-1}} a_N^{(j)}$$
$$= \frac{N-2}{N^2} \sum_{j=0}^{L-2} \left(\frac{2}{N}\right)^j a_N^{(j)},$$

we have

$$\frac{1}{2}\left(\frac{2}{N}\right)^L (a^* - \varepsilon) + \frac{N-2}{N^2} \sum_{j=0}^{L-2} \left(\frac{2}{N}\right)^j a_N^{(j)}$$
$$\leqslant \varliminf_{k\to\infty} (I_k + II_k) \leqslant \varlimsup_{k\to\infty} (I_k + II_k)$$
$$\leqslant \frac{1}{2}\left(\frac{2}{N}\right)^L (a^* + \varepsilon) + \frac{N-2}{N^2} \sum_{j=0}^{L-2} \left(\frac{2}{N}\right)^j a_N^{(j)}.$$

By letting $L \to \infty$, we see that

$$\lim_{k\to\infty} (I_k + II_k) = \frac{N-2}{N^2} \sum_{i=0}^{\infty} \left(\frac{2}{N}\right)^i a_N^{(i)}.$$

In the same way, the third term of the right hand side of (6.3) converges as $k \to \infty$ to $(N - 2/N^2)\Sigma_{i=0}^{\infty}(2/N)^i a_{N+2}^{(i)}$. ■

Let

$$\nu = \tilde{\nu} \cdot \phi_+^{-1} \circ \psi^{-1}. \tag{6.4}$$

ν is a probability measure on $\psi \circ \phi_+(0, 2N] \subset [\psi(N), \psi(N+2)]$.

COROLLARY. *$\nu^{(k)}$ defined by (5.4) converges weakly to ν as $k \to \infty$.*

In Section 4, we have introduced the Laplacian $\Delta^{(n)}$ on the expanded gasket $K^{(n)}$ with Dirichlet boundary condition (see (4.16)). If we let

$$\mathfrak{N}^{(n)}(x) = \frac{\#\{\text{eigenvalues of } -\Delta^{(n)} \text{ not exceeding } x\}}{\mu(K^{(n)})}, \tag{6.5}$$

then we have from Proposition 4.5

$$\mathfrak{N}^{(n)}(x) = \frac{\rho((N+2)^n x)}{N^n} \tag{6.6}$$

where $\rho(x)$ is the number of eigenvalues of $-\Delta$ not exceeding x. Of course, we take the multiplicity of each eigenvalue into account in the above definitions. Denote by F_k and F the distribution functions of ν_k and ν respectively. Finally we let

$$\mathfrak{N}(x) = \begin{cases} 1 + (N/2)F(x) & x \in [\psi(N), \psi(N+2)), \\ N & x \in [\psi(N+2), (N+2)\psi(N)), \end{cases} \tag{6.7}$$

and, for $k = \pm 1, \pm 2, \ldots$,

$$\mathfrak{N}(x) = N^k \mathfrak{N}\left(\frac{x}{(N+2)^k}\right), \quad x \in [(N+2)^k \psi(N), (N+2)^{k+1}\psi(N)). \tag{6.8}$$

THEOREM 6.1.

$$\lim_{n\to\infty} \mathfrak{N}^{(n)}(x) = \mathfrak{N}(x) \tag{6.9}$$

at each point x where $\mathfrak{N}(x)$ is continuous and at $x = (N+2)^k\psi(N+2), k = 0, \pm 1, \pm 2, \ldots$.

Proof. If one can show (6.9) on $[\psi(N), (N+2)\psi(N))$, then (6.6) obviously converges to (6.8) on other intervals.

For $x \in [\psi(N), \psi(N+2))$, we have from Theorem 5.1

$$\rho((N+2)^n x) = \rho((N+2)^{n-1}\psi(N+2)) + N_n F_n(x)$$

and consequently

$$\mathfrak{N}^{(n)}(x) = 1 + \frac{N_n}{N^n} F_n(x),$$

which converges as $n \to \infty$ to $1 + (N/2)F(x)$ by Corollary. For $x \in [\psi(N+2), (N+2)\psi(N))$,

$$\rho((N+2)^n x) = \rho((N+2)^n \psi(N+2)) = N^{n+1}$$

and

$$\lim_{n \to \infty} \mathfrak{N}^{(n)}(x) = N. \qquad \blacksquare$$

By this theorem, we may well call $\mathfrak{N}(x)$ the *integrated density of states* of $-\Delta$ on the infinite Sierpinski gasket. $\mathfrak{N}(x)$ behaves similarly to $\rho(x)$ when $x \to \infty$. But $\mathfrak{N}(x)$ exhibits a similar behaviour also when $x \downarrow 0$. By (6.7) and (6.8), $N^k \leqslant \mathfrak{N}(x) \leqslant N^{k+1}$ for $x \in [(N+2)^k \psi(N), (N+2)^{k+1}\psi(N))$, $k = 0, \pm 1, \pm 2, \ldots$, and hence

$$\frac{1}{N}\psi(N)^{-d_s/2} \leqslant \frac{\mathfrak{N}(x)}{x^{d_s/2}} \leqslant N\psi(N)^{-d_s/2}, \quad 0 < x < \infty.$$

Further we have for $x_k = (N+2)^k \psi(N+2)$, $k = 0, \pm 1, \pm 2, \ldots$,

$$\begin{cases} \dfrac{\mathfrak{N}(x_k -)}{x_k^{d_s/2}} = \left(1 + \dfrac{N}{2}\right)\psi(N+2)^{-d_s/2} \\ \dfrac{\mathfrak{N}(x_k)}{x_k^{d_s/2}} = N\psi(N+2)^{-d_s/2}. \end{cases} \tag{6.10}$$

$\mathfrak{N}(x)$ increases with pure jumps and its jump points are describable by those of the measure $\tilde{\nu}$ of (6.1). If we let

$$\tilde{\nu}_1 = \frac{N-2}{N^2} \sum_{i=0}^{\infty} \left(\frac{2}{N}\right)^i \nu_{N+2}^{(i)}$$

$$\tilde{\nu}_2 = \frac{N-2}{N^2} \sum_{i=0}^{\infty} \left(\frac{2}{N}\right)^i \nu_N^{(i)} + \frac{N-2}{N}\delta_{\{2N\}},$$

$$\mathscr{P}_w = \bigcup_{n=0}^{\infty} \mathscr{P}_w^{(n)},$$

then $\tilde{\nu} = \tilde{\nu}_1 + \tilde{\nu}_2$ and the set of jump points of $\tilde{\nu}_1$ (resp. $\tilde{\nu}_2$) is $\mathscr{P}_{N+2}$ (resp. $\mathscr{P}_N \cup \{2N\}$). $\mathscr{P}_{N+2}$ is a subset of $\mathscr{J}$ the Julia set of Φ, while $\mathscr{P}_N \cup \{2N\} \subset \mathscr{J}^c$. In fact, $\mathscr{J} = \cap_{n=0}^{\infty} E_n$, where

$$
\begin{aligned}
E_0 &= [0, N+2],\\
E_1 &= \Phi^{-1}(E_0) = [0, \phi_-(N+2)] \cup [\phi_+(N+2), N+2],\\
&\vdots\\
E_n &= \Phi^{-1}(E_{n-1}),\\
&\vdots
\end{aligned}
$$

Hence $\mathscr{P}_{N+2} = \cup_k \partial E_k \setminus \{0\} \subset \mathscr{J}$. It is known however that the set of the accumulating points of $\mathscr{P}_N$ coincides with $\mathscr{J}$. Rammal [11] has discovered a similar phenomena for the jump points of the integrated density of states of the discrete Laplacian on the infinite pregasket.

7. Eigenvalues of $-\Delta^{(\infty)}$

We now work with the infinite Sierpinski gasket $K^{(\infty)}$. Denote by $\Delta^{(\infty)}$ the self-adjoint operator on $L^2(K_0^{(\infty)}; \mu)$ associated with the Dirichlet form $(\mathscr{E}_{K^{(\infty)}}, \mathscr{F}_0^{(\infty)})$ studied at the end of Section 4. In the preceding section, we have seen that the integrated density of states $\mathfrak{N}(x)$ increases only with jumps. The totality of those jump points is the sum of D_1 and D_2 where

$$
\begin{cases}
D_1 = \bigcup_{k=-\infty}^{\infty} ((N+2)^k \psi \circ \phi_+(\mathscr{P}_N) \cup \{(N+2)^k \psi(N)\})\\
D_2 = \bigcup_{k=-\infty}^{\infty} ((N+2)^k \psi \circ \phi_+(\mathscr{P}_{N+2}) \cup \{(N+2)^k \psi(N+2)\}).
\end{cases}
\tag{7.1}
$$

We aim at proving the following:

THEOREM 7.1. *Each point of $D_1 \cup D_2$ is an eigenvalue of $-\Delta^{(\infty)}$ with infinite multiplicity.*

Proof for D_1. First take $\kappa = \psi(x)$ with $x \in \phi_+(\mathscr{P}_N)$ or $x = N$. x is then a predecessor of $2N$ with order, say, k_0. Let $s \in \mathfrak{S}$ be the associated sequence and $\{\lambda_m\}_{m=1,2,\ldots}$ be the corresponding series of eigenvalues of $-H_m^0$: $s = (+, \varepsilon_2, \ldots, \varepsilon_{k_0-1}, +, -, -, \cdots)$, $\lambda_1 = 2N$, $\lambda_m = \phi_{s_{m-1}}(2N)$, $m = 2, 3, \ldots, x = \lambda_{k_0+1}$. Then

$$
\begin{aligned}
\lim_{m\to\infty} (N+2)^m \lambda_m &= \lim_{m\to\infty} (N+2)^m (\phi_-)^{(m-k_0-1)}(x)\\
&= (N+2)^{k_0}\kappa.
\end{aligned}
$$

Just as in the beginning of Section 5, we choose a series $\{f_m\}_{m=1,2,\ldots}$, $f_m \in \ell_0(V_m^{(0)})$, of eigenfunctions associated with $\{\lambda_m\}$. We have $-H_m^0 f_m(p) = \lambda_m f_m(p)$, $p \in V_m^{(0)} \setminus V_0^{(0)}$. Extend f_m to $V_m^{(\infty)}$ by defining its values on $V_m^{(\infty)} \setminus V_m^{(0)}$ to be zero. The extended function is denoted by f_m again. We would like to have for any $p \in V_m^{(\infty)} \setminus \{O\}$, $-H_{m,p} f_m$ $(= -\Sigma_{q \in V_{m,p}^{(\infty)}} f(q) + 2(N-1)f(p)) = \lambda_m f(p)$. But since $\lambda_1 = 2N$, we can invoke Lemma

3.2 to see that this is true for $m = 1$. By the way of the construction of successive f_m's, we can then see that this is valid for all m.

Let f be the function on $V_*^{(\infty)}$ with $f|_{V_m^{(\infty)}} = f_m$, $m = 1, 2, \ldots$. Then $f \in \mathscr{F}_0^{(\infty)}$ because $\sup_m \mathscr{E}_{K^{(\infty)}}^{(m)}(f, f) = \sup_m \mathscr{E}_{K^{(0)}}^{(m)}(f, f)$ is finite by Proposition 5.1. Further by letting m tend to infinity in

$$\mathscr{E}_{K^{(\infty)}}^{(m)}(f, g) = (N + 2)^m \lambda_m (f, g)_{\mu_m}, \quad g \in \mathscr{F}_0^{(\infty)},$$

we get

$$\mathscr{E}_{K^{(\infty)}}(f, g) = (N + 2)^{k_0} \kappa (f, g)_\mu, \quad g \in \mathscr{F}_0^{(\infty)},$$

which means that $(N + 2)^{k_0}\kappa$ is an eigenvalue of $-\Delta^{(\infty)}$ and f is an eigenfunction belonging to it.

In the present construction, f is supported by $K^{(0)}$. Similarly we can construct an eigenfunction of $(N + 2)^{k_0}\kappa$ supported by subset of $K^{(\infty)}$ obtained by any shift of $K^{(0)}$. Therefore the eigenspace of $(N + 2)^{k_0}\kappa$ is of infinite dimension.

Next define a transformation $\sigma_n : R^{K^{(\infty)}} \to R^{K^{(\infty)}}$ by $(\sigma_n u)(x) = u(2^n x)$, $x \in K^{(\infty)}$, $n = 0, \pm 1, \pm 2, \ldots$. If u is supported by $E \subset K^{(\infty)}$, then the support of $\sigma_n u$ is $2^{-n}E$. $\mathscr{E}_{K^{(\infty)}}$ and L^2-inner product enjoy with respect to σ_n the similar scaling properties to Lemma 4.2. Hence we can conclude analogously to Proposition 4.5 that, if κ is an eigenvalue of $-\Delta^{(\infty)}$ with an eigenfunction f, then $(N + 2)^n \kappa$ is also an eigenvalue of $-\Delta^{(\infty)}$ and $\sigma_n f$ is an eigenfunction belonging to it, $n = 0, \pm 1, \pm 2, \ldots$. We have shown the assertion of Theorem 7.1 for any $\kappa \in D_1$. ∎

The assertion of Theorem 7.1 for $\kappa \in D_2$ can be proved in the similar way once the following lemma is shown:

LEMMA 7.1. *There exists a function f on $V_2^{(\infty)}$ such that*

$$\begin{aligned} &H_{2,p} f = -(N + 2) f(p), \quad p \in V_2^{(\infty)} \backslash \{O\} \\ &supp[f] \subset V_2^{(0)} \backslash V_1^{(0)}. \end{aligned} \tag{7.2}$$

Proof. When $N = 3$, f can be chosen as in Figure 2 which can be also found in Rammal [11]. In general, it suffices to construct f in such a way that (7.2) is satisfied on $V_2^{(0)} \backslash V_1^{(0)}$ and

$$f(p) = 0, \quad \sum_{q \in V_{2,p}^{(0)}} f(q) = 0, \quad p \in V_1^{(0)}. \tag{7.3}$$

Take $M \in F_1$ and consider a function g on $Son(M) \cup V(M)$ as in Lemma 3.1. g vanishes on $V(M)$ and satisfies (7.2) for $p \in Son(M)$. If we evaluate the value

$$\sum_{\substack{q \in Son(M) \\ |q - p| = 2^{-2}}} g(q)$$

on each vertex $p \in V(M)$, it takes $N-2$ on one vertex, $-(N-2)$ on another vertex and 0 on other $N-2$ vertices. We associated with each $p \in V(M)$ the label $+$, $-$ and 0 accordingly.

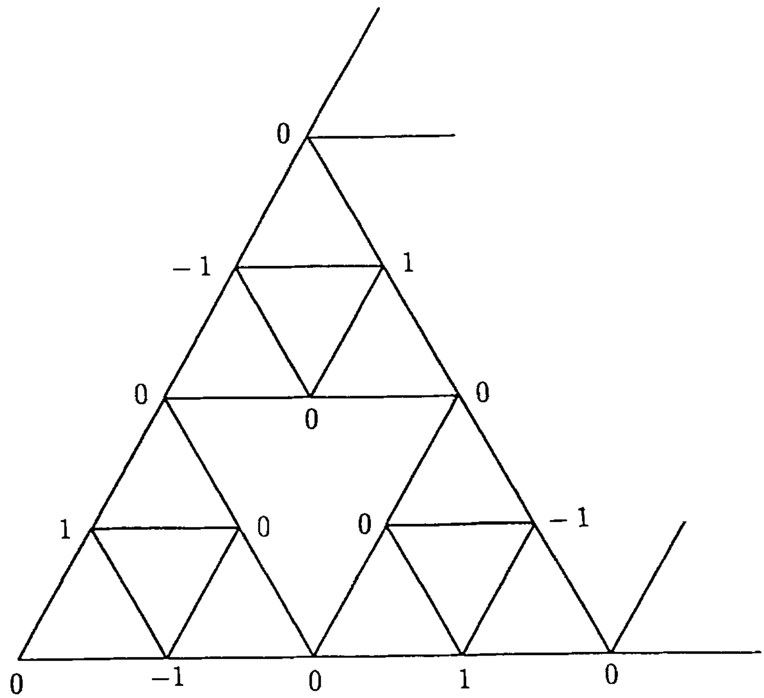

Fig. 2.

We prepare N-simplices $\{M_k\}_{k=1,2,\ldots,N}$ such that the first three M_1, M_2, M_3 have vertices labeled as above and each of the other simplex has all its vertices labeled zero. We can connect those N-simplices in the way that two of them are connected by one and only one vertex and the pairing of the labels of the connected vertex is required to be one of $\{+,-\}$, $\{-,+\}$, and $\{0,0\}$, and further each vertex of the resulting larger simplex is required to be of label 0. This can be achieved by induction. For instance, if $N=4$, $\{M_1, M_2, M_3, M_4\}$ can be connected so that the resulting larger simplex has one side looking like Figure 2 and all labels not sitting on this side are 0.

Each M_k carries an associated function g ($g=0$ for $k \geqslant 4$) which gives rise to a function f on $V_2^{(0)}$. Obviously f is a non-trivial function satisfying (7.2) on $V_2^{(0)} \setminus V_1^{(0)}$ and (7.3) on $V_1^{(0)}$. ■

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64. *On a Strict Decomposition of Additive Functionals for Symmetric Diffusion Processes*

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1. Introduction. Let X be a locally compact separable metric space and m be a positive Radon measure on X with full support. For an m-symmetric Hunt process $\boldsymbol{M}=(X_t, P_x)$ on X with the associated Dirichlet form being regular, the following decomposition of additive functionals (AF's in abbreviation) has been known ([1], [3]):

(1) $$u(X_t)-u(X_0)=M_t^{[u]}+N_t^{[u]},\ P_x\text{-almost surely}$$

which holds for quasi every (q. e. in abbreviation) $x \in X$. Here u is a function in the Dirichlet space, $M_t^{[u]}$ is a martingale AF of finite energy, $N_t^{[u]}$ is a continuous AF of zero energy and 'for q.e. $x \in X$' means 'for every $x \in X$ outside a set of zero capacity'. $N_t^{[u]}$ is then of zero quadratic variation on each finite time interval P_m-a.s. but not necessarily of bounded variation. In this sense, (1) is beyond a semimartingale decomposition and it is a prototype of the so called Dirichlet process. However we can not tell in general where the exceptional set of zero capacity involved in the decomposition (1) is located. This ambiguity imposes a limitation on its applicability especially to the finite dimensional analysis.

If we assume the absolute continuity of the transition function $p_t(x, B)$ of the process $\boldsymbol{M}$:

(2) $$p_t(x,\cdot) \prec m,\ \forall t>0,\ \forall x \in X,$$

then it is possible to refine the decomposition (1) by giving conditions on the function u so that the AF's on the right hand side of (1) make sense for every starting point $x \in X$ (namely, they are converted into AF's in the strict sense) and further (1) holds for every $x \in X$ as well. Some sufficient conditions for this are presented in the book [3]. In [2], the author shows that a necessary and sufficient condition for this is that the energy measure of u is smooth in the strict sense:

(3) $$\mu_{\langle u\rangle} \in S_1.$$

In this paper, we start with conditions (2) and (3) and investigate some basic properties of the corresponding AF $N_t^{[u]}$ in the strict sense. To simplify the presentation, we assume that the Dirichlet form is strongly local and $\boldsymbol{M}$ is a conservative diffusion process. We can then deal with functions belonging locally to the Dirichlet space.

2. A strict decomposition. We use those notions and notations in [3] concerning Dirichlet forms, diffusion processes and additive functionals (AF's in abbreviation). Let $(\mathcal{E}, \mathcal{F})$ be a strongly local, regular Dirichlet form on $L^2(X\,; m)$. We assume that there exists a conservative diffusion process

$\boldsymbol{M} = (\Omega, \mathscr{F}_t, X_t, P_x)$ on X associated with $\mathscr{E}$ whose transition function p_t satisfies the absolute continuity condition (2). We note that, although the existence of a diffusion process associated with the present Dirichlet form $\mathscr{E}$ follows automatically from general theorems in [3], the existence of such a process possessing the additional property (2) is highly non-trivial, and in many cases we have to work with other methods like PDE in the construction. See [4] for a prototype of such a construction.

We say that a function u is locally in $\mathscr{F}$ ($u \in \mathscr{F}_{loc}$ in notation) if for any relatively compact open set G there exists a function $w \in \mathscr{F}$ such that $u = w$ m-a.e. on G. Any $u \in \mathscr{F}$ admits a unique positive Radon measure $\mu_{\langle u \rangle}$ called the *energy measure* of u, which satisfies $\mathscr{E}(u, u) = \frac{1}{2}\mu_{\langle u \rangle}(X)$. Under the present locality assumption on $\mathscr{E}$, the energy measure can be also associated with $\mathscr{F}_{loc}$ uniquely.

A real valued function $A_t(\omega)$ of $t \geq 0$ and $\omega \in \Omega$ is called a *continuous AF* (CAF in abbreviation) *in the strict sense* if it is $\{\mathscr{F}_t\}$-adapted and the following properties in $t \geq 0$ hold P_x-a.s. $\forall x \in X$: $A_0(\omega) = 0$, $A_t(\omega)$ is continuous on $[0, \infty)$ and additive: $A_{t+s}(\omega) = A_s(\omega) + A_t(\theta_s \omega)$, $s, t \geq 0$. Two CAF's in the strict sense $A^{(1)}, A^{(2)}$ are regarded to be equivalent if they are *indistinguishable* in the sense that $A_t^{(1)} = A_t^{(2)}$ $\forall t > 0$ P_x-a.s. $\forall x \in X$.

A CAF in the strict sense is called *positive* (a PCAF in the strict sense in abbreviation) if it is non-negative $\forall t \geq 0$ P_x-a.s. $\forall x \in X$. The equivalence classes of all PCAF's in the strict sense is denoted by $\boldsymbol{A}_{c1}^{+}$.

We add the phrase 'in the strict sense' to an AF in order to distinguish it from the somewhat relaxed notion of an AF employed in [3] which admits an exceptional set of starting points $x \in X$ of zero capacity and fits more in the Dirichlet form setting. Under the present absolute continuity assumption (2) however, we can handle AF's in the strict sense equally in a systematic way.

(2) implies that the resolvent kernel $R_\alpha(x, E)$ of $\boldsymbol{M}$ admits a symmetric density $r_\alpha(x, y)$ with respect to m which is α-excessive in each variable ([3, Lemma 4.2.4]). For a positive Borel measure μ on X, its α-potential is defined by $R_\alpha\mu(x) = \int_X r_\alpha(x, y)\mu(dy)$, $x \in X$. Denote by S_{00} the family of positive Borel measures μ such that $\mu(X) < \infty$ and $\sup_{x \in X} R_\alpha\mu(x) < \infty$. An increasing sequence $\{E_l\}$ of finely open sets with $\cup_{l=1}^{\infty} E_l = X$ will be called an *exhaustive sequence* of finely open sets. A positive Borel measure μ is said to be *smooth in the strict sense* if there exists an exhaustive sequence of finely open sets $\{E_l\}$ such that $I_{E_l}\cdot\mu \in S_{00}$, $l = 1, 2, \cdots$.

Denote by S_1 the totality of smooth measures in the strict sense. S_1 is known to stand in one to one correspondence with the family $\boldsymbol{A}_{c1}^{+}$ of functionals by the Revuz correspondence ([3, Th. 5.1.7]). S_1 is contained in the family S of smooth measures introduced in [3, §2.2].

The next theorem is formulated and proven in [2, Theorem 2] without the present conservativeness assumption. We use the notations $\mathring{\mathcal{M}}_{loc}$ (resp.

$\mathcal{N}_{c,loc}$) from [3, §5.5] standing for the family of all martingale AF's locally of finite energy (resp. all continuous AF's locally of zero energy).

Theorem 2.1. *Suppose*

(4) u is finite valued, finely continuous and $u \in \mathcal{F}_{loc}$.

The condition (3) *is then necessary and sufficient for* u *to admit the decomposition*

(5) $$u(X_t) - u(X_0) = M_t^{[u]} + N_t^{[u]}, \ \forall t \geq 0 \ P_x\text{-a. s.}, \ \forall x \in X,$$

with $M^{[u]}$, $N^{[u]}$ *possessing the following properties:*

(a) $M^{[u]} \in \mathring{\mathcal{M}}_{loc}$, $M^{[u]}$ *is a CAF in the strict sense admitting an* $A \in \mathbf{A}_{c1}^{+}$ *and an exhaustive sequence* $\{E_l\}$ *of finely open sets such that, for each* l,

$$E_x((M_{t\wedge\tau_l}^{[u]})^2) = E_x(A_{t\wedge\tau_l}) < \infty, \ E_x(M_{t\wedge\tau_l}^{[u]}) = 0 \ \forall x \in E_l.$$

Here τ_l *denotes the first leaving time from the set* E_l.

(b) $N^{[u]} \in \mathcal{N}_{c,loc}$, $N^{[u]}$ *is a CAF in the strict sense.*

In this case, the energy measure $\mu_{\langle u\rangle}$ *and* A *in* (a) *are related by the Revuz correspondence. Any exhaustive sequence of finely open sets associated with* $\mu_{\langle u\rangle}$ *works as a sequence appearing in* (a).

The above decomposition is unique up to the indistinguishability.

Theorem 2.1 (a) means that $M^{[u]}$ is a local martingale CAF (in the strict sense) with the quadratic variation being the PCAF in the strict sense corresponding to the energy measure $\mu_{\langle u\rangle}$.

3. Properties of $N^{[u]}$. In this section, we consider those functions u satisfying conditions (3) and (4) and investigate the locality, the support and the absolute variation of the corresponding CAF's $N^{[u]}$ in the strict sense, locally of zero energy, produced by Theorem 2.1.

Theorem 3.1. *Suppose that* u_1, u_2 *satisfy conditions* (3) *and* (4) *and that* $u_1 - u_2$ *is constant on a nearly Borel finely open set* G. *Then*

(6) $$M_{t\wedge\tau_G}^{[u_1]} = M_{t\wedge\tau_G}^{[u_2]}, \ N_{t\wedge\tau_G}^{[u_1]} = N_{t\wedge\tau_G}^{[u_2]}$$

up to indistinguishability. Here τ_G *denotes the first leaving time from* G.

Proof. By virtue of [3, Lemma 5.5.1], (6) holds $\forall t \geq 0$ P_x-a.s. for q.e. $x \in X$. This can be strengthened to "$\forall x \in X$" in the same way as in [2, Proof of uniqueness].

The *sprectrum* $\sigma(u)$ of a function $u \in \mathcal{F}_{Loc}$ is defined in [3] as the complement of the largest open set G such that $\mathcal{E}(u, v) = 0$ $\forall v = \mathcal{C}_G$, $\mathcal{C}$ being any special standard core of $\mathcal{E}$ and $\mathcal{C}_G = \{v \in \mathcal{C} : \mathrm{supp}[u] \subset G\}$. The α-spectrum $\sigma_\alpha(u)$ is defined by replacing $\mathcal{E}$ with $\mathcal{E}_\alpha$ in the above.

For a closed set F, we write $\{t > 0 : X_t(\omega) \in X - F\} = \cup_\eta I_\eta(\omega)$, $\{I_\mu\}$ being countable number of disjoint open intervals, which can be enumerated in a way that ends points are measurable. We call $\{I_\eta\}$ excursions of the sample path X_t out of F.

Theorem 3.2. *Suppose that* $u \in \mathcal{F}$ *and* u *satisfies conditions* (3) *and* (4). *Let* $\{I_\eta\}$ *be excursions out of the spectrum* $\sigma(u)$ *of* u. *Then*

(7) $$P_x(N_t^{[u]} \text{ is constant on } I_\eta \ \forall \eta) = 1, \ \forall x \in X.$$

Let $\{I_\eta\}$ *be excursions out of the* α-*spectrum* $\sigma_\alpha(u)$ *of* u. *Then*

(8) $$P_x(N_t^{[u]} - \alpha \int_0^t u(X_s)ds \text{ is constant on } I_\eta, \ \forall \eta) = 1, \ \forall x \in X.$$

Proof. A weaker version of the above theorem with '$\forall x \in X$' being replaced by 'for q.e. $x \in X$' is implied in the proof of [3, Theorem 5.4.1], where the Beurling-Deny theorem on the spectral synthesis is invoked. Then we can utilize the condition (2) to get for any $x \in X$ and $\varepsilon > 0$

$$\begin{aligned} &P_x(N_t^{[u]} \text{ is not constant on } [\varepsilon, \infty) \cap I_\eta \ \exists \eta) \\ &= P_x(N_t^{[u]}(\theta_\varepsilon \omega) \text{ is not constant on } I_\eta(\theta_\varepsilon \omega) \ \exists \eta) \\ &= E_x(P_{X_\varepsilon}(N_t^{[u]} \text{ is not constant on } I_\eta \ \exists \eta)) = 0 \end{aligned}$$

arriving at (7). The proof of (8) is the same.

Remark. On account of Theorem 3.1 and the derivation property of the energy measure ([3, §3.2]), We can prove that Theorem 3.2 extends to $u \in \mathscr{F}_{b,loc}$ provided that there exist an exhaustive sequence $\{G_k\}$ of relatively compact open sets and functions $\{\phi_k\}$ in $\mathscr{F} \cap C_0(X)$ such that ϕ_k equals 1 on G_k, vanishes outside G_{k+1} and $\mu_{\langle \phi_k \rangle} \in S_1$, $k = 1,2, \cdots$.

For a subset $B \subset X$, we let $\mathscr{F}_{b,B} = \{v \in \mathscr{F} \cap L^\infty : \tilde{v} = 0 \text{ q.e. on } X - B\}$.

Theorem 3.3. *The next two conditions* (3.a) *and* (3.b) *are equivalent for a function u satisfying* (3) *and* (4):

(3.a) *There exists a signed measure μ expressible as $\mu = \mu_1 - \mu_2$ for some μ_1, $\mu_2 \in S_1$ and*

$$\mathscr{E}(u, v) = \langle \mu, \tilde{v} \rangle \ \forall v \in \bigcup_k \mathscr{F}_{b,G_k}$$

where $\{G_k\}$ is an exhaustive sequence of finely open sets commonly associated with μ_1, μ_2.

(3.b) *$N_t^{[u]}$ is of bounded variation on each compact interval of $[0, \infty)$ P_x-a.s. $\forall x \in X$.*

In this case, $N^{[u]} = -A^{(1)} + A^{(2)}$ for $A^{(i)} \in \mathbf{A}_{c,1}^+$ with Revuz measure μ_i, $i =$ 1,2.

Proof of (3.a) ⇒ (3.b). Let $A^{(i)}$, $i = 1,2$, be as in the last assertion above and set $A = A^{(1)} - A^{(2)}$. Then, we get from (3.a) and [3, Lemma 5.4.4]

$$P_x(N_t^{[u]} = -(I_{G_k} \cdot A)_t (= -A_t), \ t < \tau_{G_k}) = 1 \text{ q.e. } x \in X.$$

By letting $k \to \infty$, $P_x(N_t^{[u]} = -A_t \ \forall t \geq 0) = 1$ q.e. $x \in X$, and by virtue of (2)

$$P_x(N_{t+\varepsilon}^{[u]} - N_\varepsilon^{[u]} = -A_{t+\varepsilon} + A_\varepsilon \ \forall t \geq 0) = 1 \ \forall x \in X.$$

Since both $N^{[u]}$ and A are AF's in the strict sense, we may let $\varepsilon \downarrow 0$ to get

$$P_x(N_t^{[u]} = -A_t \ \forall t \geq 0) = 1 \ \forall x \in X.$$

Proof of (3.b) ⇒ (3.a). As in the proof of [3, Theorem 5.4.2], it suffices to use the following variant of [3, Lemma 5.4.3]:

Lemma 3.1. *For any $N \in \mathscr{N}_c$ and for any nearly Borel finely open set G,*

$$\lim_{t \downarrow 0} \frac{1}{t} E_{v \cdot m}(N_t ; t > \tau_G) = 0, \ \forall v = R_\alpha^G f, f \in L_b^1(X ; m).$$

Since the functional $N^{[u]}$ is strict version of that appearing in [3], we can restate [3, Theorem 5.5.4] as follows:

Theorem 3.4. *The following two conditions* (3.c) *and* (3.d) *are equivalent for a function u satisfying* (3) *and* (4):

(3.c) *For some smooth signed measure μ and a generalized compact nest $\{F_k\}$ associated with μ,*

$$\mathcal{E}(u, v) = \langle \mu, \tilde{v} \rangle \ \forall v \in \bigcup_k \mathcal{F}_{b,F_k}.$$

(3.d) $N_t^{[u]}$ *is of bounded variation on each compact interval of* $[0, \infty)$ P_x*-a.s. for q.e.* $x \in X$.

In this case, the following is true:

(3.e) $N_t^{[u]}$ *is of bounded variation on each compact interval of* $(0, \infty)$ P_x*-a.s.* $\forall x \in X$.

The last assertion is due to the absolute continuity condition (2).

Corollary 3.1. *Let* u *be a function satisfying* (3) *and* (4). *Suppose there exists a signed Radon measure* μ *satisfying for a special standard core* $\mathcal{C}$ *of* $\mathcal{E}$

$$\mathcal{E}(u, v) = \langle \mu, v \rangle \ \forall v \in \mathcal{C}.$$

(i) *If* μ *charges no set of zero capacity, then* (3.d) *and* (3.e) *hold.*

(ii) *If the total variation of* μ *is in* S_1, *then* (3.b) *holds.*

In fact, as [3, Corollary 5.5.1] we can see that (i) (resp. (ii)) implies (3.c) (resp. (3.a)).

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On a decomposition of additive functionals in the strict sense for a symmetric Markov process

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Abstract. For a symmetric Markov process, the decomposition theorem of the associated additive functional into its martigale part and energy zero part has been formulated admitting exceptional starting points of zero capacity. Under the assumption of the absolute continuity of the transition probability, we now present a necessary and sufficient condition for the decomposition to be refined to a strict one holding for every starting point.

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1. Introduction

Let X be a locally compact separable Hausdorff metric space and m be a positive Radon measure on X with full support. We consider an m-symmetric Hunt process $\mathbf{M} = (X_t, P_x)$ on X with the associated Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X; m)$ being regular. Then the composite process $u(X_t)$ for any quasi continuous function $u \in \mathcal{F}$ is known to admit the following decomposition ([1]):

$$u(X_t) - u(X_0) = M_t^{[u]} + N_t^{[u]}, \quad t > 0, \quad P_x\text{-a.s.}, \tag{1}$$

holding for q.e. $x \in X$, where $M_t^{[u]}$ is a martingale additive functional of finite energy and N_t is a continuous additive functional of zero energy. Here "q.e." or "quasi everywhere" means "except for a set of zero capacity". Since the above decomposition is formulated depending on the potential theory of the Dirichlet form $\mathcal{E}$, we have to adopt the notion of an additive functional in a more relaxed sense than usual by admitting an exceptional set of zero capacity. The second process $N_t^{[u]}$ on the right hand side of (1) is known to be of zero quadratic variation in $L^1(P_m)$-sense on each finite time interval but it is not necessarily of bounded variation. (1) is beyond a semi-martingale decomposition in this sense. However we can not tell in general where the exceptional set involved in the decomposition (1) is located. This ambiguity imposes a limitation on its applicability.

The aim of the present paper is to demonstrate how generally (1) can be converted into decomposition theorems of additive functionals in the strict sense (namely, admitting no exceptional set of zero capacity) thus holding for every $x \in X$, under the assumption that

$$p_t(x,\cdot) \text{ is absolutely continuous with respect to } m \tag{2}$$

for the transition function $p_t(x,B)$ of the process $\mathbf{M}$. Such a strengthening of the quasi everywhere statement (1) into everywhere ones is not only important theoretically but also useful in applications as we shall indicate it at the end of the next section for the case that the Dirichlet form $\mathcal{E}$ corresponds to a uniformly elliptic second order differential operator of divergence form.

2. Statements of the theorems

Let us fix the notions more precisely. A set $N \subset X$ is said to be *properly exceptional* if it is a nearly Borel measurable m-negligible set and its complement $X - N$ is $\mathbf{M}$-invariant. A set is of $\mathcal{E}_1$-capacity zero if and only if it is contained in a properly exceptional set([1]). Let Ω, $\{\mathcal{F}_t\}_{t\in[0,\infty]}$, ζ, θ_t be the sample space, the minimum completed admissible filtration, the life time and the shift operator respectively associated with the given Hunt process $\mathbf{M}$. An extended real valued function $A_t(\omega)$ of $t \geq 0$ and $\omega \in \Omega$ is called an *additive functional* (AF in abbreviation) if it is $\{\mathcal{F}_t\}$-adapted and there exist $\Lambda \in \mathcal{F}_\infty$ with $\theta_t\Lambda \subset \Lambda$, $\forall t > 0$ and a properly exceptional set $N \subset X$ with $P_x(\Lambda) = 1$, $\forall x \in X - N$, such that, for each $\omega \in \Lambda, A_0(\omega) = 0$, $A_t(\omega)$ is cadlag and finite on $[0,\zeta(\omega))$, $A_t(\omega) = A_{\zeta(\omega)}$, $t \geq \zeta(\omega)$, and

$$A_{s+t}(\omega) = A_s(\omega) + A_t(\theta_s\omega), \quad s,t \geq 0. \tag{3}$$

Λ (resp. N) in the above definition is called a *defining* (resp. an *exceptional*) set for the AF A. We regard two AF's to be equivalent if

$$P_x\left(A_t^{(1)} = A_t^{(2)}\right) = 1, \quad t \geq 0, \tag{4}$$

for q.e. $x \in X$.

An AF $A_t(\omega)$ is said to be *finite, cadlag* and *continuous* respectively if it satisfies the respective property at every $t \in [0,\infty)$ for each ω in its defining set. A $[0,\infty]$-valued continuous AF is called a *positive continuous* AF (PCAF in abbreviation). If $A_t(\omega)$ satisfies all the requirements for an AF except that the additivity (3) is required only for non-negative t,s with $t+s<\zeta$, then $A_t(\omega)$ is called a *local* AF. Hence a local AF is a synonym for an AF provided that the process $\mathbf{M}$ is conservative. We call a local AF *continuous* if it is continuous in $t \in [0,\zeta(\omega))$ for each ω in its defining set. Two local AF's are regarded to be equivalent if

$$P_x(A_t^{(1)} = A_t^{(2)}, \quad \forall t \in [0,\zeta(\omega)) = 1, \tag{5}$$

for q.e. $x \in X$.

An *additive functional in the strict sense* (AF in the strict sense in abbreviation) is by definition an AF admitting a defining set Λ with $P_x(\Lambda) = 1$, $\forall x \in X$, namely an AF without exceptional set N. Two AF's in the strict sense are regarded to be equivalent if they are indistinguishable in the sense that the relation (4) holds for every $x \in X$. The equivalence of two local AF's in the strict sense is similarly defined by the validity of the relation (5) for every $x \in X$.

Denote by $\mathbf{A}_c^+$ the family of all PCAF's of $\mathbf{M}$ and by S the family of all smooth measures on X. It is known that the equivalence classes of $\mathbf{A}_c^+$ and S are in one to one correspondence by the *Revuz correspondence* specified by the following formula ([2; Theorem 5.1.4]):

$$\lim_{t\downarrow 0} \frac{1}{t} E_{h\cdot m}\left(\int_0^t f(X_t)dA_t\right) = \int_X h\cdot f d\mu, \quad A \in \mathbf{A}_c^+,\ \mu \in S,$$

holding for any non-negative Borel function f and γ-excessive function h, $\gamma \geq 0$.

Any $u \in \mathcal{F}_b$ admits a unique positive Radon measure $\mu_{\langle u \rangle}$ such that

$$\int_X f(x)\mu_{\langle u\rangle}(dx) = 2\mathcal{E}(u\cdot f, u) - \mathcal{E}(u^2, f), \quad \forall f \in \mathcal{C}, \tag{6}$$

where $\mathcal{C}$ is a special standard core of $\mathcal{E}$ ([2]). $\mu_{\langle u\rangle}$ is extended to $u \in \mathcal{F}$ and called the *energy measure* of u. It charges no set of zero capacity and hence belongs to S ([2; Lemma 3.2.4]). The energy measure can be also associated with $u \in \mathcal{F}_{\text{loc}}$ uniquely whenever $\mathcal{E}$ is local.

The *energy* $e(A)$ of an AF A is defined by $e(A) = \lim_{t\downarrow 0} \frac{1}{2t} E_m(A_t^2)$. The families of martingale AF's of finite energy and CAF's of zero energy are introduced respectively by

$$\overset{\circ}{\mathcal{M}} = \left\{M : \text{finite cadlag AF } E_x(M_t^2) < \infty,\ E_x(M_t) = 0 \text{ q.e. and } e(M) < \infty\right\},$$

$$\mathcal{N}_c = \{N : \text{CAF } E_x(|N_t|) < \infty \text{ q.e. and } e(N) = 0\}.$$

Any quasi continuous $u \in \mathcal{F}$ is known to admit a decomposition (1) uniquely up to the equivalence of AF's, where $M^{[u]} \in \overset{\circ}{\mathcal{M}}$ and $N^{[u]} \in \mathcal{N}_c$ ([2; Theorem 5.2.2]). Further the quadratic variation $\langle M^{[u]}\rangle (\in \mathbf{A}_c^+)$ has as its Revuz measure just the energy measure of u ([2; Theorem 5.2.3]).

Keeping these facts from [2] in mind, let us make from now on the absolute continuity assumption (2) for $\mathbf{M}$. Then the resolvent kernel $R_\alpha(x, E)$ of $\mathbf{M}$ admits a symmetric density $r_\alpha(x, y)$ with respect to m which is α-excessive in each variable ([2; Lemma 4.2.4]). For a positive Borel measure μ on X, its α-potential is defined by

$$R_\alpha\mu(x) = \int_X r_\alpha(x, y)\mu(dy), \quad x \in X.$$

Denote by S_{00} the family of positive Borel measures μ such that $\mu(X) < \infty$, $\sup_{x\in X} R_\alpha\mu(x) < \infty$. An increasing sequence $\{E_\ell\}$ of finely open sets with $\bigcup_{\ell=1}^\infty E_\ell = X$ will be called an *exhaustive sequence* of finely open sets. We note that each finely open set E_ℓ in the above sequence can be assumed to be relatively compact if necessary. A positve Borel measure μ is said to be *smooth in the strict* *sense* if there exists an exhaustive sequence of finely open sets $\{E_\ell\}$ such that

$$I_{E_\ell} \cdot \mu \in S_{00}, \quad \ell = 1, 2, \dots. \tag{7}$$

We quote once more the theorem from [2]. Denote by $\mathbf{A}_{c1}^{+}$ and S_1 the totality of PCAF's in the strict sense and the smooth measures in the strict sense respectively. Then they are in one to one correspondence again by the Revuz correspondence ([2; Theorem 5.1.7]). In particular, $S_1 \subset S$. Furthermore, it holds for $A \in \mathbf{A}_{c1}^{+}$ that

$$E_x\left(\int_0^\infty e^{-s} I_{E_\ell}(X_s)dA_s\right) = R_1\left(I_{E_\ell} \cdot \mu\right)(x), \quad x \in X,\ \ell = 1, 2, \dots, \tag{8}$$

where $\mu \in S_1$ is the Revuz measure of A and $\{E_\ell\}$ is an exhaustive sequence associated with μ ([2; Theorem 5.1.6]).

We are now in a position to formulate the main theorem. For an exhaustive sequence $\{E_\ell\}$ of finely open sets, τ_ℓ will denote the first leaving time from E_ℓ: $\tau_\ell = \inf\{t > 0 : X_t \in X_\Delta - E_\ell\}$, $\ell = 1, 2, \dots$. It follows from the quasi left continuity of the Hunt process $\mathbf{M}$ that

$$P_x\left(\lim_{\ell\to\infty} \tau_\ell = \zeta\right) = 1, \quad \forall x \in X. \tag{9}$$

In order to simplify the presentation, we make an additional assumption that the Hunt process $\mathbf{M}$ admits *no killing inside* in the sense that

$$P_x(\zeta < \infty, X_{\zeta-} \in X) = 0, \quad \forall x \in X. \tag{10}$$

This condition is equivalent to the absence of the killing part in the Beurling–Deny representation of the Dirichlet form $\mathcal{E}$ ([2; Theorem 4.5.4]). The condition (10) implies

$$P_x(t \wedge \tau_\ell < \zeta) = 1, \quad \forall t > 0,\ \forall x \in X, \tag{11}$$

provided that E_ℓ is relatively compact.

Theorem 1. *Assume the absolute continuity condition* (2) *and the condition* (10) *for* $\mathbf{M}$. *The next two conditions are then equivalent to each other for a finite valued, finely continuous function* $u \in \mathcal{F}$:

(i) $\mu_{\langle u \rangle} \in S_1$.

(ii) *The decomposition*

$$u(X_t) - u(X_0) = M_t + N_t, \quad 0 \le t < \zeta,\ P_x\text{-a.s.},\ \forall x \in X, \tag{12}$$

holds with

(a) $M \in \overset{\circ}{\mathcal{M}}$, *$M$ is a local AF in the strict sense admitting an* $A \in \mathbf{A}_{c1}^{+}$ *and an exhaustive sequence* $\{E_\ell\}$ *of relatively compact finely open sets such that, for each* ℓ,

$$E_x\left(M_{t\wedge\tau_\ell}^2\right) = E_x\left(A_{t\wedge\tau_\ell}\right) < \infty, \quad E_x\left(M_{t\wedge\tau_\ell}\right) = 0 \quad \forall x \in E_\ell. \tag{13}$$

(b) $N \in \mathcal{N}_c$, *N is a local CAF in the strict sense.*

In this case, $\mu_{\langle u\rangle}$ in (i) *and A in* (ii) *are related by the Revuz correspondence. Any exhaustive sequence of relatively compact finely open sets associated with $\mu_{\langle u\rangle}$ works as a sequence appearing in* (ii)(a).

The decomposition in (ii) *is unique up to the equivalence of local AF's in the strict sense.*

The requirement of the relative compactness for finely open sets E_ℓ in the condition (ii)(a) *can be dispensed with, provided that* $\mathbf{M}$ *is conservative in the sense that*

$$P_x(\zeta = \infty) = 1 \quad \forall x \in X. \tag{14}$$

If moreover $\mathcal{E}$ is local, Theorem 1 can be readily extended to a function $u \in \mathcal{F}_{\rm loc}$ on X which admits for each relatively compact open set G a function $v \in \mathcal{F}$ such that $u = v$ on G. The decomposition (1) has been extended under the locality of $\mathcal{E}$ to $u \in \dot{\mathcal{F}}_{\rm loc}(\supset \mathcal{F}_{\rm loc})$, $M^{[u]} \in \overset{\circ}{\mathcal{M}}_{\rm loc}$, $N^{[u]} \in \mathcal{N}_{c,{\rm loc}}$. We refer the readers to [2; Theorem 5.5.1] for such an extension and the notations just described.

To formulate an extension of Theorem 1, we assume the *strong local property* of $\mathcal{E}$: for some special standard core $\mathcal{C}$ of $\mathcal{E}$, $\mathcal{E}(u,v) = 0$ if $u, v \in \mathcal{C}$ and v is constant on a neighbourhood of Supp$[u]$. The strong locality of $\mathcal{E}$ is equivalent to the condition that $\mathbf{M}$ is a diffusion and admits no killing inside in the sense of (10) ([2; Theorem 4.5.4]).

Theorem 2. *Assume the absolute continuity condition* (2) *for* $\mathbf{M}$ *and the strong local property of $\mathcal{E}$. The next two conditions are equivalent to each other for a finite valued finely continuous function $u \in \mathcal{F}_{\rm loc}$.*
(i) $\mu_{\langle u\rangle} \in S_1$.
(ii) *The decomposition* (12) *holds with*
  (a)′ $M \in \overset{\circ}{\mathcal{M}}_{\rm loc}$, *$M$ is a local CAF in the strict sense admitting an $A \in \mathbf{A}_{c1}^+$ and an exhaustive sequence $\{E_\ell\}$ of relatively compact finely open sets such that* (13) *is satisfied.*
  (b)′ $N \in \mathcal{N}_{c,{\rm loc}}$, *$N$ is a local CAF in the strict sense.*

The assertions in the latter half of Theorem 1 *except for the last one remain valid with* (ii)(a) *being replaced by* (ii)(a)′.

The local CAF N in the strict sense appearing in Theorem 1 and Theorem 2 is not of bounded variation unless u is a potential of a signed smooth measure in the strict sense. Sufficient conditions of this type ensuring N to be expressible as a difference of PCAF's in the strict sense are given in [2; Theorem 5.2.5 and Theorem 5.5.5]. In these cases, N can be easily defined first and then M defined simply by the formula (12) is shown to have the property (ii)(a) (resp. (ii)(a)′) under a stronger condition on $\mu_{\langle u\rangle}$ than (i) of Theorem 1 (resp. Theorem 2).

Example. Let $X = \mathbb{R}^d$ the Euclidean d-space, $m(dx) = dx$ the Lebesgue measure and

$$\mathcal{E}(u,v) = \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx, \quad \mathcal{F} = H^1(\mathbb{R}^d),$$

where $H^1(\mathbb{R}^d)$ is the Sobolev space of order 1 and a_{ij} are Borel functions on $\mathbb{R}^d$ with $a_{ij} = a_{ji}$ and

$$\frac{1}{\lambda} \sum_{i=1}^{d} \xi_i^2 \leq \sum_{i,j=1}^{d} a_{ij}(x) \xi_i \xi_j \leq \lambda \sum_{i=1}^{d} \xi_i, \quad \forall \xi \in \mathbb{R}^d,\ \forall x \in \mathbb{R}^d,\ \exists \lambda > 1.$$

$\mathcal{E}$ is a strongly local regular Dirichlet form on $L^2(\mathbb{R}^d) = L^2(\mathbb{R}^d; dx)$. It is known that there is an associated conservative diffusion process $\mathbf{M} = (X_t, P_x)$ on $\mathbb{R}^d$ satisfying the absolute continuity condition (2). The energy measure $\mu_{\langle u \rangle}$ of $u \in H^1_{\text{loc}}(\mathbb{R}^d)$ is given by

$$d\mu_{\langle u \rangle} = 2 \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx$$

according to the formula (6).

Consider the coodinate functions $\varphi_i(x) = x^{(i)}$, $1 \leq i \leq d$. Then $d\mu_{\langle \varphi_i \rangle} = 2a_{ii}(x)dx$ is in S_1 because $\mathbb{R}_1 \mu_{\langle \varphi_i \rangle} \leq \sup_{x \in \mathbb{R}^d} a_{ii}(x)$ and $E_\ell = \{|x| < \ell\}$, $\ell = 1, 2, \ldots$, constitute an associated exhaustive sequence of open sets. $A_t^{(i)} = 2\int_0^t a_{ii}(X_s)ds$ is the PCAF in the strict sense with Revuz measure $\mu_{\langle \varphi_i \rangle}$.

By virtue of Theorem 2, there exist unique CAF's $M^{(i)}$ and $N^{(i)}$ in the strict sense such that

$$X_t^{(i)} - X_0^{(i)} = M_t^{(i)} + N_t^{(i)},$$

$M^{(i)}$ is a continuous local martingale with quadratic variation $A^{(i)}$ with respect to P_x for each $x \in \mathbb{R}^d$ and $N^{(i)}$ is in $\mathcal{N}_{\text{c,loc}}$. The co-variation $\langle M^{(i)}, M^{(j)} \rangle_t$ is given by $2\int_0^t a_{ij}(X_s)ds$.

$N^{(i)}$ is not of bounded variation unless $a_{ij}(x)$ is smooth in a certain sense. For instance, if the distribution derivatives $\frac{\partial}{\partial x_j} a_{ij}$, $1 \leq j \leq d$, are locally bounded functions, say $a_{ij,j}$, then

$$N_t^{(i)} = \int_0^t \sum_{j=1}^{d} a_{ij,j}(X_s) ds$$

by virtue of [2; Theorem 5.5.5].

3. Proof of Theorem 1

Proof of the implication (i) $\Rightarrow$ (ii). This will be carried out in four steps.

1°. We assume the absolute continuity condition (2) together with the condition (10) of the absence of killing inside and consider a finite valued, finely continuous function $u \in \mathcal{F}$ satisfying condition (i). Let $\{E_\ell\}$ be an exhaustive sequence of relatively compact finely open sets associated with $\mu_{\langle u \rangle} \in S_1$. Since u is quasi continuous, the decomposition (1) holds for q.e. $x \in X$ for some AF's $M^{[u]} \in \overset{\circ}{\mathcal{M}}$ and $N^{[u]} \in \mathcal{N}_c$ by virtue of [2; Theorem 5.2.2]. We are going to construct a strict version of $M^{[u]}$ roughly speaking by redefining it as

$$M_t = \lim_{\ell \to \infty} \lim_{\epsilon \downarrow 0} M^{[u]}_{(t-\epsilon)\wedge\tau_\ell}(\theta_{\epsilon\wedge\tau_\ell}\omega).$$

M_t then turns out to be an extension of $M^{[u]}$ with a suitably enlarged defining set. The strict version of $N^{[u]}$ will be defined similarly.

Let $\Lambda \in \mathcal{F}_\infty$ (resp. N) be a common defining set (resp. exceptional set) for $M^{[u]}$ and $N^{[u]}$. In particular,

$$\theta_t\Lambda \subset \Lambda, \ \forall t \geq 0 \quad \text{and} \quad P_x(\Lambda) = 1, \ \forall x \in X - N. \tag{15}$$

We may assume that

$$\Lambda = \Lambda \cap \{X_0 \in X\} + \{X_0 = \Delta\} \tag{16}$$

by replacing Λ with the right hand side if necessary. In view of (9) and (11), we may further assume that

$$\epsilon \wedge \tau_\ell(\omega) < \zeta(\omega), \quad \ell = 1, 2, \ldots, \quad \lim_{\ell \to \infty} \tau_\ell(\omega) = \zeta(\omega) \tag{17}$$

for any $\epsilon > 0$ and for any $\omega \in \Omega$ such that $X_0(\omega) \in X$. Let us fix a sequence ϵ_n of positive numbers decreasing to zero and put

$$\Lambda_{n,\ell} = \theta^{-1}_{\epsilon_n\wedge\tau_\ell}\Lambda, \quad \Lambda_\ell = \bigcap_{n=1}^{\infty} \Lambda_{n,\ell}, \quad \Lambda_0 = \bigcup_{\ell=1}^{\infty} \Lambda_\ell. \tag{18}$$

It follows from (15) that $\Lambda_{n,\ell}$ decreases as n increases and $\Lambda \subset \Lambda_\ell \subset \Lambda_{\ell+1}$.

Lemma 1.

$$\theta_s\Lambda_0 \subset \Lambda \subset \Lambda_0, \quad \forall s > 0. \tag{19}$$

$$P_x(\Lambda_\ell) = 1, \quad \forall x \in E_\ell, \quad \ell = 1, 2, \ldots, \quad \textit{and} \quad P_x(\Lambda_0) = 1, \quad \forall x \in X. \tag{20}$$

Proof. To prove the inclusion (19), take $\omega \in \Lambda_0$ and $s > 0$. Suppose $s < \zeta(\omega)$, then, by (17), $\epsilon_n < s < \tau_\ell(\omega)$ and $\omega \in \Lambda_\ell$ for some ℓ and sufficiently large n's. Hence

$$\theta_s\omega = \theta_{s-\epsilon_n} \circ \theta_{\epsilon_n}\omega = \theta_{s-\epsilon_n} \circ \theta_{\epsilon_n\wedge\tau_\ell}\omega$$

which belongs to Λ along with $\theta_{\epsilon_n\wedge\tau_\ell}\omega$ because of (15). Therefore $\theta_s\omega \in \Lambda_{n,\ell}$ and $\theta_s\omega \in \Lambda_0$. If $s \geq \zeta(\omega)$, then $X_0(\theta_s\omega) = \Delta$ and $\theta_s\omega \in \Lambda$ by (16).

By the assumption of the absolute continuity (2), the exceptional set N becomes polar in the sense that $P_x(\sigma_N < \infty) = 0\ \forall x \in X$. Here σ_N denotes the hitting time of N. If $x \in E_\ell$, then $P_x(\tau_\ell > 0) = 1$ and

$$\begin{aligned} P_x\left(\theta^{-1}_{\epsilon_n\wedge\tau_\ell}\Lambda\right) &= E_x\left(E_{X_{\epsilon_n\wedge\tau_\ell}}(\Lambda)\right) \\ &= E_x\left(P_{X_{\epsilon_n\wedge\tau_\ell}}(\Lambda); \epsilon_n\wedge\tau_\ell < \zeta, X_{\epsilon_n\wedge\tau_\ell} \in X - N\right) = 1. \end{aligned}$$

2°. Let $A \in \mathbf{A}^+_{c1}$ be a PCAF in the strict sense with Revuz measure $\mu_{\langle u\rangle}$. Then

$$E_x\left(A_{t\wedge\tau_\ell}\right) \leq e^t R_1 I_{E_\ell}\mu_{\langle u\rangle}(x) < \infty, \quad \forall x \in X, \tag{21}$$

because the left side is dominated by the left side of (8) multiplied by e^t. Since $M^{[u]}_t$ and $\left(M^{[u]}_t\right)^2 - A_t$ are P_x-martingale for any $x \in X - N$, we have for any stopping time σ that

$$\begin{aligned} E_x\left(\left(M^{[u]}_{t\wedge\sigma}\right)^2\right) &= E_x(A_{t\wedge\sigma}) \leq E_x(A_t) < \infty \\ E_x\left(M^{[u]}_{t\wedge\sigma}\right) &= 0 \quad \forall x \in X - N. \end{aligned} \tag{22}$$

We now put

$$M^{n,\ell}_t(\omega) = M^{[u]}_{(t-\epsilon_n)\wedge\tau_\ell}\left(\theta_{\epsilon_n\wedge\tau_\ell}\omega\right), \quad t > \epsilon_n,\ \omega \in \Lambda_0. \tag{23}$$

Lemma 2. *For any $x \in E_\ell$*

$$\begin{aligned} E_x\left(\left(M^{n,\ell}_t\right)^2\right) &= E_x\left(A_{t\wedge\tau_\ell}\right) - E_x\left(A_{\epsilon_n\wedge\tau_\ell}\right) \\ E_x\left(M^{n,\ell}_t\right) &= 0. \end{aligned} \tag{24}$$

For any $x \in E_\ell$, the following identities are valid P_x-a.s.:

$$M^{n,\ell}_t(\omega) - M^{m,\ell}_t(\omega) = M^{n,\ell}_{\epsilon_m}(\omega) \quad \text{for any } \epsilon_n < \epsilon_m < t, \tag{25}$$

$$M^{n,\ell}_{t+s}(\omega) = M^{n,\ell}_s(\omega) + M^{[u]}_{t\wedge\tau_\ell}\left(\theta_{s\wedge\tau_\ell}\omega\right), \quad \text{for any } t \geq 0,\ s > \epsilon_n. \tag{26}$$

Proof. It follows from the polarity of N and the second identity of (22) that

$$\begin{aligned} E_x\left(M^{n,\ell}_t\right) &= E_x\left(E_{X_{\epsilon_n\wedge\tau_\ell}}\left(M_{(t-\epsilon_n)\wedge\tau_\ell}\right); X_{\epsilon_n\wedge\tau_\ell} \in X - N\right) \\ &= 0, \quad x \in E_\ell. \end{aligned}$$

In the same way, we get from the first identity of (22) that

$$E_x\left(\left(M^{n,\ell}_t\right)^2\right) = E_x\left(E_{X_{\epsilon_n\wedge\tau_\ell}}\left(A_{(t-\epsilon_n)\wedge\tau_\ell}\right)\right). \tag{27}$$

Here we note that generally, for an AF B_t with an exceptional polar set N, the relation

$$B_{(t+s)\wedge\tau_\ell}(\omega) = B_{s\wedge\tau_\ell}(\omega) + B_{t\wedge\tau_\ell}\left(\theta_{s\wedge\tau_\ell}\omega\right), \quad P_x\text{-a.s.} \tag{28}$$

holds for any $x \in X - N$ provided that N is enlarged to include the polar set $(X - E_\ell) \cap (X - E_\ell)^{\text{irr}}$. In particular, if B is an AF in the strict sense, then we may set $N = (X - E_\ell) \cap (X - E_\ell)^{irr}$ and hence (28) holds for any $x \in E_\ell$. Therefore the first identity of (24) follows from (27).

As for the identity (25), observe that

$$t - \epsilon_n = (\epsilon_m - \epsilon_n) + (t - \epsilon_m), \theta_{\epsilon_m - \epsilon_n)\wedge\tau_\ell} \circ \theta_{\epsilon_n\wedge\tau_\ell}\omega = \theta_{\epsilon_m\wedge\tau_\ell}\omega.$$

We then see from the definition (23), the relation (28) for $M^{[u]}$ and the polarity of N that

$$\begin{aligned} 1 &= E_x\left(P_{X_{\epsilon_n\wedge\tau_\ell}}\left(M^{[u]}_{(t-\epsilon_n)\wedge\tau_\ell} = M^{[u]}_{(\epsilon_m-\epsilon_n)\wedge\tau_\ell} + M_{(t-\epsilon_m)\wedge\tau_\ell}\left(\theta_{(\epsilon_m-\epsilon_n)\wedge\tau_\ell}\right)\right)\right) \\ &= P_x\left(M^{n,\ell}_t = M^{n,\ell}_{\epsilon_m} + M^{m,\ell}_t\right), \quad x \in E_\ell. \end{aligned}$$

The identity (26) holding P_x-a.s for any $x \in E_\ell$ can be proved in the same way. □

Lemma 3.

$$E_x\left(\sup_{t\in[\epsilon_m,\tau_\ell)} |M^{n,\ell+1}_t - M^{m,\ell}_t|\right) \le E_x\left(A_{\epsilon_m\wedge\tau_\ell}\right), \quad \forall x \in E_\ell,\ m < n. \tag{29}$$

Proof. By (24) and (25)

$$\begin{aligned} E_x\left(\sup_{t\ge\epsilon_m} |M^{n,\ell}_t - M^{m,\ell}_t|\right) &\le E_x\left(|M^{n,\ell}_{\epsilon_m}|\right) \\ &\le E_x\left(\left(M^{n,\ell}_{\epsilon_m}\right)^2\right) \le E_x\left(A_{\epsilon_m\wedge\tau_\ell}\right). \end{aligned}$$

Now, for $t \in [\epsilon_m, \tau_\ell(\omega))$,

$$M^{n,\ell}_t(\omega) = M^{n,\ell+1}_t(\omega) = M_{t-\epsilon_n}\left(\theta_{\epsilon_n}\omega\right),$$

and hence we conclude that the quantity inside the braces of the left side of (29) equals $\sup_{t\in[\epsilon_m,\tau_\ell)} |M^{n,\ell}_t - M^{m,\ell}_t|$. □

3°. By virtue of the finiteness (21), we can choose a sequence

$$n_0(x) < n_1(x) < n_2(x) < \cdots < n_\ell(x) < \cdots$$

by

$$\begin{aligned} &n_0(x) = 1 \\ &n_\ell(x) = \min\left\{m > n_{\ell-1}(x) : E_x\left(A_{\epsilon_m\wedge\tau_\ell}\right) < 2^{-\ell}\right\}. \end{aligned} \tag{30}$$

$n_\ell(x)$ is measurable in $x \in X$. We let

$$\tilde{M}_t^\ell(\omega) = M_t^{n_\ell(X_0(\omega)),\ell}(\omega). \tag{31}$$

Then we have

$$E_x\left(\sup_{t\in[\epsilon_{n_\ell(x)},\tau_\ell)} |\tilde{M}_t^{\ell+1} - \tilde{M}_t^\ell|\right) \le 2^{-\ell}, \tag{32}$$

because the left side is equal to

$$E_x\left(\sup_{t\in[\epsilon_{n_\ell(x)},\tau_\ell)} |M_t^{n_{\ell+1}(x),\ell+1} - M_t^{n_\ell(x),\ell}|\right)$$

which is not greater than $E_x\left(A_{\epsilon_{n_\ell(x)}\wedge\tau_\ell}\right) < 2^{-\ell}$ on account of the preceding lemma.

The estimate (32) combined with the relation (17) leads us to

$$P_x(\tilde{\Lambda}) = 1, \quad \forall x \in X, \tag{33}$$

where

$$\begin{aligned}\tilde{\Lambda} =\{\, \omega \in \Lambda_0 : \tilde{M}_t^\ell(\omega) &\text{ converges as } \ell \to \infty \text{ uniformly} \\ &\text{on each compact subinterval of } (0,\zeta(\omega))\,\}.\end{aligned} \tag{34}$$

Let us put for $\omega \in \tilde{\Lambda}$

$$M_t(\omega) = \begin{cases} \lim_{\ell\to\infty} \tilde{M}_t^\ell(\omega), & 0 < t < \zeta(\omega); \\ 0, & t = 0. \end{cases} \tag{35}$$

$M_t(\omega)$ is cadlag on $(0,\zeta(\omega))$ for each $\omega \in \tilde{\Lambda}$.

Lemma 4. (1) $\Lambda \subset \tilde{\Lambda}$ *and*

$$M_t(\omega) = M_t^{[u]}(\omega), \quad \forall \omega \in \Lambda,\ \forall t \in [0,\zeta(\omega)). \tag{36}$$

(2) *Fix an* ℓ_0. *Then for any* $x \in E_{\ell_0}$

$$E_x\left(M_{t\wedge\tau_{\ell_0}}^2\right) \le E_x\left(A_{t\wedge\tau_{\ell_0}}\right) \tag{37}$$

$$E_x\left(M_{t\wedge\tau_{\ell_0}}\right) = 0. \tag{38}$$

(3) *The following identities hold* P_x*-a.s.* $\forall x \in X$:

$$M_{t+s}(\omega) = M_s(\omega) + M_t(\theta_s\omega), \quad 0 \le t, s,\ t+s < \zeta(\omega), \tag{39}$$

$$\lim_{t\downarrow 0} M_t(\omega) = 0. \tag{40}$$

Proof. (1) If $\omega \in \Lambda$ and $0 < t < \zeta(\omega)$, then, by (17) and the additivity of $M_t^{[u]}(\omega)$, we have

$$M_t^{n,\ell}(\omega) = M_{t-\epsilon_n}^{[u]}(\theta_{\epsilon_n}\omega) = M_t^{[u]}(\omega) - M_{\epsilon_n}(\omega)$$

for large ℓ and n. Hence $\tilde{M}_t^{\ell}(\omega) = M_t^{[u]}(\omega) - M_{\epsilon_{n_\ell(X_0)}}^{[u]}(\omega)$ for a sufficiently large ℓ, which tends to $M_t^{[u]}(\omega)$ uniformly on each compact subinterval of $(0, \zeta(\omega))$.

(2) When τ_{ℓ_0} is positive and ℓ is large enough, $\epsilon_{n_\ell(X_0)} < t \wedge \tau_{\ell_0} < \tau_\ell$ and $\tau_{\ell_0}(\omega) = \epsilon_{n_\ell(X_0)} + \tau_{\ell_0}\left(\theta_{\epsilon_{n_\ell(X_0)}}\omega\right)$. Accordingly

$$\begin{aligned}\tilde{M}_{t\wedge\tau_{\ell_0}}(\omega)^{\ell} &= M_{t\wedge\tau_{\ell_0}-\epsilon_{n_\ell(X_0)}}^{[u]}\left(\theta_{\epsilon_{n_\ell(X_0)}}\omega\right) \\ &= M_{(t-\epsilon_{n_\ell(X_0)})\wedge\tau_{\ell_0}}^{[u]}\left(\theta_{\epsilon_{n_\ell(X_0)}\wedge\tau_{\ell_0}}\omega\right) = M_t^{n_\ell(X_0),\ell_0}.\end{aligned}$$

Hence

$$M_{t\wedge\tau_{\ell_0}} = \lim_{\ell\to\infty} \tilde{M}_{t\wedge\tau_{\ell_0}}^{\ell} = \lim_{\ell\to\infty} M_t^{n_\ell(X_0),\ell_0}$$

P_x-a.s. for $x \in E_\ell$. On the other hand, we see from the first equality of (24) that $E_x\left(\left(M_t^{n_\ell(x),\ell_0}\right)^2\right)$ is bounded by $E_x\left(A_{t\wedge\tau_{\ell_0}}\right)$ uniformly in ℓ. Hence we get the inequality (37) by Fatou's lemma. We also obtain the equality (38) by the second equality of (24) and the P_x-uniform integrability of $M_t^{n_\ell(X_0),\ell_0}$.

(3) By (26), we have

$$\tilde{M}_{t+s}^{\ell}(\omega) = \tilde{M}_s^{\ell}(\omega) + M_{t\wedge\tau_\ell}^{[u]}(\theta_{s\wedge\tau_\ell}\omega)$$

for $t, s > 0$ and sufficiently large ℓ. By letting ℓ tend to infinity and by taking (19) and (36) into account, we arrive at the additivity (39) for M_t holding P_x-a.s. for every $x \in X$.

On account of (17) and the observation made right after the relation (28), the identity (39) leads us to

$$M_{(t+s)\wedge\tau_{\ell_0}}(\omega) = M_{s\wedge\tau_{\ell_0}}(\omega) + M_{t\wedge\tau_{\ell_0}}\left(\theta_{s\wedge\tau_{\ell_0}}\omega\right) \tag{41}$$

holding P_x-a.s. for every $x \in E_{\ell_0}$. This together with (37) and (38) implies that $\left\{M_{t\wedge\tau_{\ell_0}}\right\}_{t>0}$ is a square integrable P_x-martingale. Consequently the limit $\lim_{t\downarrow 0} M_{t\wedge\tau_{\ell_0}}$ exists and it must be zero P_x-a.s. for $x \in E_{\ell_0}$ in virtue of the estimate (37) and Fatou's lemma. Since ℓ_0 is arbitrary, we see that (40) is valid P_x-a.s. for every $x \in X$. □

4°. We finally set

$$\hat{\Lambda} = \{\omega \in \tilde{\Lambda} : \text{ (39) and (40) are valid}\}. \tag{42}$$

By virtue of (19) and the preceding lemma, we can conclude that

$$\Lambda \subset \hat{\Lambda}, \quad \theta_t\hat{\Lambda} \subset \hat{\Lambda}, \quad \forall t \geq 0, \quad P_x(\hat{\Lambda}) = 1, \quad \forall x \in X.$$

We complete the definition (35) of M_t by setting

$$M_t(\omega) = \begin{cases} M_{\zeta(\omega)-} & \omega \in \Lambda \quad t \geq \zeta(\omega) \\ 0 & \omega \in \hat{\Lambda} - \Lambda \quad t \geq \zeta(\omega). \end{cases} \tag{43}$$

We further extend the function u to X_Δ by setting $u(\Delta) = 0$ and let, for $\omega \in \hat{\Lambda}$ and $t \geq 0$,

$$A_t^{[u]}(\omega) = u(X_t(\omega)) - u(X_0)) \tag{44}$$

$$N_t(\omega) = A_t^{[u]}(\omega) - M_t(\omega). \tag{45}$$

At this stage, we make the following consideration on the common defining set Λ and exceptional polar set N for $M^{[u]}$ and $N^{[u]}$ we have started with. Since u is a finite valued finely continuous function belonging to the Dirichlet space $\mathcal{F}$, $A_t^{[u]}$ is a finite cadlag AF and

$$\lim_{t' \uparrow t} A_{t'}^{[u]} = u(X_{t-}) - u(X_0) \quad P_x\text{-a.s.}$$

for q.e. $x \in X$ ([2; Lemma 4.2.2]). However under the present assumption (10) of no killing inside

$$X_{\zeta-} = \Delta \quad P_x\text{-a.s. on } \{\zeta < \infty\},$$

and consequently $A_t^{[u]}$ is continuous at $t = \zeta$ P_x-a.s. on $\{\zeta < \infty\}$ for q.e. x. Since $N_t^{[u]}(\omega)$ is continuous on $[0, \infty)$ and the decomposition (1) holds, we could have assumed by choosing Λ and N appropriately from the beginning that

$$A_t^{[u]}(\omega) = M_t^{[u]}(\omega) + N_t^{[u]}(\omega), \quad \forall \omega \in \Lambda, \ \forall t \geq 0, \tag{46}$$

$$A_t^{[u]}(\omega) = A_{\zeta(\omega)-}^{[u]}(\omega), \ M_t^{[u]}(\omega) = M_{\zeta(\omega)-}^{[u]}(\omega), \quad \forall \omega \in \Lambda, \ \forall t \in [\zeta(\omega), \infty). \tag{47}$$

We assume that (46) and (47) are fulfilled already.

By the next lemma, we complete the proof of the implication (i) $\Rightarrow$ (ii) of Theorem 1. We notice that the second statement of the next lemma says that the constructed functionals M and N are not only strict versions of $M^{[u]}$ and $N^{[u]}$ respectively but also their extensions by an enlargement of the defining set from Λ to $\hat{\Lambda}$.

Lemma 5. (1) *M_t is a local AF in the strict sense with defining set $\hat{\Lambda}$. N_t is a local CAF in the strict sense with defining set $\hat{\Lambda}$.*

(2) *It holds that*

$$M_t(\omega) = M_t^{[u]}(\omega), \ N_t(\omega) = N_t^{[u]}(\omega), \ \forall \omega \in \Lambda, \quad \forall t \geq 0. \tag{48}$$

(3) *Equality takes place in* (37):

$$E_x\left(M_{t\wedge\tau_{\ell_0}}^2\right) = E_x\left(A_{t\wedge\tau_{\ell_0}}\right), \quad \forall x \in E_{\ell_0}. \tag{49}$$

Proof. The first equality of (48) follows from (36), (43) and (47). The second is immediate from (45) and (46). The first statement in (1) is clear from the construction of M_t and $\hat{\Lambda}$. $N_t(\omega)$ defined by (45) is therefore again a local AF in the strict sense with defining set $\hat{\Lambda}$.

To see the continuity of N_t, we put for $\omega \in \hat{\Lambda}$,

$$N_t^{n,\ell}(\omega) = N^{[u]}_{(t-\epsilon_n)\wedge\tau_\ell}(\theta_{\epsilon_n\wedge\tau_\ell}\omega)$$
$$A_t^{n,\ell}(\omega) = A_{(t-\epsilon_n)\wedge\tau_\ell}(\theta_{\epsilon_n\wedge\tau_\ell}\omega).$$

Since $\theta_{\epsilon_n\wedge\tau_\ell}\omega \in \Lambda$ for a sufficiently large ℓ and all n, (46) implies

$$A_t^{n,\ell}(\omega) = M_t^{n,\ell}(\omega) + N_t^{n,\ell}(\omega). \tag{50}$$

If $0 < t < \zeta(\omega)$, then by (17) we have

$$A_t^{n,\ell}(\omega) = u(X_t(\omega)) - u(X_{\epsilon_n}(\omega))$$

for a large enough ℓ and n. Hence $A_t^{n_\ell(X_0),\ell}(\omega)$ converges as $\ell \to \infty$ to $A_t^{[u]}(\omega)$ uniformly on each compact subinterval of $(0, \zeta(\omega))$ and we see from (50) that $N_t(\omega) = \lim_{\ell\to\infty} N_t^{n_\ell(X_0),\ell}(\omega)$ must be continuous on $(0, \zeta(\omega))$.

To show the equality (49), we make use of the first equality of (22), the relations (41) and (48) and the polarity of N to obtain for $0 < \delta < t$ $E_x\left(\left(M_{t\wedge\tau_{\ell_0}} - M_{\delta\wedge\tau_{\ell_0}}\right)^2\right) = E_x\left(E_{X_{\delta\wedge\tau_{\ell_0}}}\left(\left(M^{[u]}_{(t-\delta)\wedge\tau_\ell}\right)^2\right); X_{\delta\wedge\tau_{\ell_0}} \in X - N\right) =$ $E_x\left(E_{X_{\delta\wedge\tau_{\ell_0}}}\left(A_{(t-\delta)\wedge\tau_\ell}\right)\right) = E_x\left(A_{t\wedge\tau_{\ell_0}} - A_{\delta\wedge\tau_{\ell_0}}\right)$. It then suffices to let $\delta \downarrow 0$ in the above equality by observing that the following estimate holds by the inequality(37):

$$\left|\left[E_x\left(\left(M_{t\wedge\tau_{\ell_0}} - M_{\delta\wedge\tau_{\ell_0}}\right)^2\right)\right]^{\frac{1}{2}} - \left[E_x\left(\left(M_{t\wedge\tau_{\ell_0}}\right)^2\right)\right]^{\frac{1}{2}}\right|$$
$$\leq \left[E_x\left(\left(M_{\delta\wedge\tau_{\ell_0}}\right)^2\right)\right]^{\frac{1}{2}} \leq \left[E_x\left(A_{\delta\wedge\tau_{\ell_0}}\right)\right]^{\frac{1}{2}}.$$

□

Proof of the implication (ii) ⇒ (i). Let M, N and A be functionals satisfying condition (ii) of Theorem 1. Then M and N are strict versions of $M^{[u]}$ and $N^{[u]}$ respectively. We can further see that A is a strict version of the quadratic variation $\langle M^{[u]}\rangle \in \mathbf{A}_c^+$ of $M^{[u]}$ in the following way. The relation (13) combined with the equation (28) for M and A holding P_x-a.s. $\forall x \in E_\ell$ implies that $A_{t\wedge\tau_\ell}$ is the quadratic variation of the martingale $M_{t\wedge\tau_\ell}$ with respect to P_x for any $x \in E_\ell$. On the other hand, by the optional sampling theorem, $\langle M^{[u]}\rangle_{t\wedge\tau_\ell}$ is the quadratic variation of $M^{[u]}_{t\wedge\tau_\ell}$ with respect to P_x for q.e. $x \in E_\ell$. Since ℓ is arbitrary, A must be a strict version of $\langle M^{[u]}\rangle$.

Hence $\mu_{\langle u\rangle}$ is the Revuz measure of $A \in \mathbf{A}_{c1}^+$ and belongs to the class S_1. □

Proof of the uniqueness of the decomposition (12). Consider two triplets $M^{(i)}$, $N^{(i)}$, $A^{(i)}$, $i = 1, 2$, satisfying condition (ii) of Theorem 1. Since both $M^{(1)}$ and $M^{(2)}$ are versions of $M^{[u]}$, they coincide P_x-a.s. for all $x \in X$ except for a polar set N. We can then proceed along the same lines as in the preceding proof of (49): for any $x \in E_\ell$

$$E_x\left(\left[\left(M^{(1)}_{t\wedge\tau_\ell} - M^{(2)}_{t\wedge\tau_\ell}\right) - \left(M^{(1)}_{\delta\wedge\tau_\ell} - M^{(2)}_{\delta\wedge\tau_\ell}\right)\right]^2\right)$$

$$= E_x\left(E_{X_{\delta\wedge\tau_\ell}}\left(\left(M^{(1)}_{(t-\delta)\wedge\tau_\ell} - M^{(2)}_{(t-\delta)\wedge\tau_\ell}\right)^2\right);\ X_{\delta\wedge\tau_\ell} \in X - N\right) = 0,$$

$$E_x\left(\left(M^{(1)}_{\delta\wedge\tau_\ell} - M^{(2)}_{\delta\wedge\tau_\ell}\right)^2\right) \le 2E_x\left(A^{(1)}_{\delta\wedge\tau_\ell}\right) + 2E_x\left(A^{(2)}_{\delta\wedge\tau_\ell}\right).$$

We can let $\delta \downarrow 0$ in the above identity to obtain

$$E_x\left(\left(M^{(1)}_{t\wedge\tau_\ell} - M^{(2)}_{t\wedge\tau_\ell}\right)^2\right) = 0.$$

Since ℓ is arbitrary, we can conclude from this that

$$P_x\left(M_t^{(1)} = M_t^{(2)} \quad \forall t < \zeta\right) = 1, \quad \forall x \in X.$$

□

Notice that, if $\mathbf{M}$ is conservative, then the property (11) is trivially satisfied and the relative compactness requirement for E_ℓ is unnecessary in the above proof. The proof of Theorem 1 is completed.

4. Proof of Theorem 2

The proof of Theorem 2 is exactly the same as that of Theorem 1 except for the following necessary modifications in proving the implication (i) $\Rightarrow$ (ii):

1. Instead of [2; Theorem 5.2.2], we can now start with [2; Theorem 5.5.1] which provides us with $M^{[u]} \in \overset{\circ}{\mathcal{M}}_{\mathrm{loc}}$ and $N^{[u]} \in \mathcal{N}_{c,\mathrm{loc}}$ for $u \in \mathcal{F}_{\mathrm{loc}}$.
2. In place of (22), $M^{[u]}$ now satisfies for relatively compact E_ℓ

$$E_x\left(\left(M^{[u]}_{t\wedge\tau_\ell}\right)^2\right) = E_x\left(A_{t\wedge\tau_\ell}\right) < \infty, \quad E_x\left(M^{[u]}_{t\wedge\tau_\ell}\right) = 0, \quad \forall x \in X - N.$$

3. Instead of (43), $M_t(\omega)$ is defined to be 0 for $\omega \in \hat{\Lambda}, t \ge \zeta(\omega)$. The identities (48) now hold only for $t < \zeta$.

The relative compactness requirement for E_ℓ can not be removed in general because Theorem 2 is dealing with u in $\mathcal{F}_{\mathrm{loc}}$.

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Probability Theory and Related Fields

Construction and decomposition of reflecting diffusions on Lipschitz domains with Hölder cusps

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Summary. We consider a d-dimensional Euclidean domain D whose boundary is Lipschitz continuous but admits locally finite number of outward or inward Hölder cusp points. Using a method of Stampacchia and Moser for PDE, we first construct a conservative diffusion process on the Euclidean closure of D possessing a strong Feller resolvent and associated with a second order uniformly elliptic differential operator of divergence form with measurable coefficients a_{ij}. The sample path of the constructed diffusion can be uniquely decomposed as a sum of a martingale additive functional and an additive functional locally of zero energy. The second additive functional will be proved to be of bounded variation with a Skorohod type expression whenever a_{ij} is weakly differentiable and the Hölder exponent at each outward cusp boundary point is greater than $1/2$ regardless the dimension d.

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1 Introduction

Let D be a domain in the d-dimensional Euclidean space R^d and $\overline{D} = D \cup \partial D \subset R^d$ be its closure. The d-dimensional Lebesgue measure is denoted by $m = m(dx)$ or simply by dx. Given measurable functions $a_{ij}(x)$, $1 \le i, j \le d$, on D such that

$$a_{ij} = a_{ji}, \quad \Lambda^{-1}|\xi|^2 \le \sum_{1 \le i,j \le d} a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2, \ x \in D, \ \xi \in R^d, \tag{1.1}$$

for some constant $\Lambda \ge 1$, we consider a Dirichlet form $\mathscr{E}$ on $L^2(D) = L^2(D; m)$ defined by

$$\mathscr{D}[\mathscr{E}] = H^1(D), \quad \mathscr{E}(u,v) = \int_D \sum_{1\le i,j\le d} a_{ij}(x)\partial_i u(x)\partial_j v(x)dx, \; u,v \in H^1(D), \tag{1.2}$$

where $H^1(D) = \{u \in L^2(D) : \partial_i u \in L^2(D), 1 \le i \le d\}$ the Sobolev space of order 1. Let $\{T_t, t > 0\}$ be the strongly continuous semigroup of Markovian symmetric operators on $L^2(D)$ associated with the Dirichlet form $\mathscr{E}$.

We denote by $C_0(D)$ [resp. $B_0(D)$] the space of continuous functions [resp. bounded measurable functions] on D with compact support [resp. vanishing outside a bounded set]. We further denote by $C(\overline{D})$ [resp. $C_0(\overline{D})$] the space of bounded continuous functions on $\overline{D}$ [resp. the restrictions to $\overline{D}$ of functions in $C_0(R^d)$]. Suppose that the Dirichlet form $\mathscr{E}$ is *regular* on $L^2(\overline{D})$ rather than on $L^2(D)$ in the sense that $H^1(D) \cap C_0(\overline{D})$ is dense in the space $H^1(D)$. This is the case for instance when the domain D is of class C in the sense that ∂D is locally expressible as a graph of a continuous function of $d-1$ variables ([16]). According to general theorems ([13]), there exists then a conservative diffusion process $\mathbf{M} = (X_t, P_x)$ on $\overline{D}$ *associated with the Dirichlet form* $\mathscr{E}$ in the sense that the transition probability $p_t(x,E) = P_x(X_t \in E)$ of $\mathbf{M}$ satisfies that

$$p_t f \text{ is a version of } T_t f \text{ for any } f \in B_0(D). \tag{1.3}$$

However we are now concerned with a highly non-trivial problem of constructing the process $\mathbf{M}$ on $\overline{D}$ with a strong Feller resolvent:

$$G_\lambda(B_0(D)) \subset C(\overline{D}), \tag{1.4}$$

which particularly implies the absolute continuity of the transition probability:

$$p_t(x,\cdot) \prec m \quad \text{for any } t > 0 \text{ and } x \in \overline{D}. \tag{1.5}$$

If both the conditions (1.3) and (1.5) are fulfilled, then we can invoke a general decomposition theorem in [13] of additive functionals (AF's in abbreviation) in the strict sense to conclude that the sample path $X_t = (X_t^1, \cdots, X_t^d)$ of $\mathbf{M}$ admits the unique decomposition

$$X_t^i - X_0^i = M_t^i + N_t^i, \quad 1 \le i \le d, \quad P_x\text{−a.s. for any } x \in \overline{D}, \tag{1.6}$$

where M_t^i are martingale additive functionals (MAF's in abbreviation) in the strict sense with covariations

$$\langle M^i, M^j\rangle_t = 2\int_0^t a_{ij}(X_s)ds, \quad 1 \le i,j \le d, \quad P_x\text{−a.s. for any } x \in \overline{D}, \tag{1.7}$$

and N_t^i are continuous additive functionals (CAF's in abbreviation) in the strict sense locally of zero energy. N_t^i are not necessarily of bounded variation (on each finite time interval) but locally of zero quadratic variation in a certain sense ([13]).

Natural questions arise:

(I) Under what condition on the domain D, there exists a conservative diffusion process $\mathbf{M} = (X_t, P_x)$ on $\overline{D}$ satisfying (1.3) and (1.5) ?
(II) Under what additional conditions on the domain D and coefficients a_{ij}, the second terms N_t^i of X_t^i are of bounded variation P_x-a.s. for any $x \in \overline{D}$?

When D is a general bounded Lipschitz domain and $a_{ij} = \frac{1}{2}\delta_{ij}, 1 \le i,j \le d$, Bass and Hsu gave affirmative answer to the both questions (I) and (II) in [2] and [3] respectively. Actually [2] and its refinement [13; Example 5.2.2] gave an explicit expression of N_t^i as

$$N_t^i = \frac{1}{2}\int_0^t n_i(X_s)dL_s, \quad 1 \le i \le d, \quad P_x-\text{a.s. for any } x \in \overline{D}, \tag{1.8}$$

where $\mathbf{n} = (n_1, \cdots, n_d)$ is the inward unit normal vector at the boundary ∂D and L_t is a positive continuous additive functional (a PCAF in abbreviation) in the strict sense associated with the surface measure on ∂D; the local time of X_t on the boundary. In this case, the diffusion $\mathbf{M}$ is called the (normally) *reflecting Brownian motion on* $\overline{D}$ and the decomposition (1.6) with (1.8) is called its *Skorohod representation*. The first process $(M_t^1, \cdots, M_t^d)$ appearing in (1.6) is the standard d-dimensional Brownian motion starting at the origin in this case.

On the other hand, by extending a work of S.R.S.Varadhan and R.J.Williams on an infinite two-dimensional wedge [22], DeBlassie and Toby [7] have formulated under a submartingale problem a normally reflecting Brownian motion on a two-dimensional standard outward cusp domain

$$C = \left\{(x,y) \in R^2 : y \ge |x|^\gamma\right\}, \quad 0 < \gamma < 1,$$

and constructed it from the normally reflecting Brownian motion on the upper half plane by means of a conformal map and a random time change. They have also shown in [8] that the constructed process admits the Skorohod representation if $\gamma > \frac{1}{2}$ but otherwise the process starting at the origin fails to be a semimartingale. By thinking of the direct product of the DeBlassie-Toby reflecting Brownian motion on C with the standard $d-2$-dimensional Brownian motion, we see that $\frac{1}{2}$ is still the critical value of the Hölder exponent for the semi-martingale property of the reflecting Brownian motion on the special Hölder domain $C \times R^{d-2} \subset R^d$.

It is therefore tempting to consider the problem (I) for a general Hölder domain D and further look for a critical value of the Hölder exponent γ with regard to the question (II). In this paper, we do not deal with a most general Hölder domain. However we assume that D is a general (not necessarily bounded) Lipschitz domain allowing locally finite number of outward or inward cusp boundary points with Hölder exponents uniformly bounded away from zero. Our first aim is to give an affirmative answer to the problem (I) (Theorem 2.1 and Theorem 2.2) by employing the PDE methods of Stampacchia and Moser. We then assume that

$$\partial_j a_{ij} \in L^\infty_{loc}(D), \quad 1 \le i,j \le d, \tag{1.9}$$

and give an affirmative answer to the question (II) under the condition that the Hölder exponent at each outward cusp boundary point is greater than $\frac{1}{2}$ regardless the dimension d. Actually an explicit expression of N_t^i using the boundary local time L_t will be derived in this case by invoking an extended version of a general theorem in [13] to characterize N_t^i and by combining the Sobolev inequalities obtained in Sect. 3 with the upper bounds of transition functions due to Carlen-Kusuoka-Stroock [5] (Theorem 2.3).

Furthermore, we shall see that the diffusion process constructed in Theorem 2.2 can be, under the condition that $\partial_j a_{ij} \in L^\infty(D)$, related to a submartingale problem (Theorem 2.4), and accordingly, identified in law with Varadhan-Williams's [resp. DeBlassie-Toby's] normally reflecting Brownian motion when $a_{ij}(x) = \frac{1}{2}\delta_{ij}$ and $\overline{D}$ is a wedge [resp. a cusp C] in R^2.

The present paper is an essential improvement of the previous one [14] where we gave affirmative answers to questions (I) and (II) only under the restriction that the Hölder exponents at cusps are uniformly greater than $\frac{d-1}{d}$, which was technically required in getting a modified Sobolev inequality of Moser's type - a key inequality in our construction of a strong Feller resolvent. This requirement now turns out to be unnecessary thanks to a specific transformation of a standard cusp domain onto a rectangular set exhibited in the last section.

In the next section, we shall formulate a precise condition on the domain D and state main theorems answering the questions (I) and (II). Their proof will be carried out in the subsequent sections.

2 Statement of main theorems

Let F be a real valued function defined on a set E $\left(\subset R^k\right)$ including the origin such that $F(x) = \alpha|x|^\gamma + f(x)$, where $0 < \gamma < 1,\ \alpha \in R$, and f is a k-dimensional Lipschitz continuous function vanishing at the origin. Here $|\cdot|$ denotes the Euclidean norm. In this paper we call such F *a Hölder function* and we denote its Hölder exponent, Hölder constant and Lipschitz constant respectively by

$$\mathrm{Exp}(F) = \gamma, \qquad \mathrm{H\ddot{o}l}(F) = \alpha,$$
$$\mathrm{Lip}(F) = \mathrm{Lip}(f) = \min\left\{K > 0 : |f(x) - f(y)| \le K|x-y|,\ x, y \in E\right\}.$$

For $x = (x_1, \cdots, x_d) \in R^d$, we let $x' = (x_1, \cdots, x_{d-1})$ so that $x = (x', x_d)$.

Let us now consider the following condition (H) on a domain $D \subset R^d$ with $d \ge 2$:

(H) There are four constants $\gamma \in (0,1),\ \delta > 0,\ A \ge 1,\ M > 0$ and a locally finite open covering $\{U_j\}_{j\in J}$ of ∂D satisfying the following properties:

(i) For each $j \in J$, there are a Hölder function F_j of $d-1$ variables and a constant $r_j > \delta$ such that
 F_j is defined on the $d-1$-dimensional ball centered at the origin with radius r_j,
 $\mathrm{Exp}(F_j) \ge \gamma$,

$\text{Höl}(F_j) = 0$, or $1/A \le \text{Höl}(F_j) \le A$, or $-A \le \text{Höl}(F_j) \le -1/A$,
$\text{Lip}(F_j) \le M$,
$U_j \cap D = \{\zeta = (\zeta', \zeta_d) : |\zeta| < r_j,\ F_j(\zeta') < \zeta_d\}$, for some Cartesian coordinate system $\zeta = (\zeta', \zeta_d)$.
(ii) $\partial D \subset \bigcup_{j\in J} \widetilde{U_{j,\delta}}$, where $\widetilde{U_{j,\delta}} = \{x \in U_j : \text{dist}(x, \partial U_j) > \delta\}$.

When D is bounded, condition (H) reduces to a simple one that every point x of ∂D has a neighbourhood U_x such that $\partial D \cap U_x$ is the graph of a Hölder function of $d-1$ variables.

For later convenience, we let

$$\begin{aligned} J_+ &= \{j \in J : \text{Höl}(F_j) > 0\}, \\ J_0 &= \{j \in J : \text{Höl}(F_j) = 0\}, \\ J_- &= \{j \in J : \text{Höl}(F_j) < 0\}. \end{aligned}$$

For $j \in J$, denote by a_j $(\in \partial D)$ the origin of U_j with respect to the coordinate system ζ. a_j is called an *outward* [resp. *inward*] *cusp boundary point* of D if $j \in J_+$ [resp. $j \in J_-$].

In what follows, we work with the Dirichlet form $(\mathcal{E}, H^1(D))$ on $L^2(D)$ given by (1.1) and (1.2). Let $\{G_\lambda, \lambda > 0\}$ be the associated resolvent on $L^2(D)$. It is then Markovian in the sense that $0 \le \lambda G_\lambda f \le 1$ whenever $0 \le f \le 1$ and it is well defined as a bounded linear operator on $L^p(D)$ for any $p \in [1, \infty]$. Denote by $C_\infty(\overline{D})$ the space of those functions in $C(\overline{D})$ vanishing at infinity.

Theorem 2.1 *Assume that a domain $D \subset R^d$ satisfies condition (H). Then G_λ enjoys the following properties :*
(i) $G_\lambda\left(L^2(D) \cap L^p(D)\right) \subset C(\overline{D}), \qquad p > 1 + (d-1)/\gamma.$
(ii) $G_\lambda\left(C_\infty(\overline{D})\right)$ *is a dense subspace of* $C_\infty(\overline{D})$.
(iii) *There is a function $G_\lambda(x, y)$ continuous on $\overline{D} \times \overline{D}$ off diagonal such that*

$$G_\lambda f(x) = \int_{\overline{D}} G_\lambda(x, y) f(y)\, dy, \qquad x \in \overline{D},\ f \in C_\infty(\overline{D}). \tag{2.1}$$

As will be seen in Sect. 4, Theorem 2.1 is still valid under condition (A) stated in Sect. 3. Condition (A) is weaker but less concrete than (H) so that we employ (H) in formulating main theorems.

Theorem 2.1 (i) means that G_λ has a strong Feller property. By virtue of Theorem 2.1 (ii) and the Hille-Yosida theorem, there exists a strongly continuous Markovian semigroup $\{T_t, t > 0\}$ on $C_\infty(\overline{D})$ such that $G_\lambda f = \int_0^\infty e^{-\lambda t} T_t f\, dt,\ f \in C_\infty(\overline{D})$. We have then a Feller transition function by $T_t f(x) = \int_{\overline{D}} p_t(x, dy) f(y)$, which gives rise to a Hunt process (cf. [13; Theorem A.2.2]) $\mathbf{M} = (X_t, P_x)$ on $\overline{D}$ such that

$$P_x(X_t \in A) = p_t(x, A), \qquad t > 0,\ x \in \overline{D},\ A \in \mathcal{B}(\overline{D}).$$

$\mathbf{M}$ is associated with the Dirichlet form $(\mathcal{E}, H^1(D))$ of (1.2) since the resolvent G_λ is. Since $G_\lambda\left(C_0(\overline{D})\right)$ is dense in the Dirichlet space, Theorem 2.1

(ii) implies that the Dirichlet form $\mathscr{E}$ is regular. Therefore we can apply general theorems in [13] to the associated pair $\mathscr{E}$ and $\mathbf{M}$. In particular, $p_t(x,\cdot)$ is absolutely continuous because $G_\lambda(x,\cdot)$ is ([13; Theorem 4.2.4]). Since $\mathscr{E}$ has the strong local property and a_{ij} are uniformly bounded, we can invoke [13; Theorem 4.5.4]) and [13; Theorem 5.7.2, Example 5.7.1] to conclude that $\mathbf{M}$ is a conservative diffusion process on $\overline{D}$. Summing up what has been mentioned, we get

Theorem 2.2 *Under condition (H), there exists a conservative diffusion process* $\mathbf{M} = (X_t, P_x)$ *on* $\overline{D}$ *with resolvent* G_λ *of Theorem 2.1.* $\mathbf{M}$ *is associated with the Dirichlet form given by (1.1) and (1.2), and the transition function* $p_t(x,\cdot)$ *of* $\mathbf{M}$ *satisfies (1.3) and (1.5).*

We next formulate a decomposition of the sample path of $\mathbf{M}$ and its Skorohod representation.

Theorem 2.3 *Consider a domain* $D \subset R^d$ *satisfying condition (H).*
(i) *The sample path* $X_t = (X_t^1, \cdots, X_t^d)$ *of the conservative diffusion process* $\mathbf{M}$ *on* $\overline{D}$ *constructed in Theorem 2.2 admits a unique decomposition (1.6) with MAF's* M_t^i *in the strict sense satisfying (1.7) and CAF's* N_t^i *in the strict sense locally of zero energy.*
(ii) *Assume condition (1.9) for* a_{ij}. *We also require the condition that*

$$\mathrm{Exp}(F_j) > \frac{1}{2}, \qquad j \in J_+, \tag{2.2}$$

for the domain D. *Then* N_t^i *has the following representation :*

$$\begin{aligned} N_t^i = \sum_{j=1}^d \int_0^t \left(\partial_j a_{ij}\right)(X_s)\, ds + \sum_{j=1}^d \int_0^t a_{ij}(X_s)\, n_j(X_s)\, dL_s, \\ 1 \le i \le d,\ t \ge 0,\ P_x - a.s. \text{ for any } x \in \overline{D}, \end{aligned} \tag{2.3}$$

where L_t *is a unique PCAF in the strict sense with Revuz measure being the surface measure on* ∂D.

Note that (2.3) reduces to (1.8) when $a_{ij} = \frac{1}{2}\delta_{ij}$. The above three theorems extend those results of R.F.Bass and P.Hsu in [2] and [3] formulated for a general bounded Lipschitz domain D and for $a_{ij} = \frac{1}{2}\delta_{ij}$.

Let us denote by Ξ_+ the set of all outward cusp boundary points. $C_b^2\left(\overline{D}\right)$ will stand for the set of twice continuously differentiable functions on R^d that are together with their first and second partial derivatives bounded on $\overline{D}$.

Theorem 2.4 *Under condition (H) for the domain and the assumption that*

$$\partial_i a_{ij} \in L^\infty(D), \qquad 1 \le i, j \le d, \tag{2.4}$$

the conservative diffusion process $\mathbf{M} = (X_t, P_x)$ *of Theorem 2.2 enjoys the following properties : for each* $x \in \overline{D}$, *1.* $P_x\,(X_0 = x) = 1,$

2. $f(X_t) - \int_0^t \sum_{i,j=1}^d \left[\partial_i \left(a_{ij}\,\partial_j f\right)\right](X_s)\,ds$ *is a* P_x*-submartingale,*

whenever $f \in C_b^2\left(\overline{D}\right)$*,* f *is constant in a neighbourhood of* Ξ_+ *and*

$$\sum_{i,j=1}^d \partial_i f(x)\,a_{ij}(x)\,n_j(x) \geq 0 \qquad \sigma\text{-a.e. on } \partial D, \tag{2.5}$$

3. $E_x\left[\int_0^\infty I_{\Xi_+}(X_s)\,ds\right] = 0.$

When $d = 2$, $a_{ij}(x) = \frac{1}{2}\delta_{ij}$ and $\overline{D} = C$ the standard outward cusp domain, R.D. DeBlassie and E.H. Toby [7] have shown the existence and the uniqueness of the corresponding submartingale problem for a probability measure P_x on $\Omega = \{\,\omega : \omega$ is a continuous function from $[0, \infty)$ into $C\,\}$: for each fixed $x \in C$,

1. $P_x\,(\omega(0) = x) = 1,$
2. $f(\omega(t)) - \frac{1}{2}\int_0^t \Delta f\left(\omega(s)\right)ds$ is a P_x-submartingale whenever $f \in C_b^2(C)$, f is constant in a neighbourhood of the origin and $\nabla f(x)\cdot \mathbf{n}(x) \geq 0$ on ∂C,
3. $E_x\left[\int_0^\infty I_0\left(\omega(s)\right)ds\right] = 0.$

Hence, by virtue of Theorem 2.4, the diffusion process of Theorem 2.2 coincides in law with DeBlassie-Toby's one in [7] in this special case.

In exactly the same way, we see that, when $d = 2$, $a_{ij}(x) = \frac{1}{2}\delta_{ij}$ and $\overline{D}$ is a wedge $\{\theta : 0 \leq \theta \leq \xi\} \subset R^2$ for a fixed $\xi \in (0, 2\pi)$, the diffusion process of Theorem 2.2 is identical in law with Varadhan-R.Williams's normally reflecting Brownian motion [22].

3 L^p-estimate, local estimate and Harnack inequality

In this and the next sections, we shall work under another condition (A) on a domain $D \subset R^d$ which will be seen to be more general than (H) (Proposition 4.1). In this section, we derive some estimates for harmonic solutions of equations associated with $\left(\mathscr{E}, H^1(D)\right)$ under condition (A).

In order to state condition (A), we employ the following notations:

$$\begin{aligned}
B(a,\rho) &= \{x \in R^d : |x-a| < \rho\},\\
B(\rho) &= B(0,\rho),\\
B_+(\rho) &= \{(x', x_d) \in B(\rho) : x_d > 0\},\\
C_\gamma(\rho) &= \{(x', x_d) \in B(\rho) : |x'|^\gamma < x_d\},\\
Q_\gamma(\rho) &= \{(x', x_d) \in B(\rho) : -|x'|^\gamma < x_d\},
\end{aligned}$$

for $a \in R^d,\ \rho > 0,\ \gamma \in (0,1)$. For a Lipschitz mapping Φ from a set $E \subset R^k$ into R^l such that $|\Phi(x) - \Phi(y)| \le K|x-y|,\ x,y \in E$, for some constant $K > 0$, we also denote by $\mathrm{Lip}(\Phi)$ the smallest constant K of this property.

We now state condition (A) on a domain $D \subset R^d$:

(A) The following properties hold for an at most countable index set I, a constant $\gamma^* \in (0,1)$ and positive constants $\rho^*,\ r^*,\ M^*$:

(i) There are a point $a_k \in \partial D$ and its neighbourhood V_k associated with each $k \in I$ such that

(i-1) $D \cap V_k \cap V_l = \emptyset, \quad k,\ l \in I,\ k \neq l$;

(i-2) there are a constant $\gamma_k \in [\gamma^*, 1)$ and a one to one mapping Φ_k from $B(\rho^*)$ onto V_k with $\Phi_k(0) = a_k,\ V_k \cap D$ equals either $\Phi_k\left(C_{\gamma_k}(\rho^*)\right)$ or $\Phi_k\left(Q_{\gamma_k}(\rho^*)\right)$, $\mathrm{Lip}(\Phi_k)\ \le M^*$, $\mathrm{Lip}(\Phi_k^{-1}) \le M^*$.

(ii) For any $a \in \partial D \setminus \bigcup_{k\in I} V_k$, there are its neighbourhood W_a and a one to one mapping Ψ_a from $B(r^*)$ onto W_a such that $\Psi_a(0) = a,\ \Psi_a(B_+(r^*)) = W_a \cap D,\ \mathrm{Lip}(\Psi_a) \le M^*,\ \mathrm{Lip}(\Psi_a^{-1}) \le M^*$.

In our previous paper [14], we considered the same condition as above, but we assumed $\gamma^* > (d-1)/d$. Further we did not consider the case $V_k \cap D = \Phi_k\left(Q_{\gamma_k}(\rho^*)\right)$, namely, we assumed in [14] that every a_k is an outward cusp boundary point but not an inward one. Under those assumptions, we got the same estimates as in this section following the PDE argument due to Stampacchia [18] and Moser [17]. The PDE argument is based on a Sobolev inequality of Moser's type formulated in Proposition 3.1 below. As will be proved in the last section, we need not the previous assumption $\gamma^* > (d-1)/d$ for the validity of Proposition 3.1. Once it is established, we can follow the PDE argument developed in [14] without any change so that we shall state the results of this section omitting the proof and only referring to the corresponding results in [14].

In the rest of this section, we assume (A).

First of all, we note the following easily verifiable observation : if ψ is a one to one mapping from an open set $U \subset R^d$ onto an open set in R^d with $\mathrm{Lip}\left(\psi^{-1}\right) \le M$ and if $B(\widetilde{a}, r) \subset U$ and $\psi(\widetilde{a}) = a$, then

$$B(a, r/M) \subset \psi\left(B(\widetilde{a}, r)\right). \tag{3.1}$$

This observation particularly leads us to the following property of the domain D (see [14; Lemma 3.1]). We set

$$\begin{aligned} I_C &= \left\{k \in I : V_k \cap D = \Phi_k\left(C_{\gamma_k}(\rho^*)\right)\right\}, \\ I_Q &= \left\{k \in I : V_k \cap D = \Phi_k\left(Q_{\gamma_k}(\rho^*)\right)\right\}. \end{aligned}$$

a_k for $k \in I_C$ [resp. I_Q] may be called an outward [resp. inward] cusp boundary point. A collection of open sets is said to have a finite intersection property if there exists an integer M such that any subcollection of cardinality greater than M has an empty intersection.

Lemma 3.1 *For any $\rho \in (0, \rho^*]$, there exist positive constants r_ρ and m_ρ for which D satisfies the following : For every $a \in \partial D^\#_\rho \equiv \partial D \setminus \bigcup_{k\in I} \Phi_k\,(B(\rho))$, there are a neighbourhood $V_{\rho,a}$ of a and a one to one mapping $\psi_{\rho,a}$ from $B\left(r_\rho\right)$ onto $V_{\rho,a}$ such that $\psi_{\rho,a}(0) = a$, $\psi_{\rho,a}\left(B_+\left(r_\rho\right)\right) = V_{\rho,a} \cap D$, $\mathrm{Lip}(\psi_{\rho,a}) \le m_\rho$, $\mathrm{Lip}(\psi^{-1}_{\rho,a}) \le m_\rho$.*

Furthermore, for any $r \in (0, r_\rho]$ we have a subset $A \subset \partial D^\#_\rho$ and a positive constant $\eta = \eta(r)$ such that $\{\psi_{\rho,a}(B_+(r))\}_{a\in A}$ has a finite intersection property, and for every $b \in \partial D$, the set $B(b,\eta) \cap D$ is contained in one of the following sets: $\Phi_k\left(C_{\gamma_k}(\rho)\right)$ for $k \in I_C$, $\Phi_k\left(Q_{\gamma_k}(\rho)\right)$ for $k \in I_Q$, $\psi_{\rho,a}\,(B_+(r))$ for $a \in A$.

In the following we set

$$\begin{aligned} C^*_k(\rho) &= \Phi_k\left(C_{\gamma_k}(\rho)\right), && k \in I_C, \\ Q^*_k(\rho) &= \Phi_k\left(Q_{\gamma_k}(\rho)\right), && k \in I_Q, \\ B^*_a(r) &= \psi_{\rho,a}\,(B_+(r)), && a \in \partial D^\#_\rho, \end{aligned}$$

for $0 < \rho \le \rho^*$, $0 < r \le r_\rho$. Since a Sobolev inequality of Moser's type formulated in Lemma 2 in [17] is valid for $u \in H^1\,(B_+(r))$, we immediately obtain by means of the map $\psi_{\rho,a}$ in Lemma 3.1 that for any $\rho \in (0, \rho^*]$, $\kappa \in (0,1]$, $q \in [2, 2d/(d-2)]$ ($q \in [2,\infty)$ if $d = 2$) there is a positive constant $C_1 = C_1(\rho,\kappa,q)$ such that

$$\left(\int_{B^*_a(r)} |u|^q dx\right)^{1/q} \le C_1\, r^{d\left(\frac{1}{q}-\frac{1}{2}\right)} \left\{\int_N |u|^2 dx + r^2 \sum_{i=1}^d \int_{B^*_a(r)} |\partial_i u|^2 dx\right\}^{1/2}, \tag{3.2}$$

for $u \in H^1\left(B^*_a(r)\right)$, $N \subset B^*_a(r)$ with $|N| \ge \kappa |B^*_a(r)|$, $0 < r \le r_\rho$, and $a \in \partial D^\#_\rho$. Here $|E|$ denotes the Lebesgue measure for measurable sets E. (C_1 and the other constants C_2, C_3 etc. below also depend on d, ρ^*, r^*, M^* and in some cases on γ^* and Λ. However we omit indicating them.)

Actually (3.2) also holds for $u \in H^1\left(C^*_k(\rho)\right)$ and $u \in H^1\left(Q^*_k(\rho)\right)$:

Proposition 3.1 (i) *For any $\kappa \in (0,1]$ there is a positive constant $C_2 = C_2(\kappa)$ such that*

$$\left(\int_{C^*_k(\rho)} |u|^q\, dx\right)^{1/q} \le C_2 \rho^{\frac{d-1+\gamma_k}{\gamma_k}\left(\frac{1}{q}-\frac{1}{2}\right)} \times \left\{\int_N |u|^2\, dx + \rho^2 \sum_{i=1}^d \int_{C^*_k(\rho)} |\partial_i u|^2\, dx\right\}^{1/2}, \tag{3.3}$$

for $u \in H^1\left(C^*_k(\rho)\right)$, $N \subset C^*_k(\rho)$ with $|N| \ge \kappa |C^*_k(\rho)|$, $0 < \rho \le \rho^*$, $2 \le q \le 2(d-1+\gamma_k)/(d-1-\gamma_k)$, and $k \in I_C$.

(ii) *Let $2 \le q < \infty$ in case $d = 2$ or $2 \le q \le 2d/(d-2)$ in case $d \ge 3$. Then for any $\kappa \in (0,1]$ there is a positive constant $C_3 = C_3(\kappa, q)$ such that*

$$\left(\int_{Q_k^*(\rho)} |u|^q\, dx\right)^{1/q} \leq C_3 \rho^{d\left(\frac{1}{q}-\frac{1}{2}\right)} \left\{\int_N |u|^2\, dx + \rho^2 \sum_{i=1}^{d} \int_{Q_k^*(\rho)} |\partial_i u|^2\, dx\right\}^{1/2}, \tag{3.4}$$

for $u \in H^1\left(Q_k^*(\rho)\right)$, $N \subset Q_k^*(\rho)$ *with* $|N| \geq \kappa |Q_k^*(\rho)|$, $0 < \rho \leq \rho^*$ *and* $k \in I_Q$.

The proof of Proposition 3.1 will be carried out in the last section by employing a specific transformation of a standard cusp domain onto a rectangular set.

The following Sobolev inequality in an ordinary sense follows from (3.2), (3.3), (3.4) and Lemma 3.1 ([14; Proposition 3.2 (ii)]).

Proposition 3.2 (i) *There is a positive constant* C_4 *such that*

$$\left(\int_D |u|^q dx\right)^{1/q} \leq C_4 \left\{\int_D |u|^2 dx + \sum_{i=1}^{d} \int_D |\partial_i u|^2 dx\right\}^{1/2}, \tag{3.5}$$

for $u \in H^1(D)$, $2 \leq q \leq 2(d-1+\gamma^*)/(d-1-\gamma^*)$.

(ii) *Assume the absence of outward cusp boundary point* : $I_C = \emptyset$. *Then the above statement is valid for* $2 \leq q \leq 2d/(d-2)$ *in case* $d \geq 3$ *and for* $2 \leq q < \infty$ *in case* $d = 2$.

We denote the norm of the Sobolev space $H^1(E)$ by $\|\cdot\|_{H^1(E)}$. For an open set $E \subset D$, let us consider the following spaces :

$$\widehat{C}(E) = \left\{u \in C^1(E) : \|u\|_{H^1(E)} < \infty,\ u = 0 \text{ on } \partial E \cap D\right\}, \tag{3.6}$$

$$\widehat{H}(E) = \text{the completion of } \widehat{C}(E) \text{ with respect to the norm } \|\cdot\|_{H^1(E)}. \tag{3.7}$$

Note that $\widehat{H}(E)$ coincides with $H_0^1(E)$ if $\overline{E} \subset D$. When $E = C_k^*(\rho)$, $Q_k^*(\rho)$ or $B_a^*(r)$, we can derive the following Sobolev inequalities from Proposition 3.1 ([14; Proposition 3.3]).

Proposition 3.3 (i) *For any* $\delta \in (0,1)$, *there is a positive constant* $C_5 = C_5(\delta)$ *such that*

$$\left(\int_{C_k^*(\rho)} |u|^q\, dx\right)^{1/q} \leq C_5 \left(\sum_{i=1}^{d} \int_{C_k^*(\rho)} |\partial_i u|^2\, dx\right)^{1/2}, \tag{3.8}$$

for $u \in \widehat{H}(C_k^*(\rho))$, $0 < \rho \leq \delta\rho^*$, $2 \leq q \leq 2(d-1+\gamma_k)/(d-1-\gamma_k)$, $k \in I_C$.
(ii) *For any* $\delta \in (0,1)$ *and for any* $2 \leq q \leq 2d/(d-2)$ ($2 \leq q < \infty$ *if* $d = 2$), *there is a positive constant* $C_6 = C_6(\delta, q)$ *such that*

$$\left(\int_{Q_k^*(\rho)} |u|^q\, dx\right)^{1/q} \leq C_6 \left(\sum_{i=1}^{d} \int_{Q_k^*(\rho)} |\partial_i u|^2\, dx\right)^{1/2}, \tag{3.9}$$

for $u \in \widehat{H}(Q_k^*(\rho))$, $0 < \rho \leq \delta\rho^*$, $k \in I_Q$.
(iii) *Let* $0 < \rho \leq \rho^*$, $0 < \delta < 1$ *and* $2 \leq q \leq 2d/(d-2)$ ($2 \leq q < \infty$ *if* $d = 2$). *Then there is a positive constant* $C_7 = C_7(\rho, \delta, q)$ *satisfying*

$$\left(\int_{B_a^*(r)} |u|^q\,dx\right)^{1/q} \leq C_7 \left(\sum_{i=1}^{d} \int_{B_a^*(r)} |\partial_i u|^2\,dx\right)^{1/2}, \tag{3.10}$$

for $u \in \widehat{H}(B_a^*(r)),\ 0 < r \leq \delta r_\rho,\ a \in \partial D_\rho^\#$.

We now turn to the Dirichlet form $\left(\mathscr{E}, H^1(D)\right)$ given by (1.1) and (1.2). We also consider the following form $\left(\mathscr{E}_E, \widehat{H}(E)\right)$ for an open set $E \subset D$.

$$\mathscr{E}_E(u,v) = \sum_{i,j=1}^{d} \int_E \partial_i u(x)\partial_j v(x) a_{ij}(x)\,dx, \qquad u,\ v \in \widehat{H}(E), \tag{3.11}$$

with $a_{ij},\ 1 \leq i,j \leq d$ satisfying (1.1). Since $\left(\mathscr{E}_E, \widehat{H}(E)\right)$ is a Dirichlet form on $L^2(E)$, we have the associated Markovian resolvent $\{G_{E,\lambda}, \lambda > 0\}$ on $L^2(E)$.

Let T be a functional defined by

$$\langle T, \varphi\rangle = \int_D f_0 \varphi\,dx + \sum_{i=1}^{d} \int_D f_i \partial_i \varphi\,dx, \qquad \varphi \in H^1(D), \tag{3.12}$$

for $f_i \in L^2(D),\ i = 0, 1, \cdots, d$. Since T is a continuous linear functional on $H^1(D)$, there is for each $\lambda > 0$ a unique element $u \in H^1(D)$ such that

$$\mathscr{E}_\lambda(u,\varphi) = \langle T, \varphi\rangle, \qquad \varphi \in H^1(D). \tag{3.13}$$

Here $\mathscr{E}_\lambda(\ ,\ ) = \mathscr{E}(\ ,\ ) + \lambda(\ ,\ )_{L^2(D)}$. We denote this function u by $G_\lambda T$. If T is defined by (3.12) with D and $H^1(D)$ replaced by E and $\widehat{H}(E)$ respectively and if every f_i belongs to $L^2(E)$, then we have for each $\lambda > 0$ a unique $u \in \widehat{H}(E)$ denoted by $G_{E,\lambda}T$ such that

$$\mathscr{E}_{E,\lambda}(u,\varphi) = \langle T, \varphi\rangle, \qquad \varphi \in \widehat{H}(E), \tag{3.14}$$

where $\mathscr{E}_{E,\lambda}(\ ,\ ) = \mathscr{E}_E(\ ,\ ) + \lambda(\ ,\ )_{L^2(E)}$.

Obviously $G_\lambda T$ [resp. $G_{E,\lambda}T$] coincides with $G_\lambda f$ [resp. $G_{E,\lambda} f$] in the case where $\langle T, \varphi\rangle = (f, \varphi)$ for $f \in L^2(D)$ [resp. $f \in L^2(E)$]. If $E = C_k^*(\rho),\ Q_k^*(\rho)$ or $B_a^*(r)$, then the norm $\|\cdot\|_{H^1(E)}$ is equivalent to $\|\cdot\|_{\widehat{H}(E)} \equiv \mathscr{E}_E(\cdot,\cdot)^{1/2}$ in view of Proposition 3.3. Therefore there exists $G_{E,0}T \in \widehat{H}(E)$ satisfying (3.14) with $\lambda = 0$.

Using Sobolev inequalities (3.5), (3.8), (3.9), (3.10) and following a standard argument as in [18; Theorem 4.1] (see also [10]), we can get the following L^p-estimates.

Theorem 3.1 (i) *Let* $p > (d-1)/\gamma^* + 1$ *and* $\lambda > 0$. *Then it holds that, for some* $C_8 = C_8(p, \lambda) > 0$,

$$\|G_\lambda T\|_{L^\infty(D)} \leq C_8 \sum_{i=0}^{d} \left(\|f_i\|_{L^2(D)} + \|f_i\|_{L^p(D)}\right), \tag{3.15}$$

where T is given by (3.12) with $f_i \in L^2(D) \cap L^p(D),\ i = 0, 1, \cdots, d$.
(ii) *Let $k \in I_C,\ p > (d-1)/\gamma_k + 1$ and $0 < \delta < 1$. Then there is a positive constant $C_9 = C_9(p, \delta)$ such that*

$$\|G_{E,\lambda} T\|_{L^\infty(E)} \le C_9 \rho^{\frac{1}{p}\left(p-1-\frac{d-1}{\gamma_k}\right)} \sum_{i=0}^{d} \|f_i\|_{L^p(E)}, \tag{3.16}$$

where $E = C_k^*(\rho),\ 0 < \rho \le \delta\rho^*,\ \lambda \ge 0$, and T is a continuous linear functional given by (3.12) with $f_i \in L^p(E)$ and $\varphi \in \widehat{H}(E)$.
(iii) *Let $p > d$ and $0 < \delta < 1$. Then there is a positive constant $C_{10} = C_{10}(p, \delta)$ such that*

$$\|G_{E,\lambda} T\|_{L^\infty(E)} \le C_{10} \rho^{(p-d)/p} \sum_{i=0}^{d} \|f_i\|_{L^p(E)}, \tag{3.17}$$

where $E = Q_k^*(\rho),\ k \in I_Q,\ 0 < \rho \le \delta\rho^*,\ \lambda \ge 0$, and T is a continuous linear functional given by (3.12) with $f_i \in L^p(E)$ and $\varphi \in \widehat{H}(E)$.
(iv) *Let $0 < \rho \le \rho^*,\ p > d$ and $0 < \delta < 1$. Then there is a positive constant $C_{11} = C_{11}(p, \delta)$ such that*

$$\|G_{E,\lambda} T\|_{L^\infty(E)} \le C_{11} r^{(p-d)/p} \sum_{i=0}^{d} \|f_i\|_{L^p(E)}, \tag{3.18}$$

for $\lambda \ge 0,\ E = B_a^*(r),\ 0 < r \le \delta r_\rho,\ a \in \partial D_\rho^\#$, and for T defined by (3.12) with $E,\ \widehat{H}(E),\ f_i \in L^p(E)$ instead of $D,\ H^1(D),\ f_i \in L^p(D)$ respectively.

We are next concerned with local estimates for subsolutions of the equations associated with $\mathscr{E}_E$. A function $u \in H^1(E)$ is called a subsolution if

$$\mathscr{E}_E(u, \varphi) \le 0, \qquad \varphi \ge 0,\ \varphi \in \widehat{H}(E). \tag{3.19}$$

In the same way as in [18; Theorem 5.1] or in [17; Theorem 1], we obtain the following local estimates from Proposition 3.3 or (3.2) ([14; Theorem 3.2]).

Theorem 3.2 (i) *Let $0 < \rho \le \delta\rho^*$ for some $\delta \in (0, 1)$ and $E = C_k^*(\rho)$ with $k \in I_C$. Then every nonnegative subsolution $u \in H^1(E)$ of (3.19) satisfies*

$$\operatorname*{ess\,sup}_{C_k^*(s)} u \le C_{12}(\rho - s)^{-\frac{d-1+\gamma_k}{2\gamma_k}} \left(\int_{C_k^*(\rho)} u^2\, dx \right)^{1/2}, \quad 0 < s < \rho, \tag{3.20}$$

for some $C_{12} = C_{12}(\delta) > 0$.
(ii) *Let $0 < \rho \le \delta\rho^*$ for some $\delta \in (0, 1)$ and $E = Q_k^*(\rho)$ with $k \in I_Q$. Then every nonnegative subsolution $u \in H^1(E)$ of (3.19) satisfies*

$$\operatorname*{ess\,sup}_{Q_k^*(s)} u \le C_{13}(\rho - s)^{-d/2} \left(\int_{Q_k^*(\rho)} u^2\, dx \right)^{1/2}, \quad 0 < s < \rho, \tag{3.21}$$

for some $C_{13} = C_{13}(\delta) > 0$.
(iii) *Let* $0 < \rho \leq \rho^*,\ 0 < \delta < 1,\ 0 < r \leq \delta r_\rho,\ a \in \partial D^{\#}_\rho$ *and* $E = B^*_a(r)$. *Then every nonnegative subsolution* $u \in H^1(E)$ *of (3.19) satisfies*

$$\operatorname*{ess\,sup}_{B^*_a(s)} u \leq C_{14}(r-s)^{-d/2}\left(\int_{B^*_a(r)} u^2\,dx\right)^{1/2}, \quad 0 < s < r, \tag{3.22}$$

for some $C_{14} = C_{14}(\rho, \delta) > 0$.

If $u \in H^1(E)$ satisfies

$$\mathscr{E}_{E,\lambda}(u,\varphi) = 0, \qquad \varphi \in \widehat{H}(E), \tag{3.23}$$

for some $\lambda \geq 0$, then $u \vee 0$ and $(-u) \vee 0$ are both nonnegative subsolutions of (3.19). Therefore as an immediate consequence of Theorem 3.2 we get the following result.

Corollary 3.1 (i) *Let* $0 < \rho \leq \delta\rho^*$ *for some* $\delta \in (0,1)$ *and* $E = C^*_k(\rho)$ *with* $k \in I_C$ $\left[\right.$ *resp.* $Q^*_k(\rho)$ *with* $k \in I_Q \left.\right]$. *Then every solution* $u \in H^1(E)$ *of (3.23) satisfies (3.20) [resp. (3.21)] with u being replaced by* $|u|$.
(ii) *Let* $0 < \rho \leq \rho^*,\ 0 < \delta < 1,\ 0 < r \leq \delta r_\rho,\ a \in \partial D^{\#}_\rho$ *and* $E = B^*_a(r)$. *Then every solution* $u \in H^1(E)$ *of (3.23) satisfies (3.22) with u being replaced by* $|u|$.

Finally, by means of Proposition 3.1 and Theorem 3.2, we can get the following Harnack inequality for solutions $u \in H^1(E)$ of the equation (3.23) with $\lambda = 0$ ([14; Theorem 3.3]).

Theorem 3.3 (i) *Let* $k \in I_C$ $\left[\right.$ *resp.* $k \in I_Q \left.\right]$, $0 < \rho \leq \rho^*,\ 0 < \kappa < 1$ *and* $E = C^*_k(\rho)$ $\left[\right.$ *resp.* $Q^*_k(\rho) \left.\right]$. *If* $u \in H^1(E)$ *is a nonnegative solution of (3.23) with* $\lambda = 0$ *and satisfies* $|\{x : u(x) \geq 1\} \cap C^*_k(\rho/2)| \geq \kappa|C^*_k(\rho/2)|$ $\left[\right.$ *resp.* $|\{x : u(x) \geq 1\} \cap Q^*_k(\rho/2)| \geq \kappa|Q^*_k(\rho/2)| \left.\right]$, *then there is a positive constant* $C_{15} = C_{15}(\kappa)$ *such that*

$$\operatorname*{ess\,inf}_{C^*_k(\rho/4)} u \geq C_{15} \quad \left[\textit{resp.}\quad \operatorname*{ess\,inf}_{Q^*_k(\rho/4)} u \geq C_{15}\right]. \tag{3.24}$$

(ii) *Let* $0 < \rho \leq \rho^*,\ 0 < r \leq r_\rho,\ a \in \partial D^{\#}_\rho,\ 0 < \kappa < 1$ *and set* $E = B^*_a(r)$. *If* $u \in H^1(E)$ *is a nonnegative solution of (3.23) with* $\lambda = 0$ *and satisfies* $|\{x : u(x) \geq 1\} \cap B^*_a(r/2)| \geq \kappa|B^*_a(r)|$, *then there is a positive constant* $C_{16} = C_{16}(\rho, \kappa)$ *such that*

$$\operatorname*{ess\,inf}_{B^*_a(r/4)} u \geq C_{16}. \tag{3.25}$$

4 Strong Feller resolvent

In this section, we will show that the resolvent $\{G_\lambda\}$ associated with the Dirichlet form $\left(\mathscr{E}, H^1(D)\right)$ has the same properties as those of Theorem 2.1 under condition (A). At the end of this section, we will show that condition (H) reduces to (A) and hence Theorem 2.1 follows.

Theorem 4.1 *Under condition (A), G_λ satisfies the same properties as in Theorem 2.1. Namely,*
(i) $G_\lambda\left(L^2(D)\cap L^p(D)\right) \subset C\left(\overline{D}\right), \qquad p > 1 + (d-1)/\gamma^*.$
(ii) $G_\lambda\left(C_\infty\left(\overline{D}\right)\right)$ *is a dense subspace of* $C_\infty\left(\overline{D}\right)$.
(iii) *There is a function $G_\lambda(x,y)$ continuous on $\overline{D}\times\overline{D}$ off diagonal such that*

$$G_\lambda f(x) = \int_{\overline{D}} G_\lambda(x,y) f(y)\,dy, \qquad x \in \overline{D},\ f \in C_\infty\left(\overline{D}\right). \tag{4.1}$$

Theorem 4.1 is obtained essentially by the same argument as in [14; Sect. 4] but we give the proof here for completeness.

Theorem 4.1 (i) is an immediate consequence of the following theorem.

Theorem 4.2 *Assume condition (A). Let $p > (d-1)/\gamma^* + 1$, T be a functional given by (3.12) with $f_i \in L^2(D)\cap L^p(D)$, $i = 0,1,2,\cdots,d$, and $\lambda > 0$. Then $G_\lambda T$ is uniformly continuous in D and accordingly $G_\lambda T$ can be extended to a continuous function on $\overline{D}$.*

Proof Put $u = G_\lambda T$. Fix a $k \in I_C$ and an $s \in (0, \rho^*/2]$ arbitrarily. Set $E = C_k^*(s)$. Let $v \equiv G_{E,0}(T - \lambda u) \in \widehat{H}(E)$ be the solution of the equation (3.14) with $\lambda = 0$ and $T = T - \lambda u$. We see by means of Theorem 3.1 (i), (ii),

$$\begin{aligned}\|v\|_{L^\infty(E)} &\le C_9(p,1/2)s^{\frac{1}{p}\left(p-1-\frac{d-1}{\gamma_k}\right)}\left\{\|f_0 - \lambda u\|_{L^p(E)} + \sum_{i=1}^d \|f_i\|_{L^p(E)}\right\} \\ &\le c_1 s^{\frac{1}{p}\left(p-1-\frac{d-1}{\gamma^*}\right)}\sum_{i=0}^d\left\{\|f_i\|_{L^2(D)} + \|f_i\|_{L^p(D)}\right\},\end{aligned} \tag{4.2}$$

for some positive c_1 independent of s and k. Since $w \equiv u - v$ belongs to $H^1(E)$ and satisfies (3.23) with $\lambda = 0$, following the same argument as in [17] we get by means of Theorem 3.3 (i)

$$\begin{aligned}\mathrm{Osc}\left(w; C_k^*(s/4)\right) &\le \left(1 - \frac{1}{2}C_{15}(1/2)\right)\mathrm{Osc}\left(w; C_k^*(s)\right) \\ &\le c_2 \mathrm{Osc}\left(w; C_k^*(s)\right),\end{aligned}$$

for $c_2 \in (0,1)$ independent of s and k. Here $\mathrm{Osc}(g;F)$ denotes the oscillation of a function g over a set F : $\mathrm{Osc}(g;F) = \mathrm{ess\,sup}_F\, g - \mathrm{ess\,inf}_F\, g$. Hence

$$\begin{aligned}&\mathrm{Osc}\left(u; C_k^*(s/4)\right) \le \mathrm{Osc}\left(v; C_k^*(s/4)\right) + \mathrm{Osc}\left(w; C_k^*(s/4)\right) \\ &\le 2\|v\|_{L^\infty(E)} + c_2\mathrm{Osc}\left(w; C_k^*(s)\right) \le 4\|v\|_{L^\infty(E)} + c_2\mathrm{Osc}\left(u; C_k^*(s)\right).\end{aligned}$$

Combining this with (4.2) and using [18; Lemma 7.3], we get

$$\text{Osc}\left(u; C_k^*(s)\right) \le c_3 s^{\xi_1}, \qquad 0 < s \le \rho^*/4,\ k \in I_C, \tag{4.3}$$

for some constants $c_3 > 0$ and $\xi_1 \in (0,1)$. In the same way we also get

$$\text{Osc}\left(u; Q_k^*(s)\right) \le c_4 s^{\xi_2}, \qquad 0 < s \le \rho^*/4,\ k \in I_Q, \tag{4.4}$$

for some constants $c_4 > 0$ and $\xi_2 \in (0,1)$. Recall r_ρ and $\partial D_\rho^\#$ appearing in Lemma 3.1. Similarly, for any $\rho \in (0, \rho^*/2]$, there then exist a $c_5 > 0$ and a $\xi_3 \in (0,1)$ such that

$$\text{Osc}\left(u; B_a^*(s)\right) \le c_5 s^{\xi_3}, \qquad 0 < s \le r_\rho/4,\ a \in \partial D_\rho^\#. \tag{4.5}$$

The estimate for oscillations on open balls with closures contained in D, which is due to Stampacchia [18], asserts that

$$\text{Osc}\,(u; B(a,s)) \le c_6 s^{\xi_4}, \qquad 0 < s \le \eta/4,\ a \in D \setminus D_\eta. \tag{4.6}$$

Here η is a positive number fixed arbitrarily, $D_\eta = \{x \in D : \text{dist}(x, \partial D) < \eta\}$, and constants $c_6 > 0$ and $\xi_4 \in (0,1)$ depend on η but are independent of $a \in D \setminus D_\eta$.

For an $\varepsilon > 0$ fixed arbitrarily, we see by virtue of (4.3) and (4.4) that there exists an $s_1 = s_1(\varepsilon) \in (0, \rho^*/4]$ such that

$$\text{Osc}\left(u; C_k^*(s_1)\right) < \varepsilon, \qquad k \in I_C, \tag{4.7}$$

$$\text{Osc}\left(u; Q_k^*(s_1)\right) < \varepsilon, \qquad k \in I_Q. \tag{4.8}$$

By means of (4.5), we further find an $s_2 = s_2(\varepsilon, s_1) \in (0, r_{s_1}/4]$ such that

$$\text{Osc}\left(u; B_a^*(s_2)\right) < \varepsilon, \qquad a \in \partial D_{s_1}^\#. \tag{4.9}$$

In view of Lemma 3.1, we can find an $\eta_o > 0$ such that

every pair $x,\ y \in D_{\eta_o}$ with $|x-y| < \eta_o$ is
simultaneously contained in one of sets
$C_k^*(s_1)$ with some $k \in I_C$, $Q_k^*(s_1)$ with some $k \in I_Q$,
$B_a^*(s_2)$ with some $a \in \partial D_{s_1}^\#$. (4.10)

(4.6) with this η_o leads us to

$$\text{Osc}\,(u; B(a, s_3)) < \varepsilon, \qquad a \in D \setminus D_{\eta_o/2}, \tag{4.11}$$

for some $s_3 = s_3(\varepsilon, \eta_o) \in (0, \eta_o/8]$.

We now set $\delta = (\eta_o/2) \wedge s_3$. Let $x,\ y \in D$ with $|x-y| < \delta$. If x or y belongs to $D_{\eta_o/2}$, then $|u(x) - u(y)| < \varepsilon$ by (4.10), (4.7), (4.8), (4.9). Otherwise, $|u(x) - u(y)| < \varepsilon$ by (4.11). □

Employing Corollary 3.1 in place of Theorem 3.1 (i) in getting (4.2), we obtain the following in the same way as above:

Theorem 4.3 *Let W be an open set of R^d and $E = W \cap D$. Every solution $u \in H^1(E)$ of (3.23) for some $\lambda \geq 0$ is uniformly continuous in $W_1 \cap D$ for every open set W_1 satisfying $\overline{W_1} \subset W$.*

We next give

Proof of Theorem 4.1 (ii) We first follow an argument in [20; Proposition 5.1] to show

$$G_\lambda\left(C_\infty\left(\overline{D}\right)\right) \subset C_\infty\left(\overline{D}\right). \tag{4.12}$$

Since $C_0\left(\overline{D}\right)$ is dense in $C_\infty\left(\overline{D}\right)$, it suffices to show that

$$G_\lambda\left(C_0\left(\overline{D}\right)\right) \subset C_\infty\left(\overline{D}\right) \tag{4.13}$$

in the case where D is unbounded.

Let $g \in C_0\left(\overline{D}\right)$ and $\varepsilon > 0$. Choose an $R_1 > 0$ such that

$$\mathrm{Supp}[g] \subset B(R_1) \cap \overline{D}, \qquad c_1 \|G_\lambda g\|_{L^2(D \setminus B(R_1))} < \varepsilon, \tag{4.14}$$

c_1 being a positive constant specified later. We next take an $R_2 > R_1$ satisfying the following :

$$\begin{array}{ll} C_k^*(\rho^*) \subset D \setminus B(R_1) & \text{for } k \in I_C \text{ with } a_k \in \partial D \setminus B(R_2), \\ Q_k^*(\rho^*) \subset D \setminus B(R_1) & \text{for } k \in I_Q \text{ with } a_k \in \partial D \setminus B(R_2), \\ B_a^*(r^*) \subset D \setminus B(R_1) & \text{for } a \in \partial D^\#_{\rho^*} \setminus B(R_2). \end{array}$$

We set $J_C = \{k \in I_C : a_k \in \partial D \setminus B(R_2)\}$, $J_Q = \{k \in I_Q : a_k \in \partial D \setminus B(R_2)\}$, and $A = \partial D^\#_{\rho^*/2} \setminus B(R_2)$. Then, on account of (3.1), there is a constant $\eta \in (0, R_2 - R_1)$ depending on ρ^* but not on R_1, R_2 such that

$$D_{2\eta} \setminus B(R_2) \subset \bigcup_{k \in J_C} C_k^*\left(\frac{\rho^*}{2}\right) \cup \bigcup_{k \in J_Q} Q_k^*\left(\frac{\rho^*}{2}\right) \cup \bigcup_{a \in A} B_a^*\left(\frac{r_{\rho^*/2}}{2}\right).$$

We consider the set $K = \left[\bigcup_{k \in J_C \cup J_Q} \{a_k\}\right] \cup A \cup \left[D \setminus D_{2\eta} \setminus B(R_2)\right]$ and, for each $a \in K$, we define a constant s and a set $E_a(s)$ as follows :

$$E_a(s) = \begin{cases} C_k^*(s), & s = \rho^*, \quad \text{if} \quad a = a_k,\ k \in J_C, \\ Q_k^*(s), & s = \rho^*, \quad \text{if} \quad a = a_k,\ k \in J_Q, \\ B_a^*(s), & s = r_{\rho^*/2}, \quad \text{if} \quad a \in A, \\ B(a, s), & s = \eta, \quad \text{if} \quad a \in D \setminus D_{2\eta} \setminus B(R_2). \end{cases}$$

Note that

$$D \setminus B(R_2) \subset \bigcup_{a \in K} E_a(s/2) \subset \bigcup_{a \in K} E_a(s) \subset D \setminus B(R_1), \tag{4.15}$$

and

$$\mathscr{E}_{E_a(s),\lambda}(G_\lambda g, \varphi) = (g, \varphi) = 0, \qquad \varphi \in \widehat{H}\,(E_a(s)).$$

By virtue of Corollary 3.1, we then have

$$\|G_\lambda g\|_{L^\infty(E_a(s/2))} \le c_1 \|G_\lambda g\|_{L^2(E_a(s))}, \tag{4.16}$$

where we used a local estimate due to Stampacchia [18] or Moser [17] in the case that $a \in D \setminus D_{2\eta} \setminus B(R_2)$. It should be noted that c_1 is a positive constant independent of a, R_1 and R_2. By (4.14), (4.15) and (4.16), we find that

$$\|G_\lambda g\|_{L^\infty(D \setminus B(R_2))} \le c_1 \|G_\lambda g\|_{L^2(D\setminus B(R_1))} < \varepsilon, \tag{4.17}$$

which along with Theorem 4.2 proves (4.13).

We next adopt Kunita's argument [15]. Let denote by $C_0^\infty(\overline{D})$ the space of the restrictions to $\overline{D}$ of all infinitely continuously differentiable functions on R^d with compact support. For each $u \in C_0^\infty(\overline{D})$, we define a functional Lu by

$$\langle Lu, \varphi\rangle = -\sum_{j=1}^{d} \int_D \left(\sum_{i=1}^{d} a_{ij}\, \partial_i u \right) \partial_j \varphi \, dx, \quad \varphi \in H^1(D).$$

Then $T = \lambda u - Lu$ satisfies the condition of Theorem 3.1 (i) and $u = G_\lambda T$ for each $\lambda > 0$. By virtue of Theorem 3.1 (i), there is for any $\varepsilon > 0$ a $g \in C_0^\infty(\overline{D})$ such that

$$\|u - G_\lambda g\|_{L^\infty(D)} < \varepsilon.$$

Since $C_0^\infty(\overline{D})$ is dense in $C_\infty(\overline{D})$, we thus obtain the denseness of $G_\lambda\left(C_\infty(\overline{D})\right)$ in $C_\infty(\overline{D})$. □

Proof of Theorem 4.1 (iii) Since G_λ is Markovian, there exists a function $G_\lambda(x, y)$ satisfying (4.1) by virtue of Theorem 3.1 (i) and Theorem 4.2. Hence it is enough to show that

$$G_\lambda(x, \cdot) \text{ belongs to } H^1(U) \text{ and is continuous on } \overline{U}, \tag{4.18}$$

for any open set U with $\overline{U} \subset \overline{D} \setminus \{x\}$, where $x \in \overline{D}$ and $\lambda > 0$.

Let us denote the dual space of $H^1(E)$ by $\left(H^1(E)\right)'$. There exists for each $\lambda > 0$ and $T \in \left(H^1(D)\right)'$ a unique element $u \in H^1(D)$ such that

$$\mathscr{E}_\lambda(u, \varphi) = \langle T, \varphi\rangle, \qquad \varphi \in H^1(D).$$

We denote this function u by $G_\lambda T$. (We already used this notation for T given by (3.12) which is actually a general expression of $T \in \left(H^1(D)\right)'$ (cf. [16; 1.1.14]).)

For a while we fix an $x \in \overline{D}$ arbitrarily. We define a set $E_x(s)$ according as three different cases.

(*Case 1*) x is a cusp point, that is, $x = a_k$ for some $k \in I$. In this case we take an $s \in (0, \rho^*]$.

(*Case 2*) x is a boundary point but not a cusp point, that is, $x \in \partial D \setminus \bigcup_{k\in I}\{a_k\}$. Choose $\rho \in (0, \rho^*]$ such that $x \in \partial D \setminus \bigcup_{k\in I} \Phi_k(B(\rho))$. Then for an r_ρ given in Lemma 3.1 we take an $s \in (0, r_\rho]$.

(*Case 3*) x is an interior point of D. In this case we take an $s \in (0, d_x/2]$, where $d_x = \mathrm{dist}(x, \partial D)$.

Let us put

$$E_x(s) = \begin{cases} C_k^*(s) & \text{if } k \in I_C, \text{ in Case 1,} \\ Q_k^*(s) & \text{if } k \in I_Q, \text{ in Case 1,} \\ B_x^*(s) & \text{in Case 2,} \\ B(x,s) & \text{in Case 3.} \end{cases}$$

Then there exists a unique element $g_s^{x,\lambda} \in H^1(D)$ such that

$$\mathscr{E}_\lambda\left(g_s^{x,\lambda}, \varphi\right) = \frac{1}{|E_x(s)|}\int_{E_x(s)} \varphi(y)\,dy, \quad \varphi \in H^1(D). \tag{4.19}$$

Here we note the following lemma which is obtained by the same method as in [14; Lemma 4.3].

Lemma 4.1 *Let U be an open set such that $\overline{U} \subset \overline{D} \setminus \{x\}$. Then $G_\lambda(x,\cdot)|_U \in H^1(U)$ and $g_s^{x,\lambda}|_U$ converges to $G_\lambda(x,\cdot)|_U$ weakly in $H^1(U)$ as $s \downarrow 0$.*

Take an open set V of R^d such that $\overline{U} \subset V$ and $x \notin \overline{V}$ and set $E = V \cap D$. On account of Lemma 4.1, $G_\lambda(x,\cdot)|_E \in H^1(E)$ and $g_s^{x,\lambda}|_E \to G_\lambda(x,\cdot)|_E$ weakly in $H^1(E)$ as $s \downarrow 0$. Take any $\varphi \in \widehat{C}(E)$ and extend it to D by putting $\varphi = 0$ on $D \setminus E$. Then

$$\begin{aligned} \mathscr{E}_{E,\lambda}(G_\lambda(x,\cdot)|_E, \varphi) &= \lim_{s\downarrow 0} \mathscr{E}_{E,\lambda}\left(g_s^{x,\lambda}|_E, \varphi\right) \\ &= \lim_{s\downarrow 0} \mathscr{E}_\lambda\left(g_s^{x,\lambda}, \varphi\right) = \lim_{s\downarrow 0} \frac{1}{|E_x(s)|}\int_{E_x(s)} \varphi(y)\,dy = 0. \end{aligned}$$

This implies that $G_\lambda(x,\cdot)|_E \in H^1(E)$ is a solution of (3.23) and hence, in view of Theorem 4.3, $G_\lambda(x,\cdot)$ is continuous in $\overline{U}$. □

We finally note the following proposition which along with Theorem 4.1 implies Theorem 2.1.

Proposition 4.1 *Condition (H) reduces to condition (A).*

Proof For each $j \in J$, a Hölder function F_j in (H) (i) is given by

$$F_j(x') = \alpha_j |x'|^{\gamma_j} + f_j(x'),$$

where $\gamma \le \gamma_j < 1$, $\alpha_j = 0$ or $1/A \le \alpha_j \le A$ or $-A \le \alpha_j \le -1/A$ according to $j \in J_0$ or $j \in J_+$ or $j \in J_-$, and f_j is a Lipschitz continuous function defined on the $d-1$-dimensional closed ball $\left\{x' \in R^{d-1} : |x'| \le r_j\right\}$ with $f_j(0) = 0$ and $\mathrm{Lip}(f_j) \le M$. Then

$$U_j \cap D = \left\{\left(\zeta^{(j)\prime}, \zeta_d^{(j)}\right) \in B(r_j) : F_j\left(\zeta^{(j)\prime}\right) < \zeta_d^{(j)}\right\},$$

for some Cartesian coordinate system $\zeta^{(j)} = \left(\zeta^{(j)\prime}, \zeta_d^{(j)}\right) = \left(\zeta_1^{(j)}, \zeta_2^{(j)}, \cdots, \zeta_d^{(j)}\right)$.

Let us put $I = J_+ \cup J_-$. For $k \in I$, a_k is the point of ∂D corresponding to the origin in $\zeta^{(k)}$-coordinate system. $\Xi \equiv \{a_k : k \in I\}$ is then the totality of cusp boundary points. For each $k \in I$, the neighbourhood U_k of a_k contains no cusp boundary point other than a_k, and hence $|a_k - a_l| \geq \delta,\ k \neq l,\ k, l \in I$.

For each $k \in I$, we define a mapping Φ_k from $E_k \equiv \{(x', x_d) : |x'| < r_k,\ x_d \in R\}$ into $\zeta^{(k)}$-space by

$$\Phi_k(x', x_d) = \left(\zeta^{(k)\prime}, \zeta_d^{(k)}\right),$$
$$\zeta^{(k)\prime} = x', \quad \zeta_d^{(k)} = |\alpha_k| x_d + f_k(x').$$

We then have $\mathrm{Lip}\big(\Phi_k\big) \leq 1 + A + M$ and $\mathrm{Lip}\big(\Phi_k^{-1}\big) \leq 1 + A + AM$. Put $\rho^* = \delta/2(1 + A + M)$, $V_k = \Phi_k\,(B(\rho^*))$ and $M_1^* = (1 + A)(1 + M)$. Then we see from (3.1) that $\{V_k\}_{k \in I}$ satisfies (A) (i) with $\gamma^* = \gamma$ and $M^* = M_1^*$. In particular, $V_k \cap D = \Phi_k\left(C_{\gamma_k}(\rho^*)\right)$ if $\alpha_k > 0$, $= \Phi_k\left(Q_{\gamma_k}(\rho^*)\right)$ if $\alpha_k < 0$.

We next show (A) (ii). Let ξ_o be the positive solution of the equation $\xi^{2\gamma} + \xi^2 = \big(\rho_o\big)^2$ for $\rho_o = \rho^* \wedge 1$. We then take a constant $R > 1$ satisfying

$$\left\{1 + M + A\left(\frac{R-1}{R}\xi_o\right)^{\gamma - 1}\right\}\frac{\xi_o}{R} < \delta,$$

and put $r^* = \xi_o/R$. This r^* will play the role of r^* in (A) (ii).

Let us fix a $p \in \partial D \setminus \bigcup_{k \in I} V_k$ arbitrarily. By means of (H) (ii), there is a $j \in J$ such that $p \in \widetilde{U_{j,\delta}}$. Denote the $\zeta^{(j)}$-coordinate of p by $(p^{(j)\prime}, p_d^{(j)})$. We shall define a mapping Ψ_p and a neighbourhood W_p in two cases $j \in I$ and $j \notin I$ separately.

In the case that $j \in I$, $(p^{(j)\prime}, p_d^{(j)})$ belongs to $\Phi_j(E_j)$. Putting $(\widetilde{p}', \widetilde{p}_d) = \Phi_j^{-1}\left(p^{(j)\prime}, p_d^{(j)}\right)$, we have that $(\widetilde{p}', \widetilde{p}_d) \in E_j \setminus B(\rho^*)$ and $\widetilde{p}_d = \pm|\widetilde{p}'|^{\gamma_j}$ in accordance to the sign of α_j. We then define a mapping Ψ_p from the set $G_p \equiv \{(x', x_d) : |x' + \widetilde{p}'| < r_j,\ x_d \in R\}$ into $\zeta^{(j)}$-space as follows:

$$\Psi_p(x', x_d) = \left(\zeta^{(j)\prime}, \zeta_d^{(j)}\right),$$
$$\zeta^{(j)\prime} = x' + \widetilde{p}',$$
$$\zeta_d^{(j)} = \alpha_j\,|x' + \widetilde{p}'|^{\gamma_j} + x_d + f_j\left(x' + \widetilde{p}'\right).$$

Notice that, on the region $\{x \in G_p : |x'| < \xi_o/S\}$ for $S > 1$, $\mathrm{Lip}\big(\Psi_p\big) \leq 1 + M + A\left(\frac{S-1}{S}\xi_o\right)^{\gamma_j - 1}$. Since the distance of $p = \Psi_p(0)$ from $\partial\left(\Psi_p(G_p)\right)$ is greater than δ, we can conclude from (3.1) that $B(r^*) \subset G_p$ for the above chosen $r^* = \xi_o/R$.

In the case that $j \notin I$, we define a mapping Ψ_p from the set $G_p \equiv \{(x', x_d) : |x' + p^{(j)\prime}| < r_j,\ x_d \in R\}$ into $\zeta^{(j)}$-space by

$$\Psi_p(x', x_d) = \left(\zeta^{(j)\prime}, \zeta_d^{(j)}\right),$$
$$\zeta^{(j)\prime} = x' + p^{(j)\prime}, \quad \zeta_d^{(j)} = x_d + f_j\left(x' + p^{(j)\prime}\right).$$

In both cases, $B(r^*) \subset G_p$. Accordingly we put $W_p = \Psi_p(B(r^*))$. It is easy to see that Ψ_p is one-to-one, $\Psi_p(0) = p$, $\Psi_p(B_+(r^*)) = W_p \cap D$, and $\mathrm{Lip}(\Psi_p) \leq M_2^*$, $\mathrm{Lip}(\Psi_p^{-1}) \leq M_2^*$ where $M_2^* = 1 + M + A\left(\frac{R-1}{R}\xi_o\right)^{\gamma - 1}$.

Thus (H) reduces to (A) with $I,\ \gamma^*,\ \rho^*,\ r^*$ as above and $M^* = M_1^* \vee M_2^*$. □

5 Decomposition of the sample path and additive functionals

This section is devoted to the proof of Theorem 2.3 and Theorem 2.4. To this end, we first prepare an extended version of a general theorem [13; Theorem 5.5.5] to characterize the second term in the decomposition (1.6).

Let X be a locally compact separable metric space, m be a positive Radon measure on X with full support and $(\mathscr{E}, \mathscr{F})$ be a strongly local regular Dirichlet form on $L^2(X;m)$. We assume that there exists a conservative diffusion process $\mathbf{M} = (X_t, P_x)$ on X associated with the form $\mathscr{E}$ whose transition function $p_t(x,\cdot)$ is absolutely continuous with respect to m for any $t > 0$ and $x \in X$.

Then the resolvent of $\mathbf{M}$ admits a symmetric density $G_\lambda(x,y)$ with respect to m which is λ-excessive in two variables $x,\ y$. The potential of a measure μ is denoted by $G_\lambda\mu(x) = \int_X G_\lambda(x,y)\,\mu(dy)$. The integral of a function f against a measure μ is denoted by $\langle \mu, f\rangle$ or $\langle f, \mu\rangle$. A positive Radon measure μ on X is said to be of finite energy integral if there exists a constant C_{17} such that

$$\int_X |v(x)|\mu(dx) \le C_{17}\sqrt{\mathscr{E}_1(v,v)}, \quad v \in \mathscr{C}, \tag{5.1}$$

for some special standard core $\mathscr{C}$ of $\mathscr{E}$. The totality of such measures is denoted by S_0. It is known that $\mu \in S_0$ if and only if $\langle \mu, G_\lambda\mu\rangle$ is finite and that, in this case, $G_\lambda\mu$ is a λ-excessive and quasi-continuous version of the potential $U_\lambda\mu$ considered in [13; Sect. 2.2]. We further introduce two classes of positive Radon measures on X by

$$S_{00} = \{\mu : \mu(X) < \infty,\ \sup_{x\in X} G_\lambda\mu(x) < \infty\}$$

$$S_{01} = \{\mu : \mu \in S_0,\ \ G_\lambda\mu(x) < \infty\ \forall x \in X\}.$$

Obviously $S_{00} \subset S_{01} \subset S_0$. In our later application, the family S_{01} turns out to be more useful than S_{00}.

An increasing sequence $\{E_\ell\}$ of finely open sets is said to be an exhaustive sequence if $\bigcup_{\ell=1}^\infty E_\ell = X$. A positive Borel measure μ on X is called smooth in the strict sense if there exists an exhaustive sequence $\{E_\ell\}$ of finely open sets such that $I_{E_\ell}\cdot\mu \in S_{00},\ \ \ell = 1, 2, \cdots$. Let S_1 be the totality of smooth measures in the strict sense. S_1 is known to be in one to one correspondence with the (equivalence classes of) positive continuous additive functionals (PCAF's in abbreviation) in the strict sense of $\mathbf{M}$ under the Revuz correspondence ([13; Theorem 5.1.7]).

Lemma 5.1 *$\mu \in S_1$ if and only if there exists an exhaustive sequence $\{E_\ell\}$ of finely open sets such that $I_{E_\ell}\cdot\mu \in S_{01},\ \ell = 1, 2, \cdots$.*

Proof It suffices to show that any $\mu \in S_{01}$ admits an exhaustive sequence $\{E_\ell\}$ of finely open sets such that $I_{E_\ell} \cdot \mu \in S_{00}, \quad \ell = 1, 2, \cdots$. We may choose E_ℓ as follows:

$$E_\ell = \{x \in O_\ell : G_\lambda \mu(x) < \ell\}, \quad \ell = 1, 2, \cdots,$$

where $\{O_\ell\}$ is an exhaustive sequence of relatively compact open sets. Then, $(I_{E_\ell} \cdot \mu)(X) = \mu(E_\ell)$ is finite, and further $G_\lambda(I_{E_\ell} \cdot \mu)(x) \leq \ell$ for m-a.e. $x \in X$ by the maximum principle ([13; Lemma 2.2.4]) and hence for every $x \in X$ by the absolute continuity of the transition function. □

We denote by $\mu_{\langle u \rangle}$ the energy measure of $u \in \mathscr{F}_{loc}$. The Dirichlet form $\mathscr{E}$ is expressible as

$$\mathscr{E}(u, v) = \frac{1}{2} \mu_{\langle u, v \rangle}(X), \quad u, v \in \mathscr{F}$$

by using the co-energy measure $\mu_{\langle u, v \rangle}$. The second assertion of the next proposition replaces S_{00} in [13; Theorem 5.5.5] by S_{01}.

Proposition 5.1 (i) *Suppose that a function u satisfies the following conditions:*
1. *u is finite valued, finely continuous and* $u \in \mathscr{F}_{loc}$.
2. $I_G \cdot \mu_{\langle u \rangle} \in S_{00}$ *for any relatively compact open set* G.

Then we have the unique decomposition

$$u(X_t) - u(X_0) = M_t^{[u]} + N_t^{[u]}, \quad \forall t \geq 0, \quad P_x - \text{a.e. } \forall x \in X, \tag{5.2}$$

where $M^{[u]}$ *is a CAF in the strict sense such that, for any relatively compact open set* G,

$$E_x\left(M_{t \wedge \tau_G}^{[u]}\right) = 0, \quad E_x\left(\left(M_{t \wedge \tau_G}^{[u]}\right)^2\right) = E_x\left(B_{t \wedge \tau_G}\right), \quad \forall x \in G, \tag{5.3}$$

B being the PCAF in the strict sense with Revuz measure $\mu_{\langle u \rangle}$ *and* τ_G *being the first leaving time from* G. $N_t^{[u]}$ *is a CAF in the strict sense locally of zero energy.*
(ii) *Assume further the following property of u:*
$\exists \nu = \nu^{(1)} - \nu^{(2)}$ *with* $I_G \cdot \nu^{(1)}, \ I_G \cdot \nu^{(2)} \in S_{01}$ *for any relatively compact open set* G *and*

$$\mathscr{E}(u, v) = \langle \nu, v \rangle, \quad \forall v \in \mathscr{C}, \tag{5.4}$$

for some special standard core $\mathscr{C}$ *of* $\mathscr{E}$.
Then

$$N^{[u]} = -A^{(1)} + A^{(2)}, \quad P_x - \text{a.s. } \forall x \in X, \tag{5.5}$$

where $A^{(1)}$ *and* $A^{(2)}$ *are PCAF's in the strict sense with Revuz measures* $\nu^{(1)}$ *and* $\nu^{(2)}$ *respectively.*

Proof The first assertion is a consequence of [11; Theorem 2]. Since $\nu^{(1)}, \ \nu^{(2)}$ in (ii) are in the class S_1 by the preceding lemma, we can see the validity of identity (5.5) on account of [12; Theorem 3.3, Corollary 3.1]. □

We are in a position to prove Theorem 2.3. We fix a domain $D \subset R^d$ possessing the property (H) and a data a_{ij} satisfying (1.1). We now apply the general theory prepared above to the specific Dirichlet form (1.2) on $L^2(\overline{D}; m)$, the resolvent G_λ of Theorem 2.1 and the conservative diffusion **M** of Theorem 2.2. Here m denotes the $d-$dimensional Lebesgue measure. **M** will be called the reflecting diffusion on $\overline{D}$ (associated with a_{ij}). Accordingly G_λ will be called the resolvent of the reflecting diffusion on $\overline{D}$.

Notice that, under the condition (H) for the domain D, the surface measure σ on ∂D is well defined with a local expression

$$\sigma(E) = \int_{E_*} \sqrt{1 + |\nabla F_j(\zeta')|^2} d\zeta', \quad E \subset U_j \cap \partial D, \tag{5.6}$$

where $E_* = \{\zeta' : (\zeta', F_j(\zeta')) \in E\}$. Further, with the unit inward normal vector $\mathbf{n}(\zeta) = (n_1(\zeta), \cdots, n_d(\zeta))$ making sense σ-a.e. on ∂D according as

$$\mathbf{n}(\zeta) = (-\nabla F_j(\zeta'), 1) \Big/ \sqrt{1 + |\nabla F_j(\zeta')|^2}\ , \quad \zeta \in U_j \cap \partial D,$$

we have the divergence theorem

$$\int_D \frac{\partial w}{\partial x_i} dm = -\int_{\partial D} w n_i d\sigma, \qquad 1 \le i \le d,\ w \in C_0^\infty(\overline{D}).$$

This formula extends to a wider class of functions w and in particular the condition (1.9) for a_{ij} guarantees the identity

$$\int_D \partial_j (a_{ij} \cdot v)\, dm = -\int_{\partial D} v\, a_{ij}\, n_j\, d\sigma, \quad v \in C_0^\infty(\overline{D}). \tag{5.7}$$

Denote by ϕ_i the coordinate functions: $\phi_i(x) = x_i,\ 1 \le i \le d$. Then, $\phi_i \in H^1_{loc}(D)$, and the co-energy measures $\mu_{\langle \phi_i, \phi_j \rangle}$ with respect to the Dirichlet form (1.2) are given by (cf. [13; Example 5.2.1])

$$\mu_{\langle \phi_i, \phi_j \rangle} = 2a_{ij} \cdot m, \quad 1 \le i, j \le d. \tag{5.8}$$

Let us denote by B an arbitrary ball in R^d. Since a_{ij} are bounded, we have

$$I_{B \cap \overline{D}} \cdot \mu_{\langle \phi_i \rangle} \in S_{00}.$$

The PCAF in the strict sense with Revuz measure m is just a constant functional t. Therefore Proposition 5.1 (i) implies the decomposition (1.6) where $M^i, 1 \le i \le d$, are CAF's in the strict sense satisfying (1.7) with t being replaced by $t \wedge \tau_{B \cap \overline{D}}$. Owing to the boundedness of a_{ij} however, we can let $B \uparrow R^d$ to get (1.7), proving Theorem 2.3 (i).

Turning to the proof of Theorem 2.3 (ii), we have under the condition (1.9)

$$\mathscr{E}(\phi_i, v) = \int_{\overline{D}} v(x)\, \nu(dx) \quad \text{with} \quad \nu = -\sum_{j=1}^d (\partial_j a_{ij}) \cdot m - \sum_{j=1}^d a_{ij}\, n_j \cdot \sigma, \tag{5.9}$$

holding for any $v \in C_0^\infty(\overline{D})$, because

$$\begin{aligned}\mathscr{E}(\phi_i, v) &= \sum_{j=1}^{d} \int_D a_{ij}(x)\partial_j v(x)\, m(dx) \\ &= -\sum_{j=1}^{d} \int_D \partial_j a_{ij} \cdot v\, dm + \sum_{j=1}^{d} \int_D \partial_j (a_{ij} \cdot v)\, dm,\end{aligned}$$

which equals to the right hand side of (5.9) by virtue of (5.7).

Suppose that the surface measure σ satisfies

$$I_{B\cap\partial D} \cdot \sigma \in S_{01} \quad \text{for any ball } \ B \subset R^d. \tag{5.10}$$

We then have from the boundedness of a_{ij} and condition (1.9)

$$I_{B\cap\overline{D}} \cdot |\nu| \in S_{01}.$$

Hence (5.9) and Proposition 5.1 (ii) lead us to the expression (2.3) in terms of a PCAF L in the strict sense with Revuz measure being the surface measure σ, proving Theorem 2.3 (ii).

It only remains to show (5.10) for the proof of Theorem 2.3.

Theorem 5.1 *If (2.2) holds, namely, each outward cusp boundary point is of Hölder exponent greater than $\frac{1}{2}$, then the surface measure σ on ∂D satisfies condition (5.10).*

For any ball B, the compact set $\overline{B} \cap \partial D$ can be covered by finite number of open sets $\widetilde{U_{j,\delta}}$ appearing in the condition (H) (ii) for the domain D. Besides $G_\lambda(x, y)$ is jointly continuous off diagonal by Theorem 2.1 (iii). For the proof of (5.10), it is therefore sufficient to show

$$I_\Gamma \cdot \sigma \in S_0 \quad \text{and} \quad G_\lambda I_\Gamma \cdot \sigma(x) < \infty, \quad x \in \Gamma, \tag{5.11}$$

where

$$\Gamma = \{\zeta = (\zeta', \zeta_d) : |\zeta| < \rho,\ \zeta_d = F_j(\zeta')\} \subset U_j \cap \partial D$$

for each fixed $j \in J$ and $\rho < r_j$. Exp(F_j) will be denoted by γ_j. $c_1, c_2, \cdots$ will denote some positive constants. We further let

$$\Gamma_* = \{\zeta' : (\zeta', F_j(\zeta')) \in \Gamma\} (\subset \{\zeta' : |\zeta'| < \rho\}).$$

Lemma 5.2 *Let $j \in J_+ \cup J_-$.*
(i) *$I_\Gamma \cdot \sigma \in S_0$ whenever $d \geq 3$. When $d = 2$, this is true if $\gamma_j > \frac{1}{2}$.*
(ii) *$I_{\Gamma_\delta} \cdot \sigma \in S_0$ for any $\delta > 0$ where $\Gamma_\delta = \{\zeta : \delta < |\zeta|\}$.*

Proof (i) In view of (1.1) and (5.1), it suffices to prove the inequality

$$\int_{\Gamma_*} |u(\zeta', F_j(\zeta'))|\sigma(d\zeta') \le c_1\sqrt{\mathbf{D}(u,u) + (u,u)_{L^2(D)}}, \quad u \in C_0^\infty(\overline{D}), \tag{5.12}$$

where $\mathbf{D}(u,u)$ denotes the Dirichlet integral of u on D. On account of (5.6), the surface measure σ has a density $\sigma(\zeta')$ with respect to $d\zeta'$ satisfying

$$\sigma(\zeta') \le c_2|\zeta'|^{\gamma_j - 1}. \tag{5.13}$$

Hence the square of the left hand side of (5.12) is dominated by

$$c_2^2 \int_{\Gamma_*} u(\zeta', F_j(\zeta'))^2 d\zeta' \ \cdot \int_{|\zeta'|<\rho} |\zeta'|^{2\gamma_j - 2} d\zeta'. \tag{5.14}$$

The second factor equals $\int_0^\rho r^{2\gamma_j + d - 4}dr$, which is finite under the stated condition. Consider a function $\psi \in C_0^\infty(U)$ taking value 1 on the set Γ. Then from the expression

$$u(\zeta', F_j(\zeta')) = -\int_{F_j(\zeta')}^{\sqrt{r^2 - |\zeta'|^2}} \frac{\partial}{\partial \zeta_d}\{\psi(\zeta', \zeta_d)u(\zeta', \zeta_d)\}d\zeta_d, \quad \zeta' \in \Gamma_*,$$

we see that the first factor of (5.14) is dominated by

$$c_3 \int_{U\cap D} (u^2 + |\nabla u|^2)d\zeta$$

arriving at (5.12).
(ii) Since $\sigma(\zeta')$ is bounded on $\Gamma_{\delta,*} = \{\zeta' : (\zeta', F_j(\zeta')) \in \Gamma_\delta\}$, (5.12) with Γ_* being replaced by $\Gamma_{\delta,*}$ holds for any $\delta > 0$. □

In order to complete the proof of (5.11), we prepare a lemma on a comparison of resolvent densities.

Lemma 5.3 *Let K be a compact subset of $\overline{D}$ and U be a bounded domain containing K such that the domain $D_1 = D \cap U$ possesses the property (H). Denote by $G^1_\lambda(x,y),\ x,y \in \overline{D}_1$, the resolvent density of the reflecting diffusion on $\overline{D}_1$. Then,*

$$G_\lambda(x,y) \le G^1_\lambda(x,y) + C_{18}, \qquad x,y \in K,\ x \ne y, \tag{5.15}$$

for some positive constant C_{18} depending on the set K.

Proof Consider the set $F = \overline{D} \cap U$ and the resolvent density $G^0_\lambda(x,y),\ x,y \in F$, of the part $\mathbf{M}_F$ of $\mathbf{M}$ on the set F. $\mathbf{M}_F$ is obtained from $\mathbf{M}$ by killing the sample paths upon leaving the set F. Then by Dynkin's formula

$$G_\lambda(x,y) = G^0_\lambda(x,y) + E_x\left(e^{-\lambda\tau}G_\lambda(X_\tau, y)\right),\ x,y \in F,$$

where τ denotes the leaving time from the set F. Take an open set W such that $K \subset W \subset \overline{W} \subset U$. The second term of the right side of the above identity with y being restricted to $\overline{D} \cap W$ is dominated by

$$C_{18} = \sup_{x \in D \cap \partial U,\ y \in \overline{D} \cap W} G_\lambda(x,y)$$

which is finite owing to the off diagonal continuity Theorem 2.1 (iii).

Let $\mathbf{M}_1$ be the reflecting diffusion on $\overline{D}_1$ and $\mathbf{M}_F^1$ be its part on the set $F(\subset \overline{D}_1)$. On account of [13; Theorem 4.4.3], $\mathbf{M}_F$ and $\mathbf{M}_F^1$ share a common Dirichlet form $\mathscr{E}_F$ on $L^2(F)$ given by

$$\begin{aligned} \mathscr{E}_F(u,v) &= \mathscr{E}(u,v), \qquad u,v \in \mathscr{D}[\mathscr{E}_F] \\ \mathscr{D}[\mathscr{E}_F] &= \widehat{H}(D_1), \end{aligned}$$

where $\widehat{H}(D_1)$ is defined by (3.7). Therefore we have the inequality

$$G_\lambda^0(x,y) \le G_\lambda^1(x,y)$$

holding for $m \times m$-a.e. $(x,y) \in F \times F$. In view of the continuity of G_λ and G_λ^1, we get (5.15) for every $x \in F$ and every $y \in \overline{D} \cap W$. □

We return to the set $\Gamma \subset U_j \cap \partial D$ specified before Lemma 5.2.

Lemma 5.4 *Following inequalities hold for $x,\ y \in \Gamma,\ x \neq y$ and a positive constant C_{19} depending on the set $U_j \cap D$:*
(i) *If $j \in J_+$ and $d \ge 2$, then*

$$G_\lambda(x,y) \le C_{19}|x-y|^{-\frac{d-1-\gamma_j}{\gamma_j}}. \tag{5.16}$$

(ii) *If $j \in J_0 \cup J_-$ and $d \ge 3$, then*

$$G_\lambda(x,y) \le C_{19}|x-y|^{-d+2}. \tag{5.17}$$

(iii) *If $j \in J_0 \cup J_-$ and $d = 2$, then*

$$G_\lambda(x,y) \le C_{19}|x-y|^{-\varepsilon} \qquad \textit{for any} \quad \varepsilon > 0. \tag{5.18}$$

Proof (i) Notice that the Sobolev inequality in the statement of Proposition 3.2 (i) holds with D and γ^* being replaced by $D_j = U_j \cap D$ and γ_j respectively. Since the domain D_j is bounded, we can invoke Carlen-Kusuoka-Stroock [5] to conclude in the same way as in [3; Sect. 2] that the resolvent density $G_\lambda^1(x,y)$ of the reflecting diffusion on $\overline{D_j}$ admits the estimate

$$G_\lambda^1(x,y) \le c_1|x-y|^{-\beta}, \qquad x,\ y \in \overline{D_j}, \tag{5.19}$$

for $\beta = 4/(q-2)$. In particular, by taking $q = 2(d-1+\gamma_j)/(d-1-\gamma_j)$ we see that (5.19) is valid for $\beta = (d-1-\gamma_j)/\gamma_j$. We can then use Lemma 5.3 to get (5.16).
(ii), (iii) In these cases, Proposition 3.2 (ii) is applicable to the domain $D_j = U_j \cap D$ and we see the validity of (5.19) for $\beta = d-2$ [resp. $\beta = \varepsilon > 0$] by taking $q = 2d/(d-2)$ [resp. $q = 4/\varepsilon + 2$]. We again use Lemma 5.3 to get (5.17) [resp. (5.18)]. □

Lemma 5.5 *Let $j \in J_+ \cup J_-$. Assume that $\gamma_j > \frac{1}{2}$ in case $j \in J_+$. Then $G_\lambda I_\Gamma \cdot \sigma(\zeta) < \infty,\ \zeta \in \Gamma$.*

Proof Keeping the expression

$$G_\lambda I_\Gamma \cdot \sigma(\zeta) = \int_{\Gamma_*} G_\lambda(\zeta, (\eta', F(\eta'))\sigma(\eta')d\eta'$$

and the bound (5.13) of σ in mind, we first prove the finiteness of the potential for $\zeta = 0$ in case that $j \in J_+$ and $\gamma_j > \frac{1}{2}$. From (5.16), we have the bound

$$G_\lambda I_\Gamma \cdot \sigma(0) \le c_1 \int_{\Gamma_*} \left\{ \left(|\eta'|^2 + |F(\eta')|^2\right)^{\frac{d-1-\gamma_j}{2\gamma_j}} |\eta'|^{1-\gamma_j} \right\}^{-1} d\eta'. \tag{5.20}$$

Since

$$\left(|\eta'|^2 + |F(\eta')|^2\right)^{\frac{d-1-\gamma_j}{2\gamma_j}} \ge \left(\frac{\alpha_j}{2}\right)^{\frac{d-1-\gamma_j}{\gamma_j}} |\eta'|^{d-1-\gamma_j}, \qquad |\eta'| < \delta,$$

for some $\delta > 0$, where $\alpha_j = \text{Höl}(F_j)$, we obtain

$$G_\lambda I_\Gamma \cdot \sigma(0) \le c_2 \int_0^\delta r^{d+\gamma_j-3-(d-1-\gamma_j)}\, dr + c_3 \delta^{-\frac{d-1-\gamma_j}{\gamma_j}} \int_\delta^\rho r^{d+\gamma_j-3}\, dr < \infty.$$

In the case that $j \in J_-$, we get the finiteness of $G_\lambda I_\Gamma \cdot \sigma(0)$ from (5.17) and (5.18) in a similar manner to the above.

Next take a $\zeta \in \Gamma,\ \zeta \ne 0$. We can choose a neighbourhood V of ζ such that $0 \notin V,\ V \subset U_j$ and $D_1 = V \cap D$ is a Lipschitz domain. Let $\tilde{\Gamma} = \Gamma \cap V$. Then the same reasoning as the proof of Lemma 5.3 works to see that $G_\lambda(\zeta, \eta),\ \eta \in \tilde{\Gamma}$, is dominated by $c_4|\zeta - \eta|^{-d+2}$ in case that $d \ge 3$ and by $c_5|\zeta - \eta|^{-\varepsilon},\ \varepsilon > 0$, in case that $d = 2$. Since $\sigma(\eta)$ is bounded on $\tilde{\Gamma}$, we see the finiteness of $G_\lambda I_{\tilde{\Gamma}} \cdot \sigma(\zeta)$ and hence of $G_\lambda I_\Gamma \cdot \sigma(\zeta)$. □

Proof of Theorem 5.1 We divide the situation into four cases:

(I) $j \in J_+,\ \gamma_j > 1/2$ (II) $j \in J_-,\ d \ge 3$

(III) $j \in J_-,\ d = 2$ (IV) $j \in J_0$

In view of Lemma 5.2 and Lemma 5.5, we see that (5.11) holds in cases (I) and (II). Hence it remains to prove (5.11) in cases (III) and (IV). We can instead prove a stronger property

$$\sup_{x \in \Gamma} G_\lambda I_\Gamma \cdot \sigma(x) < \infty \tag{5.21}$$

in these cases.

Indeed, when $j \in J_-$ and $d = 2$, we have the bound (5.18) of $G_\lambda(x, y)$ for any $\varepsilon > 0$, and we can proceed in a similar manner to the proof of Theorem 6.1 in our preceding paper [14] in getting (5.21) by choosing ε smaller than γ_j. When

$j \in J_0$, then we have the bound (5.17) or (5.18) of $G_\lambda(x, y)$ which, together with the uniform boundedness on Γ of the density function σ of the surface measure, readily leads us to (5.21). □

Proof of Theorem 2.4 Take any function f as is stated in the theorem and denote by W a neighbourhood of Ξ_+ on which f is constant. Then

$$I_{B\cap(\partial D\setminus W)} \cdot \sigma \in S_{01} \quad \text{for any ball } B \subset R^d. \tag{5.22}$$

To see this, it suffices to show

$$I_{\Gamma\setminus W} \cdot \sigma \in S_0 \quad \text{and} \quad G_\lambda I_{\Gamma\setminus W} \cdot \sigma(x) < \infty,\ x \in \Gamma, \tag{5.23}$$

for the set $\Gamma \subset U_j \cap \partial D$ appearing in (5.11) and exclusively for $j \in J_+$. Since $\Gamma \setminus W \subset \Gamma_\delta$ for some $\delta > 0$, the first assertion in (5.23) follows from Lemma 5.2 (ii). The second one for $x = 0$ [resp. for $x \neq 0$] is immediate from the continuity of $G_\lambda(0, y)$ [resp. from Lemma 5.5].

Now just as computations made in (5.8) and (5.9), we have

$$\mu_{\langle f,f\rangle} = 2\left(\sum_{i,j=1}^{d} a_{ij} \cdot \partial_i f \cdot \partial_j f\right) \cdot m \tag{5.24}$$

and

$$\mathscr{E}(f, v) = \int_{\overline{D}} v(x)\, \nu(dx), \qquad v \in C_0^\infty\left(\overline{D}\right),$$

with

$$\nu = -\sum_{i,j=1}^{d} \partial_i\left(a_{ij}\, \partial_j f\right) \cdot m - \sum_{i,j=1}^{d} \partial_i f \cdot a_{ij}\, n_j\, I_{\partial D\setminus W} \cdot \sigma. \tag{5.25}$$

$I_{\partial D\setminus W}$ can be inserted in the last expression because $\partial_i f$ vanishes on W.

In view of (2.4), (5.22), (5.24) and (5.25), Proposition 5.1 applies and we get

$$f(X_t) - f(X_0) = M_t^{[f]} + N_t^{[f]}, \qquad P_x\text{-a.s., } x \in \overline{D}, \tag{5.26}$$

where $M^{[f]}$ is a MAF in the strict sense with

$$\langle M^{[f]}\rangle_t = 2\int_0^t \left(\sum_{i,j=1}^{d} a_{ij}\, \partial_i f \cdot \partial_j f\right)(X_s)\, ds \tag{5.27}$$

and

$$N_t^{[f]} = \int_0^t \left(\sum_{i,j=1}^{d} \partial_i\left(a_{ij}\, \partial_j f\right)\right)(X_s)\, ds + \int_0^t \left(\sum_{i,j=1}^{d} \partial_i f \cdot a_{ij}\, n_j\right)(X_s)\, d\widetilde{L}_s. \tag{5.28}$$

Here $\widetilde{L}_t$ is a PCAF in the strict sense with Revuz measure $I_{\partial D\setminus W} \cdot \sigma$. (5.27) and (5.28) are valid P_x-a.e. for every $x \in \overline{D}$. Under the condition (2.5) for f, the second functional in the right hand side of (5.28) is a PCAF in the strict sense. Therefore the desired conclusion follows from (5.26) and (5.28). □

6 Sobolev inequality of Moser type

In this section we will show Proposition 3.1. Throughout this section we assume condition (A).

Proposition 3.1 is immediate from the following two propositions.

Proposition 6.1 *For any $\kappa \in (0,1]$ there is a positive constant $C_{20} = C_{20}(\kappa, \gamma^*, \rho^*, M^*, d)$ such that*

$$\int_{E_k^*(\rho)} |u|^2\, dx \le C_{20} \left\{ \int_N |u|^2\, dx + \rho^2 \sum_{i=1}^d \int_{E_k^*(\rho)} |\partial_i u|^2\, dx \right\}, \qquad (6.1)$$

for $E_k^*(\rho) = C_k^*(\rho)$ $\left[\text{ resp. } Q_k^*(\rho)\ \right]$, $u \in H^1\left(E_k^*(\rho)\right)$, N being a Borel subset of $E_k^*(\rho)$ with $|N| \ge \kappa |E_k^*(\rho)|$, $0 < \rho \le \rho^*$ and $k \in I_C$ $\left[\text{ resp. } k \in I_Q\ \right]$.

Proposition 6.2 (i) *Let $k \in I_C$ and $1 \le p \le (d-1+\gamma_k)/\gamma_k$. Take a q satisfying $p \le q \le p\left(d-1+\gamma_k\right)/(d-1+\gamma_k-\gamma_k p)$ if $p < (d-1+\gamma_k)/\gamma_k$, or $p \le q < \infty$ if $p = (d-1+\gamma_k)/\gamma_k$. Then there is a positive constant $C_{21} = C_{21}(p, q, \gamma^*, \rho^*, M^*, d)$ such that*

$$\left(\int_{C_k^*(\rho)} |u|^q\, dx \right)^{1/q} \le C_{21} \rho^{\frac{d-1+\gamma_k}{\gamma_k}\left(\frac{1}{q}-\frac{1}{p}\right)} \times \left\{ \int_{C_k^*(\rho)} |u|^p\, dx + \rho^p \sum_{i=1}^d \int_{C_k^*(\rho)} |\partial_i u|^p\, dx \right\}^{1/p},$$

for $u \in H^1\left(C_k^*(\rho)\right)$, $0 < \rho \le \rho^*$.

(ii) *Let $1 \le p \le d$ and take a q satisfying $p \le q \le pd/(d-p)$ if $p < d$, or $p \le q < \infty$ if $p = d$. Then there is a positive constant $C_{22} = C_{22}(p, q, \gamma^*, \rho^*, M^*, d)$ such that*

$$\left(\int_{Q_k^*(\rho)} |u|^q\, dx \right)^{1/q} \le C_{22} \rho^{d\left(\frac{1}{q}-\frac{1}{p}\right)} \times \left\{ \int_{Q_k^*(\rho)} |u|^p\, dx + \rho^p \sum_{i=1}^d \int_{Q_k^*(\rho)} |\partial_i u|^p\, dx \right\}^{1/p},$$

for $u \in H^1\left(Q_k^*(\rho)\right)$, $0 < \rho \le \rho^*$, $k \in I_Q$.

The part (i) of Proposition 6.2 is obtained by the same method as in [1; pp.128–135] if we employ a transformation $\Psi_\gamma(x)$ defined below in place of the transformation $r_k(x)$ in [1; p.130]. The part (ii) is also obtained following argument in [1; pp.103–104]. So we omit the proof of Proposition 6.2.

In order to show Proposition 6.1 with $E_k^*(\rho) = C_k^*(\rho)$, we make use of a mapping $\Psi_\gamma(x)$ from a cusp $C_\gamma(\rho)$ onto a direct product set $\Sigma(\rho)$. We begin with the definition of Ψ_γ. Let $R_+^d = \{(x', x_d) \in R^d : x_d > 0\}$ and Ξ_d be the following product space:

$$\Xi_d = \begin{cases} \{(r,t) : 0 < r < \infty,\ -\infty < t < \infty\} & \text{if } d = 2, \\ \{(r,t,\theta) : 0 < r < \infty,\ 0 \le t < \infty,\ \theta \in \Theta_{d-2}\} & \text{if } d \ge 3, \end{cases}$$

where $\Theta_{d-2} = \{(\theta_1, \theta_2, \cdots, \theta_{d-2}) : 0 \le \theta_j \le \pi\ (j = 1, \cdots, d-3),\ 0 \le \theta_{d-2} < 2\pi\}$. Given $\gamma \in (0,1)$, we define the mapping $\Psi_\gamma : R_+^d \longrightarrow \Xi_d$ as follows: When $d = 2$, we set for $x = (x_1, x_2) \in R_+^2$

$$\Psi_\gamma(x) = (r,t), \quad r = |x|\ (> 0), \quad t = x_1 x_2^{-1/\gamma}\ (\in R);$$

When $d \ge 3$, we set for $x = (x', x_d) \in R_+^d$

$$\begin{aligned} &\Psi_\gamma(x) = (r,t,\theta), \\ &r = |x|\ (> 0), \quad t = |x'| x_d^{-1/\gamma}\ (\ge 0), \\ &\theta = \begin{cases} (0, \cdots, 0)\ (\in \Theta_{d-2}) & \text{if } |x'| = 0, \\ \theta\ (\in \Theta_{d-2}) \text{ satisfying } (\phi_1(\theta), \cdots, \phi_{d-1}(\theta)) = x'/|x'| & \text{if } |x'| > 0. \end{cases} \end{aligned}$$

Here $(\phi_1(\theta), \cdots, \phi_{d-1}(\theta))$ is the spherical polar coordinate of S^{d-2}, that is, for $\theta = (\theta_1, \cdots, \theta_{d-2}) \in \Theta_{d-2}$,

$$\begin{aligned} \phi_1(\theta) &= \cos\theta_1, \\ \phi_2(\theta) &= \sin\theta_1 \cos\theta_2, \\ &\vdots \\ \phi_{d-2}(\theta) &= \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{d-3} \cos\theta_{d-2}, \\ \phi_{d-1}(\theta) &= \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{d-3} \sin\theta_{d-2}. \end{aligned}$$

For each $r > 0$, we identify the points $(r, 0, \theta)$, $\theta \in \Theta_{d-2}$ by regarding them as a same point. Under this identification, Ψ_γ is one to one from R_+^d onto Ξ_d and the inverse mapping Ψ_γ^{-1} is as follows: For $(r,t) \in \Xi_2$, let $\xi = \xi(r,t)$ be the (unique) positive solution of the equation

$$t^2 \xi^{2/\gamma} + \xi^2 = r^2. \tag{6.2}$$

Then

$$\Psi_\gamma^{-1}(r,t) = (x_1, x_2), \quad x_1 = t\xi^{1/\gamma}\ (\in R), \quad x_2 = \xi\ (> 0). \tag{6.3}$$

For $(r,t,\theta) \in \Xi_d$ with $d \ge 3$, let $\xi = \xi(r,t)$ be the positive solution of the equation (6.2). Then

$$\begin{aligned} &\Psi_\gamma^{-1}(r,t,\theta) = (x_1, x_2, \cdots, x_d), \\ &x_i = t\xi^{1/\gamma}\phi_i(\theta)\ (\in R), \quad i = 1, 2, \cdots, d-1, \quad x_d = \xi\ (> 0). \end{aligned} \tag{6.4}$$

We next note that

$$\Psi_\gamma : C_\gamma(\rho) \longrightarrow \Sigma(\rho) \quad \text{one to one, onto}$$

where $\Sigma(\rho)$ is a subset of Ξ_d given by

$$\Sigma(\rho) = \begin{cases} \{(r,t) : 0 < r < \rho,\ -1 < t < 1\} & \text{if } d = 2, \\ \{(r,t,\theta) : 0 < r < \rho,\ 0 \le t < 1,\ \theta \in \Theta_{d-2}\} & \text{if } d \ge 3. \end{cases} \tag{6.5}$$

For the sake of convenience, we write $(r,t,\theta) \in \Sigma(\rho)$ in case $d = 2$ too, and use (6.4) with the convention that $\phi_1(\theta) \equiv 1$. Since $\xi = \xi(r,t)$ is the solution of the equation (6.2), $(x_1,\cdots,x_d) = \Psi_\gamma^{-1}(r,t,\theta)$ satisfies the following relations:

$$\frac{\partial x_d}{\partial r} = \frac{r}{\xi + (1/\gamma)t^2\xi^{2/\gamma-1}}, \tag{6.6}$$

$$\frac{\partial x_d}{\partial t} = \frac{-t\xi^{2/\gamma}}{\xi + (1/\gamma)t^2\xi^{2/\gamma-1}}, \tag{6.7}$$

$$\frac{\partial x_d}{\partial \theta_j} = 0 \quad (j = 1,2,\cdots,d-2), \tag{6.8}$$

and for $i = 1,2,\cdots,d-1$,

$$\frac{\partial x_i}{\partial r} = \frac{1}{\gamma}t\xi^{1/\gamma-1}\frac{\partial x_d}{\partial r}\phi_i(\theta), \tag{6.9}$$

$$\frac{\partial x_i}{\partial t} = \left(\xi^{1/\gamma} + \frac{1}{\gamma}t\xi^{1/\gamma-1}\frac{\partial x_d}{\partial t}\right)\phi_i(\theta), \tag{6.10}$$

$$\frac{\partial x_i}{\partial \theta_j} = t\xi^{1/\gamma}\frac{\partial \phi_i(\theta)}{\partial \theta_j} \quad (j = 1,2,\cdots,d-2). \tag{6.11}$$

In the following, $A_1,\ A_2,\cdots$ denote positive constants depending only on $\gamma^*,\ \rho^*$ and d. Let $\gamma^* \le \gamma < 1,\ 0 < \rho \le \rho^*$ and $(x_1,x_2,\cdots,x_d) = \Psi_\gamma^{-1}(r,t,\theta),\ (r,t,\theta) \in \Sigma(\rho)$. Then we get the following estimates by means of (6.2), (6.3), (6.4), (6.6), (6.7):

$$A_1 r \le x_d \le r, \tag{6.12}$$

$$A_2 \le \frac{\partial x_d}{\partial r} \le A_3, \tag{6.13}$$

$$\left|\frac{\partial x_d}{\partial t}\right| \le A_4 r. \tag{6.14}$$

Further we see that the Jacobian determinant is given by

$$J(r,t,\theta) = \frac{\partial(x_1,\cdots,x_d)}{\partial(r,t,\theta_1,\cdots,\theta_{d-2})} = (-1)^{d+1}x_d^{(d-1)/\gamma}\frac{\partial x_d}{\partial r}\,t^{d-2}\,S_d(\theta), \tag{6.15}$$

where

$$S_d(\theta) = \begin{cases} 1 & \text{if } d = 2, \\ \sin^{d-3}\theta_1\ \sin^{d-4}\theta_2\cdots\sin\theta_{d-3} & \text{if } d \ge 3, \end{cases} \tag{6.16}$$

for $\theta = (\theta_1,\cdots,\theta_{d-2}) \in \Theta_{d-2}$. By means of (6.12)–(6.16), we readily get

$$|J(r,t,\theta)| \le A_3\, r^{(d-1)/\gamma}, \tag{6.17}$$

$$A_5\rho^{(d-1)/\gamma+1} \le |C_\gamma(\rho)| = \int_{\Sigma(\rho)} |J(r,t,\theta)|\,dr\,dt\,d\theta \le A_6\rho^{(d-1)/\gamma+1}. \tag{6.18}$$

We next note the following fact:

Lemma 6.1 *Let* $\gamma^* \leq \gamma < 1,\ 0 < \rho \leq \rho^*$ *and* $x^{(i)} \in C_\gamma(\rho),\ i = 0, 1,\ x^{(0)} \neq x^{(1)}$. *Set* $\left(r^{(i)}, t^{(i)}, \theta^{(i)}\right) = \Psi_\gamma\left(x^{(i)}\right),\ \theta^{(i)} = \left(\theta_1^{(i)}, \cdots, \theta_{d-2}^{(i)}\right),\ i = 0, 1$. *For* $0 \leq s \leq 1$, *put*

$$\begin{aligned} r^{(s)} &= r^{(0)} + s\left(r^{(1)} - r^{(0)}\right), \\ t^{(s)} &= t^{(0)} + s\left(t^{(1)} - t^{(0)}\right), \\ \theta^{(s)} &= \left(\theta_1^{(s)}, \cdots, \theta_{d-2}^{(s)}\right), \\ \theta_j^{(s)} &= \theta_j^{(0)} + s\left(\theta_j^{(1)} - \theta_j^{(0)}\right), \quad j = 1, 2, \cdots, d-2. \end{aligned} \tag{6.19}$$

Then $\left(r^{(s)}, t^{(s)}, \theta^{(s)}\right)$ *belongs to* $\Sigma(\rho)$ *for* $0 \leq s \leq 1$. *Moreover the following estimate holds :*

$$\left| \frac{J\left(r^{(0)}, t^{(0)}, \theta^{(0)}\right) J\left(r^{(1)}, t^{(1)}, \theta^{(1)}\right)}{J\left(r^{(s)}, t^{(s)}, \theta^{(s)}\right)} \right| \leq A_7\, \rho^{(d-1)/\gamma}, \tag{6.20}$$

$$\textit{for} \quad 0 \leq s \leq 1 \quad \textit{if} \quad d = 2, \ \textit{and}$$

$$\textit{for} \quad 0 < s < 1 \quad \textit{if} \quad d = 3 \quad \textit{and} \quad \left(t^{(0)} + t^{(1)}\right) \prod_{j=1}^{d-3} \left(\sin\theta_j^{(0)} + \sin\theta_j^{(1)}\right) \neq 0.$$

Proof In view of (6.5) it is obvious that $\left(r^{(s)}, t^{(s)}, \theta^{(s)}\right) \in \Sigma(\rho),\ 0 \leq s \leq 1$.We now assume that $\left(t^{(0)} + t^{(1)}\right) \prod_{j=1}^{d-3} \left(\sin\theta_j^{(0)} + \sin\theta_j^{(1)}\right) \neq 0$ in case $d \geq 3$. Let $0 \leq s \leq 1$ in case $d = 2$, and $0 < s < 1$ in case $d \geq 3$. Then $\left|J\left(r^{(s)}, t^{(s)}, \theta^{(s)}\right)\right| > 0$ by virtue of (6.12), (6.13), (6.15) and (6.16). Moreover

$$\begin{aligned} &\left| \frac{J\left(r^{(0)}, t^{(0)}, \theta^{(0)}\right) J\left(r^{(1)}, t^{(1)}, \theta^{(1)}\right)}{J\left(r^{(s)}, t^{(s)}, \theta^{(s)}\right)} \right| \\ &\leq \left(\frac{r^{(0)} r^{(1)}}{A_1\, r^{(s)}}\right)^{(d-1)/\gamma} \frac{A_3^2}{A_2} \left(\frac{t^{(0)} t^{(1)}}{t^{(s)}}\right)^{d-2} \frac{S_d\left(\theta^{(0)}\right) S_d\left(\theta^{(1)}\right)}{S_d\left(\theta^{(s)}\right)}. \end{aligned}$$

Note that

$$r^{(s)} \geq r^{(0)} \wedge r^{(1)}, \quad \text{and hence} \quad r^{(0)} r^{(1)} / r^{(s)} \leq \rho,$$

and if $d \geq 3$, then

$$t^{(s)} \geq t^{(0)} \wedge t^{(1)}, \quad \text{and hence} \quad t^{(0)} t^{(1)} / t^{(s)} \leq 1,$$

and for $j = 1, 2, \cdots, d-3$,

$$\sin\theta_j^{(s)} \geq \begin{cases} \sin\theta_j^{(0)} & \text{if } \left|\theta_j^{(0)} - \pi/2\right| \geq \left|\theta_j^{(1)} - \pi/2\right|, \\ \sin\theta_j^{(1)} & \text{if } \left|\theta_j^{(0)} - \pi/2\right| \leq \left|\theta_j^{(1)} - \pi/2\right|, \end{cases}$$

consequently

$$0 \leq \frac{S_d\left(\theta^{(0)}\right) S_d\left(\theta^{(1)}\right)}{S_d\left(\theta^{(s)}\right)} \leq 1.$$

We thus get (6.20). □

For an open set $E \subset R^d$, denote by $C^1(\overline{E})$ the restrictions to $\overline{E}$ of all continuously differentiable functions on R^d. c_1, c_2, etc. appearing in what follows denote positive constants depending only on γ^*, ρ^* and d.

Lemma 6.2 *For any $\kappa \in (0,1]$, there is a positive constant $C_{23} = C_{23}(\kappa, \rho^*, \gamma^*, d)$ such that*

$$\int_{C_\gamma(\rho)} |u|^2\, dx \le C_{23} \left\{ \int_N |u|^2\, dx + \rho^2 \sum_{i=1}^d \int_{C_\gamma(\rho)} |\partial_i u|^2\, dx \right\}, \tag{6.21}$$

for $u \in C^1\big(\overline{C_\gamma(\rho)}\big)$, a Borel subset $N \subset C_\gamma(\rho)$ with $|N| \ge \kappa |C_\gamma(\rho)|$, $\gamma^* \le \gamma < 1$ and $0 < \rho \le \rho^*$.

Proof Let $\gamma^* \le \gamma < 1$ and $0 < \rho \le \rho^*$. For $u \in C^1\big(\overline{C_\gamma(\rho)}\big)$, we set

$$\begin{aligned} &x = \Psi_\gamma^{-1}(r,t,\theta) \in C_\gamma(\rho), \\ &\widetilde{u}(r,t,\theta) = u \circ \Psi_\gamma^{-1}(r,t,\theta) = u(x), \\ &\left|\widetilde{\partial u}(r,t,\theta)\right| = \sum_{i=1}^d \left| \frac{\partial u}{\partial x_i} \circ \Psi_\gamma^{-1}(r,t,\theta) \right|. \end{aligned}$$

By virtue of (6.8) – (6.11),

$$\begin{aligned} \frac{\partial \widetilde{u}}{\partial r}(r,t,\theta) &= \left\{ \frac{1}{\gamma}\, t\, x_d^{1/\gamma - 1} \sum_{i=1}^{d-1} \frac{\partial u}{\partial x_i}(x)\, \phi_i(\theta) + \frac{\partial u}{\partial x_d}(x) \right\} \frac{\partial x_d}{\partial r}, \\ \frac{\partial \widetilde{u}}{\partial t}(r,t,\theta) &= x_d^{1/\gamma} \sum_{i=1}^{d-1} \frac{\partial u}{\partial x_i}(x)\, \phi_i(\theta) \\ &\quad + \left\{ \frac{1}{\gamma}\, t\, x_d^{1/\gamma - 1} \sum_{i=1}^{d-1} \frac{\partial u}{\partial x_i}(x)\, \phi_i(\theta) + \frac{\partial u}{\partial x_d}(x) \right\} \frac{\partial x_d}{\partial t}, \\ \frac{\partial \widetilde{u}}{\partial \theta_j}(r,t,\theta) &= t\, x_d^{1/\gamma} \sum_{i=j}^{d-1} \frac{\partial u}{\partial x_i}(x)\, \frac{\partial \phi_i}{\partial \theta_j}(\theta), \quad j = 1, 2, \cdots, d-2. \end{aligned}$$

Combining these with (6.12), (6.13), (6.14), we find that

$$\left| \frac{\partial \widetilde{u}}{\partial r}(r,t,\theta) \right| \le A_8 \left| \widetilde{\partial u}(r,t,\theta) \right|, \tag{6.22}$$

$$\left| \frac{\partial \widetilde{u}}{\partial t}(r,t,\theta) \right| \le A_9\, r \left| \widetilde{\partial u}(r,t,\theta) \right|, \tag{6.23}$$

$$\left| \frac{\partial \widetilde{u}}{\partial \theta_j}(r,t,\theta) \right| \le A_{10}\, rt \left| \widetilde{\partial u}(r,t,\theta) \right|, \quad j = 1, 2, \cdots, d-2. \tag{6.24}$$

Let $0 < \kappa \le 1$, $N \subset C_\gamma(\rho)$ with $|N| \ge \kappa |C_\gamma(\rho)|$, $x^{(0)} \in C_\gamma(\rho)$ and $x^{(1)} \in N$. Put $\sigma^{(i)} = \big(r^{(i)}, t^{(i)}, \theta^{(i)}\big) = \Psi_\gamma\big(x^{(i)}\big)$, $i = 0, 1$. Then

$$u\left(x^{(0)}\right) - u\left(x^{(1)}\right) = \widetilde{u}\left(\sigma^{(0)}\right) - \widetilde{u}\left(\sigma^{(1)}\right) = -\int_0^1 \frac{\partial}{\partial s}\widetilde{u}\left(\sigma^{(s)}\right) ds,$$

where $\sigma^{(s)} = \left(r^{(s)}, t^{(s)}, \theta^{(s)}\right)$, and $r^{(s)},\ t^{(s)},\ \theta^{(s)}$ are those given by (6.19). Integrating over $x^{(1)} \in N$, we find that

$$|N|\ \left|u\left(x^{(0)}\right)\right| \le \int_N |u|\, dx + \int_{\sigma^{(1)}\in\Sigma(\rho)} \left|J\left(\sigma^{(1)}\right)\right|\, d\sigma^{(1)} \int_0^1 \left|\frac{\partial}{\partial s}\widetilde{u}\left(\sigma^{(s)}\right)\right| ds.$$

Here $d\sigma^{(i)}$ denotes the product measure $dr^{(i)}dt^{(i)}d\theta^{(i)}$ for each $i = 1, 2$. From this

$$\begin{aligned}
&|N|^2 \int_{C_\gamma(\rho)} |u|^2\, dx \\
&\le 2|C_\gamma(\rho)| \left(\int_N |u|\, dx\right)^2 \\
&\quad +2\int_{\sigma^{(0)}\in\Sigma(\rho)} \left|J\left(\sigma^{(0)}\right)\right|\, d\sigma^{(0)} \left(\int_{\substack{\sigma^{(1)}\in\Sigma(\rho)\\ 0<s<1}} \left|\frac{\partial}{\partial s}\widetilde{u}\left(\sigma^{(s)}\right)\right| \left|J\left(\sigma^{(1)}\right)\right|\, d\sigma^{(1)}ds\right)^2 \\
&\equiv 2(I + I\!I). \qquad (6.25)
\end{aligned}$$

Obviously,

$$I \le |C_\gamma(\rho)|^2 \int_N |u|^2\, dx. \qquad (6.26)$$

On account of Lemma 6.1 and (6.17),

$$\begin{aligned}
I\!I \ &\le \int_{\sigma^{(0)}\in\Sigma(\rho)} d\sigma^{(0)} \int_{\substack{\sigma^{(1)}\in\Sigma(\rho)\\ 0<s<1}} \left|\frac{\partial}{\partial s}\widetilde{u}\left(\sigma^{(s)}\right)\right|^2 \left|J\left(\sigma^{(s)}\right) J\left(\sigma^{(1)}\right)\right|\, d\sigma^{(1)}ds \\
&\qquad \times \int_{\substack{\sigma^{(1)}\in\Sigma(\rho)\\ 0<s<1}} \left|\frac{J\left(\sigma^{(0)}\right)J\left(\sigma^{(1)}\right)}{J\left(\sigma^{(s)}\right)}\right| d\sigma^{(1)}ds \\
&\le A_3A_7\rho^{2(d-1)/\gamma} \int_{\substack{\sigma^{(0)}\in\Sigma(\rho)\\ \sigma^{(1)}\in\Sigma(\rho)\\ 0<s<1}} \left|\frac{\partial}{\partial s}\widetilde{u}\left(\sigma^{(s)}\right)\right|^2 \left|J\left(\sigma^{(s)}\right)\right|\, d\sigma^{(0)}d\sigma^{(1)}ds \\
&\qquad \times \int_{\substack{\sigma^{(1)}\in\Sigma(\rho)\\ 0<s<1}} d\sigma^{(1)}ds \\
&= c_1\rho^{2(d-1)/\gamma+1} \int_{\substack{\sigma^{(0)}\in\Sigma(\rho)\\ \sigma^{(1)}\in\Sigma(\rho)\\ 0<s<1}} \left|\frac{\partial}{\partial s}\widetilde{u}\left(\sigma^{(s)}\right)\right|^2 \left|J\left(\sigma^{(s)}\right)\right|\, d\sigma^{(0)}d\sigma^{(1)}ds. \qquad (6.27)
\end{aligned}$$

On the other hand, we get from (6.22)–(6.24)

$$\begin{aligned}\left|\frac{\partial}{\partial s}\widetilde{u}\left(\sigma^{(s)}\right)\right| &\le \left|\frac{\partial\widetilde{u}}{\partial r}\left(\sigma^{(s)}\right)\right|\left|r^{(0)}-r^{(1)}\right| + \left|\frac{\partial\widetilde{u}}{\partial t}\left(\sigma^{(s)}\right)\right|\left|t^{(0)}-t^{(1)}\right| \\ &\quad + \sum_{j=1}^{d-2}\left|\frac{\partial\widetilde{u}}{\partial\theta_j}\left(\sigma^{(s)}\right)\right|\left|\theta_j^{(0)}-\theta_j^{(1)}\right| \\ &\le c_2\rho\sum_{i=1}^{d}\left|\frac{\partial u}{\partial x_i}\circ\Psi_\gamma^{-1}\left(\sigma^{(s)}\right)\right|. \end{aligned} \tag{6.28}$$

Denoting $\dfrac{\partial u}{\partial x_i}\circ\Psi_\gamma^{-1}$ by v_i and substituting (6.28) into (6.27), we arrive at

$$\begin{aligned} I\!I &\le c_3\,\rho^{2(d-1)/\gamma+3}\sum_{i=1}^{d}\int_{\substack{\sigma^{(0)}\in\Sigma(\rho)\\ \sigma^{(1)}\in\Sigma(\rho)\\ 0<s<1}} v_i\left(\sigma^{(s)}\right)^2\left|J\left(\sigma^{(s)}\right)\right|\,d\sigma^{(0)}d\sigma^{(1)}ds \\ &\equiv c_3\,\rho^{2(d-1)/\gamma+3}\sum_{i=1}^{d} I\!I_i(\rho). \end{aligned} \tag{6.29}$$

By fixing $\sigma^{(0)} = \left(r^{(0)},t^{(0)},\theta^{(0)}\right)\in\Sigma(\rho)$ and $0<s<1$, we make use of the transformation $\sigma^{(1)} = \left(r^{(1)},t^{(1)},\theta^{(1)}\right)\longmapsto\sigma^{(s)} = \left(r^{(s)},t^{(s)},\theta^{(s)}\right)$. Putting $\sigma=(r,t,\theta)=\left(r^{(s)},t^{(s)},\theta^{(s)}\right)$, we find that the Jacobian determinant is given by $\partial\sigma^{(1)}/\partial\sigma = s^{-d}$. Moreover $\sigma=(r,t,s)$ exhausts a set $\Sigma\left(\sigma^{(0)},s\right)$ specified by

$$\begin{aligned} (1-s)r^{(0)} &< r < \rho s+(1-s)r^{(0)}, \\ a_d+(1-s)t^{(0)} &< t < s+(1-s)t^{(0)}, \\ (1-s)\theta_j^{(0)} &< \theta_j < \alpha_j s+(1-s)\theta_j^{(0)}, \quad j=1,2,\cdots,d-2, \end{aligned}$$

where $a_d=-1$ if $d=2$, $=0$ if $d\ge 3$, and $\alpha_j=\pi$ $(j=1,2,\cdots,d-3)$, $\alpha_{d-2}=2\pi$. So we get, for each $i=1,2,\cdots,d$,

$$\begin{aligned} I\!I_i(\rho) &= \int_{\substack{\sigma^{(0)}\in\Sigma(\rho)\\ 0<s<1}} d\sigma^{(0)}ds\int_{\sigma^{(1)}\in\Sigma(\rho)} v_i\left(\sigma^{(s)}\right)^2\left|J\left(\sigma^{(s)}\right)\right|\,d\sigma^{(1)} \\ &= \int_{\substack{\sigma^{(0)}\in\Sigma(\rho)\\ 0<s<1}} d\sigma^{(0)}ds\int_{\sigma\in\Sigma\left(\sigma^{(0)},s\right)} v_i(\sigma)^2\left|J(\sigma)\right|s^{-d}\,d\sigma. \end{aligned}$$

By exchanging the order of integration,

$$\begin{aligned} I\!I_i(\rho) &= \int_{\substack{\sigma\in\Sigma(\rho)\\ 0<s<1}} v_i(\sigma)^2|J(\sigma)|s^{-d}\left(\frac{r}{1-s}\wedge\rho-\frac{r-\rho s}{1-s}\vee 0\right) \\ &\qquad\times\left(\frac{t}{1-s}\wedge 1-\frac{t-s}{1-s}\vee a_d\right)\prod_{j=1}^{d-2}\left(\frac{\theta_j}{1-s}\wedge\alpha_j-\frac{\theta_j-\alpha_j s}{1-s}\vee 0\right)d\sigma ds \\ &\le 2^{d+1}\pi^{d-2}\rho\int_{\Sigma(\rho)} v_i(\sigma)^2|J(\sigma)|\,d\sigma. \end{aligned}$$

Combining this with (6.29) and (6.18), we have

$$\begin{aligned} I\!I &\leq c_4 \rho^{2(d-1)/\gamma+4} \sum_{i=1}^{d} \int_{\Sigma(\rho)} v_i(\sigma)^2 |J(\sigma)|\, d\sigma \\ &\leq c_5 \rho^2 |C_\gamma(\rho)|^2 \sum_{i=1}^{d} \int_{C_\gamma(\rho)} |\partial_i u|^2 \, dx. \end{aligned} \tag{6.30}$$

Since $|N| \geq \kappa |C_\gamma(\rho)|$, (6.25), (6.26) and (6.30) lead us to (6.21). □

Lemma 6.3 *Let $0 < \rho \leq \rho^*$ and $E(\rho)$ be the following subset of R^d with $d \geq 2$.*

$$E(\rho) = \{(x', x_d) \in B(\rho) : x_d > g(x')\},$$

where g is a continuous function on $\{x' \in R^{d-1} : |x'| < \rho^*\}$ such that $g(0) = 0$ and $g(x') \leq 0,\ |x'| < \rho^*$. Then the statement of Lemma 6.2 with $C_\gamma(\rho)$ replaced by $E(\rho)$ above holds.

Proof Let $u \in C^1\big(\overline{E(\rho)}\big)$ and N $(\subset E(\rho))$ be a Borel subset satisfying $|N| \geq \kappa |E(\rho)|$. For $x = (x', x_d) \in E(\rho)$, we set $\bar{x} = (x', |x_d|)$. We also use the polar coordinate with center $\bar{x}$: $\bar{y} = \bar{x} + r\omega,\ r = |\bar{y} - \bar{x}|,\ \omega = (\bar{y} - \bar{x})/r \in S^{d-1}$. The following inequality is obvious:

$$\begin{aligned} |N|\,|u(x)| &\leq \int_N |u(y)|\, dy + |N| \left|u(x) - u\big(\bar{x}\big)\right| \\ &+ \int_N \left|u(y) - u\big(\bar{y}\big)\right| dy + \int_N \left|u\big(\bar{x}\big) - u\big(\bar{y}\big)\right| dy, \end{aligned}$$

from which we obtain

$$\begin{aligned} &|N|^2 |u(x)|^2 \\ &\leq c_1 \Bigg\{ \left(\int_N |u|\, dy \right)^2 + |N|^2 \left(\int_{x_d}^{|x_d|} \partial_d u(x', s)\, ds \right)^2 \\ &\quad + \left(\int_{y \in E(\rho), y_d < 0} dy \int_{y_d}^{|y_d|} |\partial_d u(y', s)|\, ds \right)^2 \\ &\quad + \sum_{i=1}^{d} \left(\int_{y \in E(\rho), \bar{y} = \bar{x} + r\omega} dy \int_0^r \left|\partial_i u\big(\bar{x} + s\omega\big)\right| ds \right)^2 \Bigg\}, \\ &\leq c_1 \Bigg\{ |E(\rho)| \int_N |u|^2\, dy + 2|E(\rho)|^2 \rho \int_{x_d}^{|x_d|} |\partial_d u(x', s)|^2\, ds \\ &\quad + 2|E(\rho)| \rho^2 \int_{E(\rho)} |\partial_d u|^2\, dy \\ &\quad + 4|E(\rho)| \rho^{d+1} \sum_{i=1}^{d} \int_{z \in E(\rho), z_d \geq 0} \frac{|\partial_i u(z)|^2}{|\bar{x} - z|^{d-1}}\, dz \Bigg\}, \end{aligned}$$

where we used the following estimate for the last term.

$$\begin{aligned}
&\left(\int_{y\in E(\rho),\overline{y}=\overline{x}+r\omega} dy \int_0^r \left|\partial_i u\left(\overline{x}+s\omega\right)\right| ds\right)^2 \\
&\leq 4|E(\rho)|\rho \int_{y\in E(\rho),\overline{y}=\overline{x}+r\omega} d\overline{y} \int_0^r \left|\partial_i u\left(\overline{x}+s\omega\right)\right|^2 ds \\
&= 4|E(\rho)|\rho \int_{y\in E(\rho),\overline{y}=\overline{x}+r\omega} r^{d-1}\, dr d\omega \int_0^r \left|\partial_i u\left(\overline{x}+s\omega\right)\right|^2 ds \\
&\leq 4|E(\rho)|\rho^{d+1} \int_{z\in E(\rho),z_d\geq 0} \frac{|\partial_i u(z)|^2}{|\overline{x}-z|^{d-1}}\, dz .
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&|N|^2 \int_{E(\rho)} |u(x)|^2 dx \\
&\leq c_2\Bigg\{|E(\rho)|^2 \int_N |u|^2\, dy + |E(\rho)|^2\rho^2 \int_{E(\rho)} |\partial_d u|^2\, dx \\
&\qquad + |E(\rho)|\rho^{d+1} \sum_{i=1}^d \int_{z\in E(\rho),z_d\geq 0} |\partial_i u(z)|^2\, dz \int_{E(\rho)} \frac{dx}{|\overline{x}-z|^{d-1}}\Bigg\} \\
&\leq c_3\Bigg\{|E(\rho)|^2 \int_N |u|^2\, dy + |E(\rho)|^2\rho^2 \int_{E(\rho)} |\partial_d u|^2\, dx \\
&\qquad + |E(\rho)|\rho^{d+2} \sum_{i=1}^d \int_{E(\rho)} |\partial_i u|^2\, dx\Bigg\}.
\end{aligned}$$

Noting that $c_4\rho^d \leq |E(\rho)| \leq c_5\rho^d$, we get the conclusion. □

Proposition 6.1 now follows from Lemmas 6.3 and 6.4.

Proof of Proposition 6.1 Let $k \in I_C$, $0 < \rho \leq \rho^*$, and N be a Borel subset of $C_k^*(\rho)$ satisfying $|N| \geq \kappa|C_k^*(\rho)|$. In view of [16; Theorem 1.1.7], $u \circ \Phi_k \in H^1\left(C_{\gamma_k}(\rho)\right)$ provided $u \in H^1\left(C_k^*(\rho)\right)$. By means of [1 : Theorem 3.18], $u \circ \Phi_k$ is approximated by functions belonging to $C^1\left(\overline{C_{\gamma_k}(\rho)}\right)$ in H^1-norm. Noting that $\Phi_k^{-1}(N)$ is a Borel subset of $C_{\gamma_k}(\rho)$ and satisfies $\left|\Phi_k^{-1}(N)\right| \geq \kappa' \left|C_{\gamma_k}(\rho)\right|$ for some $\kappa' \in (0,1]$, we obtain (6.1) with $E_k^*(\rho) = C_k^*(\rho)$ from Lemma 6.3. Similarly (6.1) with $E_k^*(\rho) = Q_k^*(\rho)$ follows from Lemma 6.4. □

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ON SEMI-MARTINGALE CHARACTERIZATIONS OF FUNCTIONALS OF SYMMETRIC MARKOV PROCESSES

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Abstract For a quasi-regular (symmetric) Dirichlet space $(\mathcal{E}, \mathcal{F})$ and an associated symmetric standard process (X_t, P_x), we show that, for $u \in \mathcal{F}$, the additive functional $u^*(X_t) - u^*(X_0)$ is a semimartingale if and only if there exists an $\mathcal{E}$-nest $\{F_n\}$ and positive constants C_n such that $|\mathcal{E}(u,v)| \leq C_n \|v\|_\infty,\ v \in \mathcal{F}_{F_n,b}$. In particular, a signed measure resulting from the inequality will be automatically smooth. One of the variants of this assertion is applied to the distorted Brownian motion on a closed subset of R^d, giving stochastic characterizations of BV functions and Caccioppoli sets.

Keywords Quasi-regular Dirichlet form, strongly regular representation, additive functionals, semimartingale, smooth signed measure, BV function

AMS subject classification 60J45, 60J55, 31C25

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1 Introduction

We consider a Hausdorff topological space X and a σ-finite positive Borel measure m on X. Let $(\mathcal{E}, \mathcal{F})$ be a (symmetric) Dirichlet form on $L^2(X; m)$, namely, $\mathcal{E}$ is a Markovian closed symmetric form with domain $\mathcal{F}$ linear dense in $L^2(X; m)$. In this paper, we follow exclusively [MR 92] for the definition of $\mathcal{E}$-quasi notions. For a closed set $F \subset X$, we set

$$\mathcal{F}_F = \{u \in \mathcal{F} : u = 0 \ m\text{-a.e. on } X \setminus F\} \quad \mathcal{F}_{b,F} = \mathcal{F}_F \cap L^\infty(X; m).$$

An increasing family $\{F_n\}$ of closed sets is called an $\mathcal{E}$-*nest* if the space $\cup_{n=1}^\infty \mathcal{F}_{F_n}$ is $\mathcal{E}_1$-dense in $\mathcal{F}$. A set $N \subset X$ is said to be $\mathcal{E}$-*exceptional* if $N \subset \cap_{n=1}^\infty F_n^c$ for some $\mathcal{E}$-nest $\{F_n\}$. '$\mathcal{E}$-*quasi-everywhere*' or '$\mathcal{E}$-*q.e.*' will mean 'except for an $\mathcal{E}$-exceptional set'. A function defined $\mathcal{E}$-q.e. on X is said to be $\mathcal{E}$-*quasicontinuous* if there exists an $\mathcal{E}$-nest $\{F_n\}$ such that the restriction of u to each set F_n is continuous there. When the Dirichlet form is quasi-regular in the sense of [MR 92], any $u \in \mathcal{F}$ admits an $\mathcal{E}$-quasicontinuous version which will be denoted by u^*.

We consider a quasi-regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ and an associated standard process $\mathbf{M} = (X_t, P_x)$ on X. A purpose of the present paper is to prove that, for $u \in \mathcal{F}$, the additive functional(AF in abbreviation)

$$A_t^{[u]} = u^*(X_t) - u^*(X_0) \tag{1}$$

of the process $\mathbf{M}$ is a semimartingale ,namely, a sum of a martingale and a process of bounded variation, if and only if there exists an $\mathcal{E}$-nest $\{F_n\}$ and positive constants $C_n,\ n = 1, 2, \cdots$ such that u satisfies

$$|\mathcal{E}(u, v)| \leq C_n \|v\|_\infty, \quad \forall v \in \mathcal{F}_{b,F_n}, \quad n = 1, 2, \cdots, \tag{2}$$

where $\|v\|_\infty$ denotes the m-essential sup norm of $v \in L^\infty(X; m)$.

The existence of a special standard process associated with a quasi-regular Dirichlet form is well known [MR 92]. In the present paper, we give another construction of an associated tight special standard process as an image by a quasi-homeomorphism of a Hunt modification of a Ray process. The importance of the notion of quasi-regularity of a Dirichlet form is in that it is not only sufficient but also necessary for the existence of an associated right process which is m-special standard and m-tight ([AM 91],[MR 92]). For instance, given simply an m-symmetric right process $\mathbf{M}$ on a Lusin topological space X, $\mathbf{M}$ automatically becomes m-tight and m-special standard, and consequently the associated Dirichlet form $\mathcal{E}$ on $L^2(X; m)$ becomes quasi-regular and $\mathbf{M}$ can be modified outside some $\mathcal{E}$-exceptional set to be a tight special standard process ([MR 92], [Fi 97]).

We note here that, when the Dirichlet space is regular in the sense of [FOT 94], the $\mathcal{E}$-quasi notions of [MR 92] defined above can be identified with those classical quasi notions introduced for instance in [FOT 94] in terms of the ($\mathcal{E}_1$-)capacity. Indeed, an increasing sequence of closed sets is an $\mathcal{E}$-nest iff it is a generalized nest in the sense of [FOT 94], a set is $\mathcal{E}$-exceptional iff it is of zero capacity, and a function is $\mathcal{E}$-quasicontinuous iff it is quasi continuous in the sense of [FOT 94] (see Lemma 2.1). We shall take those identifications for granted for the moment.

When the Dirichlet space is regular and $\mathbf{M}$ is an associated Hunt process, the following facts are already known ([F 80], [FOT 94, Theorem 5.4.2]): the AF $A^{[u]}$ for $u \in \mathcal{F}$ can be uniquely decomposed as

$$A^{[u]} = M^{[u]} + N^{[u]}, \tag{3}$$

where $M^{[u]}$ is a martingale AF of finite energy and $N^{[u]}$ is a continuous AF of zero energy. $N^{[u]}$ needs not be of bounded variation. It is of bounded variation if and only if there exists a set function ν on X and an $\mathcal{E}$-nest $\{F_n\}$ such that $\nu|_{F_n}$ is a finite signed measure for each n and ν charges no $\mathcal{E}$-exceptional set (such a set function ν is called a *smooth signed measure with an attached $\mathcal{E}$-nest* $\{F_n\}$) and further

$$\mathcal{E}(u,v) = -\int_X v^*(x)d\nu(x), \qquad \forall v \in \cup_{n=1}^{\infty}\mathcal{F}_{b,F_n}. \tag{4}$$

In this case moreover, expressing ν as a difference $\nu^1 - \nu^2$ of two positive smooth measures and denoting by A^k the positive continuous AF associated with ν^k by the Revuz correspondence, $k = 1, 2$, it holds that

$$N^{[u]} = A^1 - A^2. \tag{5}$$

The above mentioned facts for a regular Dirichlet form and an associated Hunt process will be systematically extended in §5 to a general quasi-regular Dirichlet space and an associated standard process. Here we will make use of a regular representation and an associated quasi-homeomorphism of the underlying spaces as will be formulated in §2 following [F 71a], [F 71b], [FOT 94], [CMR 94]. Such a method was already adopted in [MR 92] in their specific context of a (local) compactification where a quasi-homeomorphism is realized by an embedding, and they called it a *transfer method*. But we would like to use this term in the present more general context. A Dirichlet space is called strongly regular if the associated resolvent admits a version possessing a Ray property (see §2 for precise definition). In §4, we crucially need a refined transfer method involving a strongly regular representation.

Thus we basically need to show for $u \in \mathcal{F}$ the equivalence between the inequality (2) and the existence of a signed smooth measure ν satisfying equation (4). This will be done in §3 and §4. While the inequality (2) follows readily from the equation (4), the proof of the converse implication, especially the derivation of the smoothness of a signed measure is a very delicate matter. It has been known however that a kind of strong Feller property of the resolvent of the associated Markov process $\mathbf{M}$ yields the desired smoothness([CFW 93], [FOT 94, Theorem 5.4.3], [F 97a]). In §3, we shall work with a strongly regular Dirichlet space to show that this requirement can be weakened to a Ray property of the resolvent of the associated process. We shall then employ in §4 a transfer method involving a strongly regular representation of the Dirichlet space prepared in §2 to complete the proof of the desired equivalence.

The condition (2) is accordingly more easily verifiable than the existence of a signed smooth measure satisfying equation (4). Actually it is enough to require inequality (2) holding for v in a more tractable dense subspace of $\mathcal{F}_{b,F_n}$. For instance, consider the simple case that the nest is trivial: $F_n = X, n = 1, 2, \cdots$. Then (2) is reduced to the condition that, for $u \in \mathcal{F}$, there exists a positive constant C such that the inequality

$$|\mathcal{E}(u,v)| \leq C\,\|v\|_\infty, \tag{6}$$

holds for any v in the space

$$\mathcal{F}_b = \mathcal{F} \cap L^\infty(X; m).$$

As will be seen in §6, the above condition is equivalent to the one obtained by replacing the space $\mathcal{F}_b$ with its subspace $\mathcal{L}$ satisfying

($\mathcal{L}$) $\mathcal{L}$ is an $\mathcal{E}_1$-dense linear subspace of $\mathcal{F}_b$, and, for any $\epsilon > 0$, there exists a real function $\phi_\epsilon(t)$ such that

$$\begin{aligned} |\phi_\epsilon(t)| &\leq 1+\epsilon,\ t \in R; \quad \phi_\epsilon(t) = t,\ t \in [-1,1]; \\ 0 &\leq \phi_\epsilon(t) - \phi_\epsilon(s) \leq t - s,\ s < t,\ s,t \in R, \end{aligned}$$

and $\phi_\epsilon(\mathcal{L}) \subset \mathcal{L}$.

We shall further see in §6 that inequality (6) holding for $v \in \mathcal{L}$ is not only sufficient but also necessary for the AF $N_t^{[u]}$ to be of bounded variation with an additional property that

$$\lim_{t\downarrow 0} \frac{1}{t} E_m \left(\int_0^t |dN_s^{[u]}| \right) < \infty, \tag{7}$$

where the integral inside the braces denotes the total variation of $N^{[u]}$ on the interval $[0,t]$. The property (7) says that this PCAF has a finite Revuz measure.

When X is an infinite dimensional vector space and $\mathcal{E}$ is obtained by closing a pre-Dirichlet form defined for smooth cylindrical functions on X, we may take as $\mathcal{L}$ the set of all smooth cylindrical functions to check inequality (6).

When the Dirichlet space is regular, a natural choice of $\mathcal{L}$ is a dense subspace of $C_0(X)$ satifying the condition ($\mathcal{L}$). In this case, inequality (6) for $\mathcal{L}$ is evidently equivalent to the existence of a unique finite signed measure ν satisfying the equation (4) for all $v \in \mathcal{L}$. Our general theorem in §4 assures that this ν is automatically smooth, namely, it charges no set of zero $\mathcal{E}_1$-capacity. When the Dirichlet space is not only regular but also strongly local, we shall extend in §6 the last statement to a function $u \in \mathcal{F}_{loc}$ satisfying equation (4) for a signed Radon measure ν and for all v belonging to a more specific subspace $\mathcal{L}$ of $\mathcal{F} \cap C_0(X)$. We will see that this property of u is equivalent to the condition that $N_t^{[u]}$ is of bounded variation and satisfies, for any compact set K,

$$\lim_{t\downarrow 0} \frac{1}{t} E_m \left(\int_0^t I_K(X_s) |dN_s^{[u]}| \right) < \infty. \tag{8}$$

In §7 we will apply the last theorem of §6 to the energy forms $\mathcal{E}$ and the associated distorted Brownian motions $\mathbf{M}$ living on closed subsets of R^d. In particular, we shall improve those results obtained in [F 97a],[F 97b] to complete stochastic characterizations of BV functions and Caccioppoli sets.

Finally we mention a celebrated paper [CJPS 80], in which it was already proved that, given a general Markov process $\mathbf{M} = (X_t, P_x)$ on a general state space X, a function u on X produces a semimartingale $u(X_t)$ under P_x for every $x \in X$ if and only if there exist finely open sets E_n with

$$\cup_{n=1}^{\infty} E_n = X, \qquad P_x(\lim_{n\to\infty} \tau_{E_n} = \infty) = 1 \ \ \forall x \in X$$

such that u is a difference of two excessive functions on each set E_n. In the present paper, we restrict ourselves to the case that $\mathbf{M}$ is symmetric and the semimartingale property of $u(X_t)$ is required to hold only for $\mathcal{E}$-q.e. starting point $x \in X$. As is clear from the above explanations, our necessary and sufficient conditions of the type (2) are more easily verifiable in many cases where X are of higher dimensions.

2 Representation and quasi-homeomorphism

We say that a quadruplet $(X, m, \mathcal{E}, \mathcal{F})$ is a *Dirichlet space* if X is a Hausdorff topological space with a countable base, m is a σ-finite positive Borel measure on X and $\mathcal{E}$ with domain $\mathcal{F}$ is a Markovian closed symmetric form on $L^2(X; m)$. The inner product in $L^2(X; m)$ is denoted by $(\cdot, \cdot)_X$ and we let

$$\mathcal{E}_\alpha(\cdot, \cdot) = \mathcal{E}(\cdot, \cdot) + \alpha(\cdot, \cdot)_X \quad \alpha > 0.$$

We note that the space $\mathcal{F}_b = \mathcal{F} \cap L^\infty(X; m)$ is an algebra.

Given two Dirichlet spaces

$$(X, m, \mathcal{E}, \mathcal{F}), \qquad (\tilde{X}, \tilde{m}, \tilde{\mathcal{E}}, \tilde{\mathcal{F}}),$$

we call them *equivalent* if there is an algebraic isomorphism Φ from $\mathcal{F}_b$ onto $\tilde{\mathcal{F}}_b$ preserving three kinds of metrics: for $u \in \mathcal{F}_b$

$$\|u\|_\infty = \|\Phi u\|_\infty, \ (u, u)_X = (\Phi u, \Phi u)_{\tilde{X}}, \ \mathcal{E}(u, u) = \tilde{\mathcal{E}}(\Phi u, \Phi u).$$

One of the two equivalent Dirichlet spaces is called a *representation* of the other.

The underlying spaces X, $\tilde{X}$ are said to be *quasi-homeomorphic* if there exist $\mathcal{E}$-nest $\{F_n\}$, $\tilde{\mathcal{E}}$-nest $\{\tilde{F}_n\}$ and a one to one mapping q from $X_0 = \cup_{n=1}^{\infty} F_n$ onto $\tilde{X}_0 = \cup_{n=1}^{\infty} \tilde{F}_n$ such that its restricition to each F_n is homeomorphic to $\tilde{F}_n$.

We say that the equivalence as above is *induced by a quasi-homeomorphism* if there exists a mapping q as above such that

$$\Phi u(\tilde{x}) = u(q^{-1}(\tilde{x})) \qquad \tilde{x} \in X_0.$$

Then $\tilde{m}$ is the image measure of m by q and $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ is the image of $(\mathcal{E}, \mathcal{F})$ by q. Furthermore q is quasi-notion preserving ([CMR 94, Cor.3.6]):

1. Let $\{E_n\}$ be an increasing sequence of closed subsets of X. It is an $\mathcal{E}$-nest if and only if $\{q(F_n \cap E_n)\}$ is an $\tilde{\mathcal{E}}$-nest.

2. $N \subset X$ is $\mathcal{E}$-exceptional if and only if $q(X_0 \cap N)$ is $\tilde{\mathcal{E}}$-exceptional.

3. A function f, $\mathcal{E}$-q.e. defined on X, is $\mathcal{E}$-quasicontinuous if and only if $f \circ q^{-1}$ is $\tilde{\mathcal{E}}$-quasicontinuous.

Let us now recall three kinds of regularity of Dirichlet spaces. We call a Dirichlet space $(X, m, \mathcal{E}, \mathcal{F})$ *quasi-regular* if there exists an $\mathcal{E}$-nest consisting of compact sets, each element in a certain $\mathcal{E}_1$ dense subspace of $\mathcal{F}$ admits its $\mathcal{E}$-quasicontinuous version and there exists an $\mathcal{E}$-nest $\{F_n\}$ such that the points of $\cup_{n=1}^{\infty} F_n$ are separated by a certain coutable familiy of $\mathcal{E}$-quasicontinuous functions belonging to $\mathcal{F}$. Every element of $\mathcal{F}$ then admits a quasicontinuous version.

When X is locally compact, we denote by $C_0(X)$ (resp. $C_\infty(X)$) the space of continuous functions on X with compact support (resp. vanishing at infinity). We call a Dirichlet space $(X, m, \mathcal{E}, \mathcal{F})$ *regular* if X is a locally compact separable metric space, m is a positive Radon measure on X with full support and the space $\mathcal{F} \cap C_0(X)$ is $\mathcal{E}_1$-dense in $\mathcal{F}$ and uniformly dense in $C_0(X)$.

A submarkovian resolvent kernel $R_\alpha(x, B)$ is said to be a *Ray resolvent* if

$$R_\alpha(C_\infty(X)) \subset C_\infty(X) \quad \alpha > 0$$

and there is a countable family C_1 of non-negative function in $C_\infty(X)$ separating points of X_Δ such that

$$\alpha R_{\alpha+1} u \leq u \quad u \in C_1 \quad \alpha > 0.$$

Such a family C_1 is said to be *attached* to the Ray resolvent $R_\alpha(x, B)$.

A Dirichlet space $(X, m, \mathcal{E}, \mathcal{F})$ is called a *strongly regular* if X is a locally compact separable metric space, m is a positive Radon measure on X with full support and the associated L^2-resolvent is generated by a Ray resolvent and a set C_1 attached to this Ray resolvent is contained in the space $\mathcal{F} \cap C_\infty(X)$. Any strongly regular Dirichlet space is regular (cf.[F 71b, Remark 2.2] and [FOT 94, Lemma 1.4.2]).

We shall prove the following two theorems by combining those results in [F 71a], [F 71b], [FOT 94] and [CMR 94].

Theorem 2.1 *Any Dirichlet space admits its strongly regular representation.*

Theorem 2.2 *A Dirichlet space is quasi-regular if and only if some (and equivalently any) of its regular representations is induced by a quasi-homeomorphism.*

We prepare a lemma about identifications of quasi-notions in the regular case. Suppose $(X, m, \mathcal{E}, \mathcal{F})$ is a regular Dirichlet space. The associated capacity Cap is defined for any open set $A \subset X$ by

$$Cap(A) = \inf\{\mathcal{E}_1(u, u) : u \in \mathcal{F},\ u \geq 1_A\} \quad \inf \phi = \infty$$

and for any set $B \subset X$ by

$$Cap(B) = \inf\{Cap(A) : B \subset A \text{ open }\}.$$

In [FOT 94], 'q.e.' means 'except for a set of zero capacity'. A family $\{F_n\}$ of increasing closed subsets of X is then said in [FOT 94] to be a *nest* if

$$\lim_{n\to\infty} Cap(X \setminus F_n) = 0$$

and to be a *generalized nest* if for any compact set $K \subset X$

$$\lim_{n\to\infty} Cap(K \setminus F_n) = 0.$$

A function u defined 'q.e.' on X is said in [FOT 94] to be *quasicontinuous* if for any $\epsilon > 0$ there exists an open set A with $Cap(A) < \epsilon$ such that $u|_{X-A}$ is continuous.

Lemma 2.1 *Suppose* $(X, m, \mathcal{E}, \mathcal{F})$ *is regular, then*

(i) *a family of increasing closed sets is an* $\mathcal{E}$*-nest iff it is a generalized nest in the sense of* [FOT 94],

(ii) $N \subset X$ *is* $\mathcal{E}$*-exceptional iff* $Cap(N) = 0$,

(iii) *a function on* X *is* $\mathcal{E}$*-quasicontinuous iff it is quasicontinuous in the sense of* [FOT 94].

Proof. Let $\mathbf{M} = (X_t, P_x)$ be a Hunt process on X which is *associated with the form* $\mathcal{E}$ in the sense that the transition semigroup $p_t f$ of the process $\mathbf{M}$ is a version of the L^2 semigroup $T_t f$ associated with $\mathcal{E}$ for any non-negative Borel function $f \in L^2(X; m)$.

(i) Denote by σ_E the hitting time of a set E:

$$\sigma_E = \inf\{t > 0 : X_t \in E\} \quad (\inf \phi = \infty).$$

In view of Lemma 5.1.6 of [FOT 94], we know that an increasing sequence $\{F_n\}$ of closed sets is a generalized nest iff

$$P_x(\lim_{n\to\infty} \sigma_{X-F_n} < \zeta) = 0 \quad \text{q.e. } x \in X, \tag{9}$$

where ζ denotes the life time of $\mathbf{M}$.

If (9) is true, then, for any bounded Borel $\varphi \in L^2(X; m)$, the function

$$R_1^{(n)}\varphi(x) = E_x\left(\int_0^{\sigma_{X-F_n}} e^{-t}\varphi(X_t)dt\right), \ x \in X, \tag{10}$$

belongs to the space $\mathcal{F}_{F_n}$ and converges as $n \to \infty$ to the 1-resolvent $R_1\varphi$ of $\mathbf{M}$ m-a.e. and in $\mathcal{E}_1$-metric as well. Here we set the value of φ at the cemetery Δ to be zero. Since the family $R_1\varphi$ is dense in $\mathcal{F}$, $\{F_n\}$ is an $\mathcal{E}$-nest.

If conversely $\{F_n\}$ is an $\mathcal{E}$-nest, then, for $\sigma = \lim_{n\to\infty} \sigma_{X-F_n}$, the function

$$u(x) = E_x\left(\int_{\sigma\wedge\zeta}^{\zeta} e^{-t}\varphi(X_t)dt\right) \tag{11}$$

must vanish m-a.e. because $u \in \mathcal{F}$ is $\mathcal{E}_1$-orthogonal to $\cup_{n=1}^{\infty}\mathcal{F}_{F_n}$. Since u is quasicontinuous, it vanishes q.e. Choosing φ in (11) to be strictly positive on X, we arrive at the property (9).

(ii) If N is $\mathcal{E}$-exceptional, then

$$N \subset \cap_{n=1}^{\infty}(X - F_n) \qquad (12)$$

for some generalized nest $\{F_n\}$ by virtue of (i). Then $Cap(K \cap N) = 0$ for any compact set K and hence $Cap(N) = 0$. Conversely, any set N of zero capacity satisfies the inclusion (12) for a certain nest $\{F_n\}$ in the sense of [FOT 94] ,which is an $\mathcal{E}$-nest by (i). Hence N is $\mathcal{E}$-exceptional.

To see the equivalence (iii), let u be $\mathcal{E}$-quasicontinuous with associated $\mathcal{E}$-nest $\{F_n\}$. Since $\{F_n\}$ is a generalized nest by (i), u is quasicontinuous in the sense of [FOT 94] on each relatively compact open subset of X, which in turn readily implies that it is quasicontinuous on X in the sense of [FOT 94]. The converse implication is trivial as in the proof of (ii). □

For a regular Dirichlet space, the notion 'q.e.' now becomes a synomym for '$\mathcal{E}$-q.e.' Further the condition (9) with 'q.e.' being replaced by '$\mathcal{E}$-q.e.' becomes a stochastic characterization of an $\mathcal{E}$-nest.

Proof of Theorem 2.1 Given a general Dirichlet space $(X, m, \mathcal{E}, \mathcal{F})$, a subalgebra L of $L^{\infty}(X; m)$ is said to satisfy condition (L) if
(L.1) L is a countably generated closed subalgebra of $L^{\infty}(X; m)$,
(L.2) $\mathcal{F} \cap L$ is dense both in $(\mathcal{F}, \mathcal{E}_1)$ and in $(L, \|\cdot\|_{\infty})$,
(L.3) $L^1(X; m) \cap L$ is dense in $(L, \|\cdot\|_{\infty})$.

Denote by $L_0^{\infty}(X; m)$ the closure of $L^2 \cap L^{\infty}$ in L^{∞} and by $\bar{G}_{\alpha}$ the extension of the L^2 resolvent operator G_{α} associated with $\mathcal{E}$ from $L^2 \cap L^{\infty}$ to L_0^{∞}. A closed subalgebra L of $L_0^{\infty}(X; m)$ is said to satisfy condition (R) if
(R.1) $\bar{G}_{\alpha}(L) \subset L$ for any $\alpha > 0$,
(R.2) L is generated by a countable subset L_0 of $\mathcal{F} \cap L$ such that each $u \in L_0$ is non-negative and satisfies

$$\alpha \bar{G}_{\alpha+1} u \leq u, \quad \alpha > 0.$$

Let L be a closed subalgebra of $L^{\infty}(X; m)$ satisfying condition (L) and $\tilde{X}$ be its character space. By virtue of [FOT 94, Th.A.4.1], there exists then a regular Dirichlet space with underlying space $\tilde{X}$ which is equivalent to the given Dirichlet space. Such a regular Dirichlet space is called a regular representation with respect to L. [F 71a, Th.3] went further asserting that there exists a subalgebra L satisfying not only (L) but also (R), and the regular representation with respect to this L becomes strongly regular.

[FOT 94, Th.A.4.1] is a reformulation of [F 71a, Th.2] just by removing the irrelevent assumption that X is locally compact and m is Radon. In the same way, [F 71a, Th.3] can be reformulated. □

Proof of Theorem 2.2 In view of [F 71b, Th.2.1] (c.f. [FOT 94, Th.A.4.2]), we see that, whenever two regular Dirichlet spaces are equivalent, then the equivalence is induced by a quasi-homeomorphism. Here, the quasi-homeomorphism was formulated by the nest defined by the associated capacities, but it is a quasi-homeomorphism in the present sense because of Lemma 2.1 (i).

On the other hand, [CMR 94, Th.3.7] shows that a Dirichlet space is quasi-regular if and only if it is equivalent to some regular Dirichlet space by means of a quasi-homeomorphism. We arrive at Theorem 2.2 by combining those two facts. □

Corollary 2.1 *If two quasi-regular Dirichlet spaces are equivalent, then the equivalence is induced by a quasi-homeomorphism.*

Corollary 2.2 *Any quasi-regular Dirichlet space is equivalent to a strongly regular Dirichlet space by means of a quasi-homeomorphsim.*

3 Smoothness of signed measures for a strongly regular Dirichlet spaces

In this section, we work with a fixed strongly regular Dirichlet space $(X, m, \mathcal{E}, \mathcal{F})$. For the associated Ray resolvent $R_\alpha(x, B)$, there exists a substochastic kernel $\mu(x, B)$ such that

$$\lim_{\alpha\to\infty} \alpha R_\alpha f(x) = \int_X f(y)\mu(x, dy) \quad \forall f \in C_\infty(X).$$

A point $x \in X$ is called a *branching point* if the measure $\mu(x, \cdot)$ is not concentrated on $\{x\}$. The set X_b of all branching points is called the *branch set.* One can then construct a Markov process $\mathbf{M} = (X_t, P_x)$ on X called a *Ray process* with resolvent $R_\alpha(x, B)$, which is known to enjoy the following specific properties ([R 59], [KW 67]). We denote by X_Δ the one point compactification of X and put $\zeta(\omega) = \inf\{t \geq 0 : X_t(\omega) = \Delta\}$.

(**M**.1) $P_x(X_0 = x) = 1 \quad x \in X - X_b$.

(**M**.2) The sample path $X_t = X_t(\omega)$ is cadlag; $X_t(\omega) \in X_\Delta$ is right continuous for all $t \geq 0$, and have the left limit $X_{t-}(\omega) \in X_\Delta$ for all $t > 0$. $X_t(\omega) = \Delta$ for any $t \geq \zeta(\omega)$.

(**M**.3) $P_x(X_t \in X_\Delta - X_b) = 1 \quad \forall x \in X,\ \forall t \geq 0$.

(**M**.4) **M** is strong Markov.

(**M**.5) (quasi-left continuity in a restricted sense)
If stopping times σ_n increase to σ, then, for any $x \in X$,

$$P_x\left(\lim_{n\to\infty} X_{\sigma_n} = X_\sigma | \sigma < \infty, \lim_{n\to\infty} X_{\sigma_n} \in X_\Delta - X_b\right) = 1.$$

A non-negative universally measurable function f on X is said to be 1-*supermedian* if

$$\alpha R_{\alpha+1} f(x) \leq f(x) \quad \alpha > 0,\ x \in X,$$

and to be 1-*excessive* if

$$\alpha R_{\alpha+1} f(x) \uparrow f(x) \quad \alpha \to \infty,\ x \in X.$$

For a 1-supermedian function f, $\hat{f}$ denotes its 1-excessive regularization :

$$\hat{f}(x) = \lim_{\alpha\to\infty} \alpha R_{\alpha+1} f(x).$$

Actually $\hat{f}$ is then 1-excessive and $\hat{f}(x) \leq f(x)$ $x \in X$. We shall use the following description of the branch set(cf. [KW 67]). For the family $C_1(\subset \mathcal{F} \cap C_\infty(X))$ of 1-supermedian functions attached to the Ray resolvent, we set

$$C_1' = \{f \wedge c : f \in C_1,\ c \text{ positive rational }\}.$$

Then

$$X_b = \bigcup_{g \in C_1'} \{x \in X : g(x) > \hat{g}(x)\}. \tag{13}$$

For a Borel set $B \subset X$, we let

$$\sigma_B(\omega) = \inf\{t > 0 : X_t \in B\} \quad \dot{\sigma}(\omega) = \inf\{t \geq 0 : X_t \in B\},$$

with the convention that $\inf \phi = \infty$ and we further let

$$p_B(x) = E_x\left(e^{-\sigma_B}\right) \quad \dot{p}_B(x) = E_x\left(e^{-\dot{\sigma}_B}\right).$$

When B is open, $p_B = \dot{p}_B$ and it is a 1-excessive Borel measurable function.

The facts stated in the next lemma are taken from [F 71b] but we shall present alternative elementary proof of them.

Lemma 3.1 (i) $Cap(X_b) = 0$.

(ii) *Let A be an open set with $Cap(A) < \infty$ and $e_A \in \mathcal{F}$ be its (1-)equilibrium potential. Then $p_A(x)$ is a Borel 1-excessive version of e_A.*

Proof. (i) Take any $g \in C_1'$. Since $\alpha R_{\alpha+1} g$ is $\mathcal{E}_1$-convergent to $g \in \mathcal{F}$, $\hat{g}$ is an $\mathcal{E}$-quasicontinuous version of g. But g is continuous and hence $g = \hat{g}$ $\mathcal{E}$-q.e., namely, $Cap(g > \hat{g}) = 0$. By (13) and the countable subadditivity of the capacity, we arrive at (i).

(ii) Since $p_A(x) = 1$ for all $x \in A - X_b$ and consequently m-a.e. on A by (**M**.6), the proof of [FOT 94, Lemma 4.2.1] (where $p_A(x) = 1$ $\forall x \in A$) works without any change in proving that p_A is a version of e_A.

□

We now proceed to the proof of a proposition which is an intermidiate but crucial step in establishing the equivalence of the inequality (2) and the exisitence of a smooth signed measure satisfying (4).

Proposition 3.1 *Let $u \in \mathcal{F}$ and $w \in \mathcal{F}_b$. Suppose there exists a finite signed measure $\nu = \nu_{u,w}$ on X such that*

$$\mathcal{E}(u, vw) = -\int_X v d\nu \quad \forall v \in \mathcal{F} \cap C_\infty(X). \tag{14}$$

Then ν is smooth, namely, ν charges no set of zero capacity. Moreover it holds that

$$\mathcal{E}(u, vw) = -\int_X v^* d\nu \quad \forall v \in \mathcal{F}_b, \tag{15}$$

where v^* is any $\mathcal{E}$-quasicontinuous version of v.

Lemma 3.2 *Assume the condition of Proposition 3.1.*

(i) *Suppose v_n satisfies the equation* (14)*, $\mathcal{E}(v_n, v_n)$ is bounded, $v_n(x)$ is uniformly bounded and v_n converges to a function $v \in \mathcal{F}$ pointwise and in $L^2(X; m)$. Then v satisfies* (14).

(ii) *The equation* (14) *holds for the product $v = v_1 \cdot v_2$ for any $v_1 \in \mathcal{F} \cap C_\infty(X)$ and for any bounded Borel* 1*-excessive function $v_2 \in \mathcal{F}$.*

Proof. (i) $w \cdot v_n$ then $\mathcal{E}$-weakly convergent to $w \cdot v$(cf.[FOT 94, Th.1.4.2]).

(ii) Fix an arbitrary non-negative $v_1 \in \mathcal{F} \cap C_\infty(X)$ and $\alpha > 0$ and let

$$H = \{f \in L^2_+(X; m) : \text{ bounded Borel, (14) holds for } v = v_1 \cdot R_\alpha f\}.$$

Since $R_\alpha(C_0(X)) \subset \mathcal{F} \cap C_\infty(X)$, $C_0(X) \subset H$. If $f_1, f_2 \in H, c_1 f_1 + c_2 f_2 \geq 0$ for some constants c_1, c_2, then clearly $C_1 f_1 + c_2 f_2 \in H$. If $f_n \in H$ increases to a bounded function $f \in L^2(X; m)$, then $v_1 \cdot R_\alpha f_n$ is $\mathcal{E}$-bounded, uniformly bounded, and convergent to $v_1 \cdot R_\alpha f$ pointwise and in L^2. Hence $f \in H$ by virtue of (i). By the monotone lemma, we see that equation (14) holds for $v = v_1 \cdot R_\alpha f$ for any nonnegative bounded Borel $f \in L^2$.

Next take any bounded Borel 1-excessive function $v_2 \in \mathcal{F}$. Since $\alpha R_{\alpha+1} v_2$ is $\mathcal{E}_1$-convergent to v_2 as $\alpha \to \infty$, $v_1 \cdot \alpha R_{\alpha+1} v_2$ is $\mathcal{E}$-bounded, uniformly bounded and convergent to $v_1 \cdot v_2$ pointwise and in L^2. Hence (14) holds for $v = v_1 \cdot v_2$ by (i) again. □

Lemma 3.3 *Assume the condition of Proposition 3.1 and denote by $\bar{\nu}$ the total variation of the finite signed measure ν.*

(i) $\bar{\nu}(X_b) = 0$.

(ii)

$$P_{\bar{\nu}}(X_{t-} \in X_b \quad \exists t \in (0, \infty)) = 0. \tag{16}$$

Proof. (i) We use the description (13) of the branch set. Take $g \in C_1'$. For any $h \in \mathcal{F} \cap C_\infty(X)$, $hg \in \mathcal{F} \cap C_\infty(X)$ and

$$\mathcal{E}(u, whg) = -\int_X hg d\nu.$$

On the other hand, $\hat{g}$ is a bounded Borel 1-excessive function and defines the same element of $\mathcal{F}$ as g because $\alpha R_{\alpha+1} g$ is $\mathcal{E}_1$-convergent to $g \in \mathcal{F}$. Therefore, by Lemma 3.2 (i),

$$\mathcal{E}(u, whg) = -\int_X h\hat{g} d\nu,$$

and consequently

$$\int_X h(g - \hat{g}) d\nu = 0 \quad \forall h \in \mathcal{F} \cap C_\infty(X),$$

which implies that $\bar{\nu}(\{x \in X : g(x) > \hat{g}(x)\}) = 0$.

(ii) Because of lemma 3.1 (i), there exists an decresing open sets A_n including X_b such that $\lim_{n\to\infty} Cap(A_n) = 0$. Due to Lemma 3.1 (ii) and Lemma 3.2 (ii), we then have

$$\mathcal{E}(u, whp_{A_n}) = -\int_X hp_{A_n} d\nu \quad \forall h \in \mathcal{F} \cap C_\infty(X). \tag{17}$$

In view of

$$Cap(A_n) = \mathcal{E}_1(p_{A_n}, p_{A_n}) \geq (p_{A_n}, p_{A_n})_{L^2},$$

p_{A_n} is $\mathcal{E}$-bounded, uniformly bounded and L^2-convergent to zero. Therefore, from (17) and Lemma 3.2 (i), we have for

$$p(x) = \lim_{n\to\infty} p_{A_n}(x)$$

the identity

$$\int_X h(x)p(x)d\nu(x) = 0 \quad \forall h \in \mathcal{F} \cap C_\infty(X),$$

which implies that

$$0 = \bar{\nu}(\{x \in X : p(x) > 0\}) = P_{\bar{\nu}}(\Lambda),$$

where

$$\Lambda = \{\omega : \lim_{n\to\infty} \sigma_{A_n}(\omega) < \infty\}.$$

Since the ω-set in the braces of the left hand side of (16) is contained in the measurable ω-set Λ, we get to (16).

□

Proof of Porposition 3.1 Take any compact set K with $Cap(K) = 0$. For the first assertion of the proposition, it suffices to show that

$$\bar{\nu}(K) = 0. \tag{18}$$

Choose a sequence $\{A_n\}$ of relatively compact open sets such that

$$A_{n+1} \supset \bar{A}_n \supset K \quad \bigcap_{n=1}^{\infty} A_n = K.$$

Since Cap is a Choquet capacity,

$$\lim_{n\to\infty} Cap(A_n) = \lim_{n\to\infty} Cap(\bar{A}_n) = Cap(K) = 0.$$

On the other hand, $\mathbf{M}$ is quasi-left continuous under $P_{\bar{\nu}}$ by virtue of ($\mathbf{M}$.5) and Lemma 3.3 (ii): for any stopping times σ_n increasing to σ,

$$P_{\bar{\nu}}\left(\lim_{n\to\infty} X_{\sigma_n} = X_\sigma,\ \sigma < \infty\right) = P_{\bar{\nu}}(\sigma < \infty).$$

Hence

$$P_{\bar{\nu}}\left(\lim_{n\to\infty}\sigma_{A_n}\neq\dot{\sigma}_K\right)=0,$$

and accordingly

$$\lim_{n\to\infty}p_{A_n}(x)=\dot{p}_K(x)\quad \text{for } \bar{\nu}-a.e.\ x\in X.$$

We now have the equation (17) for p_{A_n} by Lemma 3.1 (i) and Lemma 3.2 (ii). In the same way as in the proof of the preceding lemma, we see that the left hand side of this equation tends to zero and consequently

$$\int_X h(x)\dot{p}_K(x)d\nu(x)=0\quad \forall h\in\mathcal{F}\cap C_\infty(X),$$

which means that

$$\bar{\nu}(\dot{p}_K>0)=0.$$

But $\dot{p}_K(x)=1$ for $x\in K-X_b$ and hence $\bar{\nu}(K-X_b)=0$, which combined with Lemma 3.1 (i) proves the desired (18).

For the second assertion, take any $v\in\mathcal{F}_b$ and let $\|v\|_\infty=M$. We can then find a sequence of functions $v_n\in\mathcal{F}\cap C_\infty(X)$ $\mathcal{E}_1$-convergent to v such that $\sup_{x\in X}|v_n(x)|\leq M$ and further v_n converges $\mathcal{E}$-q.e. to an $\mathcal{E}$-quasicontinuous version v^* of v. As lemma 3.2 (i), the desired identity (15) now follows from (14) for v_n. □

We are now in a position to prove the main theorem of this section.

Theorem 3.1 *For $u\in\mathcal{F}$, the next two conditions are equaivalent:*

(i) *There exists an $\mathcal{E}$-nest $\{F_n\}$ for which the inequality (2) is valid for some positive constants C_n.*

(ii) *There exists a signed smooth measure ν with some attached $\mathcal{E}$-nest $\{F_n\}$ for which the equation (4) is valid.*

Proof of the implication (ii) $\Rightarrow$ (i). Suppose (ii) is fulfilled. Take any $v\in\mathcal{F}_{b,F_n}$ and let $M=\|v\|_\infty$. Then $v^*\leq M\ \mathcal{E}-q.e.$ and further $v^*=0\ \mathcal{E}-q.e.$ on $X\setminus F_n$. Therefore the absolute value of the right hand side of (4) is domonated by $C_n\cdot M$ with C_n being the total variation of ν on the set F_n.

□

The converse implication (i) $\Rightarrow$ (ii) will be proved in the following more specific form.

Proposition 3.2 *Suppose, for $u\in\mathcal{F}$, the inequality (1) is valid for some $\mathcal{E}$-nest $\{F_n\}$. Then there exists a smooth signed measure ν with an attached $\mathcal{E}$-nest $\{F'_n\}$ with $F'_n\subset F_n,\ n=1,2,\cdots,$ for which the equation (4) is valid.*

In the rest of this section, we assume that $u \in \mathcal{F}$ satisfies (1) for an $\mathcal{E}$-nest $\{F_n\}$. We shall construct ν and $\{F'_n\}$ of Proposition 3.2 by a series of lemmas.

First fix an arbitrary w belonging to the space $\mathcal{F}_{F_n,b}$ for some n. Then $v \cdot w \in \mathcal{F}_{F_n,b}$ for any $v \in \mathcal{F}_b$,

$$|\mathcal{E}(u, v \cdot w)| \leq C_n \|v \cdot w\|_\infty \leq C_n \|w\|_\infty \cdot \|v\|_\infty, \ v \in \mathcal{F}_b.$$

Since this inequality holds for any v in the space $\mathcal{F} \cap C_\infty(X)$ which is uniformly dense in $C_\infty(X)$, there exists a unique finite signed measure $\nu = \nu_w$ for which the equation (14) is valid. In virtue of Proposition 3.1, ν_w charges no set of zero capacity and

$$\mathcal{E}(u, \ v \cdot w) = -\int_X v^*(x) d\nu_w(x) \quad \forall v \in \mathcal{F}_b. \tag{19}$$

Lemma 3.4

$$w_1^* d\nu_{w_2} = w_2^* d\nu_{w_1} \quad w_1, \ w_2 \in \cup_{n=1}^\infty \mathcal{F}_{F_n,b}.$$

Proof. For any $v \in \mathcal{F}_b$,

$$\mathcal{E}(u, \ v w_1 w_2) = -\int_X v^* w_1^* d\nu_{w_2} = -\int_X v^* w_2^* d\nu_{w_1}.$$

□

This lemma says that, roughly speaking, $(w^*)^{-1} d\nu_w$ is independent of w and a candidate of the measure ν we want to construct. To make a rigorous construction, we need to consider a Hunt process associated with the given strongly regular Dirichlet space in order to use a general theory in [FOT 94]. Such a process can be immediately constructed from the Ray process $\mathbf{M} = (X_t, P_x)$ already being considered . In fact, Lemma 3.1 (ii) readily implies the following ([F 71b, Th.3.9]): for any set $B \subset X$ with $Cap(B) = 0$, there exists a Borel set $N \supset B$ with $Cap(N) = 0$ such that $X - N$ is $\mathbf{M}$-*invariant* in the sense that

$$P_x(X_t \in X_\Delta - N, \ X_{-t} \in X_\Delta - N \ \forall t \geq 0) = 1. \ \forall x \in X - N.$$

Since $Cap(X_b) = 0$ by Lemma 3.1 (i), the branch set is included in a set N of the above type. On account of the properties ($\mathbf{M}$.1) $\sim$ ($\mathbf{M}$.5) of the Ray process $\mathbf{M}$, we can get a Hunt process on X still associated with the form $\mathcal{E}$ first by restricting the state space of $\mathbf{M}$ to $X_\Delta - N$ and then by making each point of N to be a trap(see [FOT 94, Th.A.2.8,A.2.9] for those procedures). We may call the resulting Hunt process a *Hunt modification of the Ray process* $\mathbf{M}$.

In what follows in this section, $\mathbf{M} = (X_t, P_x)$ denotes a Hunt process on X associated with the form $\mathcal{E}$. For a strictly positive bounded Borel function φ on X with $\varphi \in L^2(X; m)$, we put

$$\rho_n(x) = R_1^{(n)} \varphi(x), \ x \in X,$$

where the right hand side is defined by (10). Here, we set $\varphi(\Delta) = 0$. By [FOT 94, Th.4.4.1], we know that ρ_n is an $\mathcal{E}$-quasicontinuous Borel function in $\mathcal{F}_{F_n,b}$. We then introduce the sets

$$E_n = \{x \in X : \rho_n(x) \geq \frac{1}{n}\}, \ n = 1, 2, \cdots, \quad N_0 = X - (\cup_{n=1}^\infty E_n). \tag{20}$$

E_n is a quasi-closed Borel set, increasing in n and $E_n \subset F_n,\ n = 1, 2, \cdots$. Since $N_0 = \{x \in X : \lim_{n\to\infty} \rho_n(x) = 0\}$, we see that

$$Cap(N_0) = 0 \tag{21}$$

owing to the stochastic charcterization (9) of the $\mathcal{E}$-nest $\{F_n\}$.
We define ν by

$$\nu(dx) = \frac{1}{\rho_n(x)}\, \nu_{\rho_n}(dx) \text{ on } E_n,\ n = 1, 2, \cdots,\ \nu(N_0) = 0. \tag{22}$$

For $m > n$, we have from Lemma 3.4

$$\frac{1}{\rho_m(x)} \nu_{\rho_m}(dx) = \frac{1}{\rho_n(x)} \nu_{\rho_n}(dx) \text{ on } E_n$$

which means that the above definition makes sense. $\nu|_{E_n}$ is then a finite signed measure for each n and ν charges no set of zero capacity. Moreover

$$\mathcal{E}(u,\ v \cdot w) = -\int_X v^* w^* \nu(dx) \quad v \in \mathcal{F}_b,\ w \in \cup_{n=1}^{\infty} \mathcal{F}_{F_n,b}. \tag{23}$$

In fact, the above definition of ν and Lemma 3.4 imply that

$$\nu_w(dx) = w^* \nu(dx) \quad w \in \cup_{n=1}^{\infty} \mathcal{F}_{F_n,b},$$

and we are led to (23) from (19).
In order to construct an appropriate $\mathcal{E}$-nest from $\{E_n\}$, we prepare a lemma.

Lemma 3.5 *Suppose* $\Gamma_n,\ n = 1, 2, \cdots$, *are quasi-closed, decreasing in* n *and*

$$Cap\,(\cap_{n=1}^{\infty} \Gamma_n) = 0.$$

Then

$$\lim_{n\to\infty} Cap(\Gamma_n \cap K) = 0 \quad \text{for any compact set } K.$$

Proof. In view of the definition of the quasi-closed set ([FOT 94, pp68]), we can find, for any $\epsilon > 0$, an open set ω with $Cap(\omega) < \epsilon$ such that $\Gamma_n - \omega$ are closed for all n. For any compact K, $(\Gamma_n - \omega) \cap K$ are decreasing compact sets. Since Cap is a Choquet capacity,

$$\lim_{n\to\infty} Cap((\Gamma_n - \omega) \cap K) = Cap(\cap_{n=1}^{\infty} (\Gamma_n - \omega) \cap K) \le Cap(\cap_{n=1}^{\infty} \Gamma_n) = 0.$$

We then get $\overline{\lim}_{n\to\infty} Cap(\Gamma_n \cap K) \le \epsilon$ from

$$Cap(\Gamma_n \cap K) \le Cap((\Gamma_n - \omega) \cap K) + Cap(\omega).$$

□

Proof of Proposition 3.2 Take a sequence $\epsilon_\ell \downarrow 0$. Since the sets E_n defined by (20) are quasi-closed, we can find decresing open sets ω_ℓ with $Cap(\omega_\ell) < \epsilon_\ell$ and $E_n - \omega_\ell$ are closed for all n and ℓ.

Let us define increasing closed sets F'_n by

$$F'_n = E_n - \omega_n \quad n = 1, 2, \cdots \tag{24}$$

and prove that

$$\lim_{n\to\infty} Cap(K - F'_n) = 0 \quad \text{for any compact } K. \tag{25}$$

Since

$$K - F'_n = (K \cap E^c_n) \cup (K \cap \omega_n),$$

we have

$$Cap(K - F'_n) \leq Cap(K \cap E^c_n) + \frac{1}{n}.$$

We let

$$\Gamma_n = \{\rho_n \leq \frac{1}{n}\}.$$

Γ_n are then quasi-closed and

$$E^c_n \subset \Gamma_n \quad \cap_{n=1}^{\infty} E^c_n = \cap_{n=1}^{\infty}\Gamma_n = N_0$$

because $E^c_n = \{\rho_n < \frac{1}{n}\}$. On account of the preceding Lemma and (21), we conclude that

$$Cap(K \cap E^c_n) \leq Cap(K \cap \Gamma_n) \to 0 \quad n \to \infty.$$

(25) is proven and $\{F'_n\}$ is an $\mathcal{E}$-nest by Lemma 2.1. Moreover

$$F'_n \subset E_n \subset F_n, n = 1, 2 \cdots .$$

For the measure ν defined by (22), $\nu|_{F'_n}$ is therefore a finite signed measure for each n. Since ν charges no set of zero capacity, it becomes a smooth measure for which the present $\mathcal{E}$-nest $\{F'_n\}$ is attached.

Finally take any w belonging to the space $\mathcal{F}_{F'_n,b}$ for some n. Let $v = (n\rho_n) \wedge 1$. Then $v = 1$ on $E_n(\supset F'_n)$ and $v \cdot w = w$. Thus the equation (23) leads us to

$$\mathcal{E}(u, w) = \int_X w^*(x)\nu(dx) \quad \forall w \in \cup_{n=1}^{\infty}\mathcal{F}_{F'_n,b}, \tag{26}$$

completing the proof of Proposition 3.2. □

Theorem 3.1 is proved. Here we add a theorem corresponding to a special case of Theorem 3.1 where the $\mathcal{E}$-nests are trivial.

Theorem 3.2 *For $u \in \mathcal{F}$, the next two conditions are equivalent:*

(i) *The inequality* (6) *holds for some constant* $C > 0$.

(ii) *There exists a finite signed measure ν charging no $\mathcal{E}$-exceptional set such that the equation* (4) *holds for any $v \in \mathcal{F}_b$.*

Proof. The proof of the implication (i) $\Rightarrow$ (ii) is the same as the corresponding proof of Theorem 3.1. The converse can be viewed as a special case of Proposition 3.1 (the case where $w = 1$). $\square$

4 Transfers of analytic theorems to a quasi-regular Dirichlet space

First, we transfer Theorem 3.1 from a strongly regular Dirichlet space to a quasi-regular Dirichlet space.

Theorem 4.1 *Let $(X, m, \mathcal{E}, \mathcal{F})$ be a quasi-regular Dirichlet space. For $u \in \mathcal{F}$, the next two conditions are equivalent:*

(i) *There exists an $\mathcal{E}$-nest $\{E_n\}$ for which the inequality* (2) *is valid for some positive constants C_n.*

(ii) *There exists a signed smooth measure ν with some attached $\mathcal{E}$-nest $\{E_n\}$ for which the equation* (4) *is valid.*

Proof. By Corollary 2.2, the quasi-regular Dirichlet space $(X, m, \mathcal{E}, \mathcal{F})$ is equaivalent to a certain strongly regular Dirichlet space $(\tilde{X}, \tilde{m}, \tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ by a quasi-homeomprphism q; there exist $\mathcal{E}$-nest $\{F_n\}$, $\tilde{\mathcal{E}}$-nest $\{\tilde{F_n}\}$, q is a one to one mapping from $X_0 = \cup_{n=1}^{\infty} F_n$ onto $\tilde{X}_0 = \cup_{n=1}^{\infty} \tilde{F}_n$, its restriction to each F_n is homeomprphic to $\tilde{F_n}$, and the map from $\mathcal{F}_b$ to $\tilde{\mathcal{F}}_b$ defined by

$$(\Phi u)(\tilde{x}) = u(q^{-1}(\tilde{x})) \quad \tilde{x} \in \tilde{X}_0$$

satisfies

$$\|u\|_\infty = \|\Phi u\|_\infty, \ (u, u)_X = (\Phi u, \Phi u)_{\dot{X}}, \ \mathcal{E}(u, u) = \tilde{\mathcal{E}}(\Phi u, \Phi u).$$

q is $\mathcal{E}$-quasi notions preserving as is explained in §2.

Suppose that the condition (i) is fulfilled. Let

$$\tilde{u} = \Phi(u) (\in \tilde{\mathcal{F}}), \quad \tilde{E}_n = q(E_n \cap F_n) \ n = 1, 2, \cdots$$

Then, $\{\tilde{E}_n\}$ is an $\tilde{\mathcal{E}}$-nest and

$$|\tilde{\mathcal{E}}(\tilde{u}, \tilde{v})| \leq C_n \|\tilde{v}\|_\infty, \quad \forall \tilde{v} \in \tilde{\mathcal{F}}_{b, \tilde{E}_n}, \ n = 1, 2, \cdots$$

for the same constant C_n as in (i).

By virtue of Theorem 3.1, there exists a smooth signed measure $\tilde{\nu}$ on X with an attached $\mathcal{E}$-nest $\{\tilde{E}'_n\}$ for which the equation

$$\tilde{\mathcal{E}}(\tilde{u}, \tilde{v}) = -\int_{\tilde{X}} \tilde{v}^*(\tilde{x}) d\tilde{\nu}(\tilde{x}), \quad \forall \tilde{v} \in \cup_{n=1}^{\infty} \tilde{\mathcal{F}}_{b,\tilde{E}_n}$$

holds. Let

$$\nu(B) = \tilde{\nu}(q(B \cap X_0)), \quad E'_n = q^{-1}(\tilde{E}'_n \cap \tilde{F}_n) \ n = 1, 2, \cdots$$

Then we can easily see that $\{E'_n\}$ is an $\mathcal{E}$-nest, ν is a smooth signed measure on X with the attached $\mathcal{E}$-nest $\{E'_n\}$. By rewiting the above equation and noting that $v^*(x) = \tilde{v}^*(qx)$ gives an $\mathcal{E}$-quasicontinuous version of $v = \Phi^{-1}(\tilde{v})$, we arrive at the equation (4) holding for $u, \nu, \{\tilde{E}'_n\}$, getting the conclusion (ii).

The converse implication (ii) $\Rightarrow$ (i) can be directly shown as the corresponding proof of Theorem 3.1. □

In exactly the same way, Theorem 3.2 can be transfered from a strongly regular Dirichlet space to a quasi-regular Dirichlet space.

Theorem 4.2 *Let $(X, m, \mathcal{E}, \mathcal{F})$ be a quasi-regular Dirichlet space. For $u \in \mathcal{F}$, the next two conditions are equivalent:*

(i) *The inequality* (6) *holds for some constant* $C > 0$.

(ii) *There exists a finite signed measure ν charging no $\mathcal{E}$-exceptional set such that the equation* (4) *holds for any $v \in \mathcal{F}_b$.*

For later convenience, we also transfer Proposition 3.1 in the following manner:

Proposition 4.1 *Let $(X, m, \mathcal{E}, \mathcal{F})$ be a quasi-regular Dirichlet space. Let $u \in \mathcal{F}$ and $w \in \mathcal{F}_b$. Suppose there exists a positive constant $C = C_{u,w}$ such that*

$$|\mathcal{E}(u, vw)| \leq C\|v\|_\infty \quad \forall v \in \mathcal{F}_b.$$

Then there exists uniquely a finite signed smooth measure ν for which the equation (15) *is valid.*

Proof. In the same way as in the proof of Theorem 4.1, the above inequality is honestly inherited to a strongly regular representation, where we can use Proposition 3.1 to obtain the conclusion as above which can be transfer back to the quasi-regular Dirichlet space also in the same way as in the proof of Theorem 4.1. □

5 Transfers of stochastic contents to a quasi-regular Dirichlet space

This section is devoted to transfers of probabilistic notions and theorems from a regular Dirichlet space to a quasi-regular Dirichlet space. In particular, we formulate the stochastic significance of the condition (ii) in Theorem 4.1 more precisely than what was mentioned in §1.

To this end, let us first consider a regular Dirichlet space $(X, m, \mathcal{E}, \mathcal{F})$ and an associated Hunt process $\mathbf{M} = (X_t, P_x)$ on X. In this context, we start with mentioning some probabilistic characterizations of $\mathcal{E}$-exceptional sets in a convenient way for later use.

A set $N \subset X$ is called $\mathbf{M}$-*exceptional* if N is contained in a nearly Borel set $\hat{N}$ such that $P_m(\sigma_{\hat{N}} < \infty) = 0$. A set N is said to be ($\mathbf{M}$-)*properly exceptional* if it is a nearly Borel measurable m-negligible set and its complement $X - N$ is $\mathbf{M}$-invariant. We call N ($\mathbf{M}$-)*properly exceptional in the standard sense* if, in the above definition of the proper exceptionality, the $\mathbf{M}$-invarance of $X - N$ is weakened to the $\mathbf{M}$-invariance up to the life time in the following sense:

$$P_x(X_t,\ X_{t-} \in X - N\ \forall t \in [0, \zeta)) = 1,\ \forall x \in X - N.$$

Remark 5.1 The last notion of the exceptionality makes sense not only for the present Hunt process but also for a standard process, and we shall later utilize it for a standard process associated with a quasi-regular Dirichlet space.

Lemma 5.1 *The following conditions for a set $N \subset X$ are equivalent each other:*

(i) *N is $\mathcal{E}$-exceptional.*

(ii) *N is $\mathbf{M}$-exceptional.*

(iii) *N is contained in a properly exceptional set.*

(iv) *N is contained in a properly exceptional set in the standard sense.*

Proof. The equivalence of the first three conditions are proven in [FOT 94]. The implication (iii)⇒(iv)⇒(ii) is also obvious. □

Remark 5.2 (i) In studying a Hunt process $\mathbf{M}$ associated with a regular Dirichlet space $(X, m, \mathcal{E}, \mathcal{F})$, the state space of $\mathbf{M}$ needs not to be the entire space X. Indeed, we may well consider a Borel subset N_0 of X with $m(N_0) = 0$ and a Hunt process $\mathbf{M}$ with state space $X - N_0$ such that it is associated with a Dirichlet form $\mathcal{E}$ on $L^2(X; m)$ in the sense that its transition function on $X - N_0$ generates the L^2-semigroup corresponding to $\mathcal{E}$. Then the set N_0 becomes automatically $\mathcal{E}$-exceptional, because it is properly exceptional with respect to the trivial extension $\mathbf{M}'$ of $\mathbf{M}$ to X ($\mathbf{M}'$ is obtained from $\mathbf{M}$ by joining every point of N_0 as a trap, see [FOT 94, Th.A.2.9]). $\mathbf{M}'$ is still associated with $\mathcal{E}$ and hence Lemma 5.1 applies.

(ii) Given a Hunt process $\mathbf{M}$ on $X - N_0$ as above, we call a set N $\mathbf{M}$-properly exceptional (resp. $\mathbf{M}$-properly exceptional in the standard sense) if $N \supset N_0$, N is nearly Borel, $m(N) = 0$ and $X - N (\subset X - N_0)$ is $\mathbf{M}$-invariant (resp. $\mathbf{M}$-invariant up to the life time). With this slight modification of the notion of proper exceptionality, not only Lemma 5.1 but also what will be stated below about additive functionals remain valid for a Hunt process $\mathbf{M}$ on $X - N_0$ as above.

(iii) We can and we shall allow an analogous freedom of choice of the state space about a standard process associated with a quasi-regular Dirichlet space, accompanyed by an analogous modification of the proper exceptionality in the standard sense.

We quote from [FOT 94] those basic notions and theorems concerning additive functionals of the Hunt process $\mathbf{M}$ on X. By convention, any numerical function f on X is extended to the one-point compactification X_Δ by setting $f(\Delta) = 0$. Let $\Omega, \{\mathcal{F}_t\}_{t\in[0,\infty]}, \zeta, \theta_t$ be the sample space, the minimum completed admissible filtration, the life time and the shift operator respectively attached to the Hunt process $\mathbf{M}$. An extended real valued function $A_t(\omega)$ of $t \geq 0$ and $\omega \in \Omega$ is called an *additive functional* (AF in abbreviation) if it is $\{\mathcal{F}_t\}$-adapted and there exist $\Lambda \in \mathcal{F}_\infty$ with $\theta_t\Lambda \subset \Lambda,\ \forall t > 0$ and a properly exceptional set $N \subset X$ with $P_x(\Lambda) = 1,\ \forall x \in X - N$, such that, for each $\omega \in \Omega,\ A_0(\omega) = 0,\ A_t(\omega)$ is cadlag and finite on $[0, \zeta(\omega)),\ A_t(\omega) = A_{\zeta(\omega)}(\omega),\ t \geq \zeta(\omega)$, and

$$A_{t+s}(\omega) = A_s(\omega) + A_t(\theta_s\omega), \quad s, t \geq 0.$$

Λ (resp. N) in the above definition is called a *defining* (resp. *exceptional*) *set* for the AF A. We regard two AF's to be *equivalent* if

$$P_x\left(A_t^{(1)} = A_t^{(2)}\right) = 1, \quad t \geq 0, \quad \mathcal{E} - \text{q.e. } x \in X.$$

Then we can find a common defining set Λ and a common properly exceptional set N of $A^{(1)}$ and $A^{(2)}$ such that $A_t^{(1)}(\omega) = A_t^{(2)}(\omega),\ \forall t \geq 0,\ \forall \omega \in \Lambda$. Here we have to use Lemma 4.1 together with the fact that the ω-set

$$\Gamma = \{\omega \in \Omega : X_t(\omega),\ X_{t-}(\omega) \in X_\Delta - N\ \forall t \geq 0\}$$

is $\mathcal{F}_\infty$-measurable.

An AF $A_t(\omega)$ is said to be *finite, cadlag* and *continuous* respectively if it satisfies the respective property at every $t \in [0, \infty)$ for each ω in its defining set. A $[0, \infty]$-valued continuous AF is called a *posiitve continuous* AF (PCAF in abbreviation). We denote by $\mathbf{A}_c^+$ the set of all PCAF's. We shall call an AF $A_t(\omega)$ *of bounded variation* if it is of bounded variation in t on each compact subinterval of $[0, \zeta(\omega))$ for every fixed ω in a defining set of A.

A positive Borel measure μ is called a *smooth measure* if μ charges no $\mathcal{E}$-exceptional set and there is an $\mathcal{E}$-nest $\{F_n\}$ such that $\mu(F_n)$ is finite for each n. The totality of smooth measures is denoted by S. There is a one to one correspondence between (the equivalence classes of) $\mathbf{A}_c^+$ and S by the *Revuz correspondence*:

$$\lim_{t\downarrow 0} \frac{1}{t} E_{h\cdot m}\left(\int_0^t f(X_t)dA_t\right) = \int_X h \cdot f d\mu, \quad A \in \mathbf{A}_c^+,\ \mu \in S, \tag{27}$$

holding for any non-negative Borel function f and γ-excessive function $h,\ \gamma \geq 0$. The smooth measure corresponding to a PCAF A in the above way is said to be the *Revuz measure of* A.

For any $u \in \mathcal{F}$, there exists a unique finite smooth measure $\mu_{\langle u\rangle}$ on X, which satisfies the following equation in the case that $u \in \mathcal{F}_b$:

$$\int_X f^*(x)\mu_{\langle u\rangle}(dx) = 2\mathcal{E}(u \cdot f, u) - \mathcal{E}(u^2.f) \quad \forall f \in \mathcal{F}_b. \tag{28}$$

$\mu_{\langle u\rangle}$ is called the *energy measure of* $u \in \mathcal{F}$.

The *energy* $e(A)$ of an AF A is defined by

$$e(A) = \lim_{t\downarrow 0} \frac{1}{2t} E_m(A_t^2).$$

For $u \in \mathcal{F}$, $A_t^{[u]}$ defined by (1) is (up to the equivalence specified above) a finite cadlag AF of finite energy. The families of *martingale AF's of finite energy* and *CAF's of zero energy* are defined respectively by

$$\overset{\circ}{\mathcal{M}}= \{M: \text{ finite cadlag AF } E_x(M_t^2) < \infty, E_x(M_t) = 0 \text{ q.e. and } e(M) < \infty\},$$

$$\mathcal{N}_c = \{N : CAF \; E_x(|N_t|) < \infty \text{ q.e. and } e(N) = 0\}.$$

For any $u \in \mathcal{F}$, the AF $A^{[u]}$ admits a decomposition (3) uniquely up to the equivalence specified above, where $M^{[u]} \in \overset{\circ}{\mathcal{M}}$ and $N^{[u]} \in \mathcal{N}_c$ ([FOT 94, Th.5.2.2]). The energy measure $\mu_{\langle u \rangle}$ of u coincides with the Revuz measure of the quadratic variation $\langle M^{[u]} \rangle \in \mathbf{A}_c^+$ of the AF $M^{[u]}$ ([FOT 94, Th. 5.2]). Furthermore [FOT 94, Th.5.4.2] asserts the following: the CAF $N^{[u]}$ is of bounded variation if and only if the condition (ii) of Theorem 4.1 is valid for some signed smooth measure ν. In this case moreover, $N^{[u]}$ admits an expression (5) for some PCAF's A^k corresponding to smooth measures ν^k, $k = 1, 2$, by the Revuz correspondence and it holds that $\nu = \nu^1 - \nu^2$.

Now we turn to a general quasi-regular Dirichlet space $(X, m, \mathcal{E}, \mathcal{F})$. Let $(\tilde{X}, \tilde{m}, \tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ be its regular representation and

$$\tilde{\mathbf{M}} = (\tilde{\Omega}, \{\tilde{\mathcal{F}}_t\}_{t \in [0,\infty]}, \tilde{X}_t, \tilde{\zeta}, \tilde{P}_{\tilde{x}})$$

be a Hunt process on $\tilde{X}$ associated with the latter (one may take for instance a strongly regular representation and a Hunt modification of the associated Ray process). By Theorem 2.2, the two Dirichlet spaces are related each other just as in the first paragraph of the proof of Theorem 4.1. We shall use the notations $\{F_n\}, X_0, \{\tilde{F}_n\}, \tilde{X}_0, q, \Phi$, appearing in that paragraph without repeating the explanation. We are ready to construct a standard process on X associated with $\mathcal{E}$ as an image of $\tilde{\mathbf{M}}$ by q^{-1} in a similar way to [F 71b].
Applying the stochastic characterization (9) to the $\tilde{\mathcal{E}}$-nest $\{\tilde{F}_n\}$, we can find an $\tilde{\mathbf{M}}$-properly exceptional Borel set $\tilde{N}_1$ including $\tilde{X} - \tilde{X}_0$ such that

$$\tilde{P}_x(\lim_{n \to \infty} \sigma_{\tilde{X} - \tilde{F}_n} < \tilde{\zeta}) = 0 \quad \forall \tilde{x} \in \tilde{X} - \tilde{N}_1. \tag{29}$$

In other words, we have

$$\tilde{P}_x(\tilde{\Lambda}_1) = 1 \quad \forall \tilde{x} \in \tilde{X} - \tilde{N}_1$$

for the set $\tilde{\Lambda}_1 = \tilde{\Lambda}_{11} \cap \tilde{\Lambda}_{12}$ where

$$\tilde{\Lambda}_{11} = \{\tilde{\omega} \in \tilde{\Omega} : \tilde{X}_t, \; \tilde{X}_{t-} \in \tilde{X}_{\tilde{\Delta}} - \tilde{N}_1 \; \forall t \geq 0\}$$

$$\tilde{\Lambda}_{12} = \{\tilde{\omega} \in \tilde{\Omega} : \lim_{t \to \infty} \sigma_{\tilde{X} - \tilde{F}_n} \geq \tilde{\zeta}(\tilde{\omega})\}.$$

Adjoin a point Δ to X as an extra point (as a point at infinity if X is locally compact) and extend q to a one to one mapping from $X_0 \cup \Delta$ onto $\tilde{X}_0 \cup \tilde{\Delta}$ by setting $q(\Delta) = \tilde{\Delta}$. We define an $\mathcal{E}$-exceptional Borel set $N_1 \subset X$ by

$$X - N_1 = q^{-1}(\tilde{X} - \tilde{N}_1). \tag{30}$$

We let

$$\Omega = \tilde{\Lambda}_1 \qquad \mathcal{F}_t = \tilde{\mathcal{F}}_t \cap \tilde{\Lambda}_1 \ t \in [0, \infty]. \tag{31}$$

The element of Ω (resp. $\{\mathcal{F}_t\}$) is denoted by ω (resp. Λ) instead of $\tilde{\omega}$ (resp. $\tilde{\Lambda}$). Finally let us define X_t, ζ, P_x by

$$X_t(\omega) = q^{-1}(\tilde{X}_t(\omega)) \quad \omega \in \Omega, \ t \geq 0, \quad \zeta(\omega) = \tilde{\zeta}(\omega), \ \omega \in \Omega, \tag{32}$$

$$P_x(\Lambda) = \tilde{P}_{qx}(\Lambda) \quad x \in X \cup \Delta - N_1, \ \Lambda \in \mathcal{F}_\infty. \tag{33}$$

With these definitions of elements, we put

$$\tilde{\mathbf{M}}_1 = (\Omega, \{\mathcal{F}_t\}_{t \in [0,\infty]}, \tilde{X}_t, \tilde{\zeta}, \tilde{P}_{\tilde{x}}),$$

$$\mathbf{M}_1 = (\Omega, \{\mathcal{F}_t\}_{t \in [0,\infty]}, X_t, \zeta, P_x).$$

$\tilde{\mathbf{M}}_1$ is a Hunt process on $\tilde{X} - \tilde{N}_1$ which is associated with the regular Dirichlet form $\tilde{\mathcal{E}}$. We shall call $\mathbf{M}_1$ *the image of the Hunt process* $\tilde{\mathbf{M}}_1$ *by the quasi-homeomorphism* q^{-1}.

Theorem 5.1 $\mathbf{M}_1$ *defined by* (30)$\sim$(33) *is a standard process on* $X - N_1$ *associated with the quasi-regular Dirichlet form* $\mathcal{E}$*. Further* $\mathbf{M}_1$ *is special and tight.*

Proof. The first assertion can be proved in the same way as in [F 71b, §4] where $(X, m, \mathcal{E}, \mathcal{F})$, $(\tilde{X}, \tilde{m}, \tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ and $\tilde{\mathbf{M}}$ were a regular Dirichlet space, its strongly regular representation, and a Hunt modification of the Ray process associated with the latter respectively, and the process $\mathbf{M}_1$ defined by (30)$\sim$(33) was shown to be a Hunt process on $X - N_1$. The only defference from the present situation was in that $\{F_n\}$, $\{\tilde{F}_n\}$ were nests in the sense of [FOT 94] rather than $\mathcal{E}$-nest and $\tilde{\mathcal{E}}$-nest, and accordingly we had the following stronger property than (29):

$$\tilde{P}_x(\lim_{n\to\infty} \sigma_{\tilde{X}-\tilde{F}_n} < \infty) = 0 \quad \forall \tilde{x} \in \tilde{X} - \tilde{N}_1. \tag{34}$$

In the present case, we can also see as in [F 71b] that $\{\mathcal{F}_t\}_{t\in[0,\infty]}$ defined by (31) is the minimum completed admissible filtration for X_t defined by (32). Thus $\mathbf{M}_1$ is special, namely, $\{\mathcal{F}_t\}$ is quasi-left continuous because so is $\{\tilde{\mathcal{F}}_t\}$. Let $\tilde{K}_n$ be compact sets increasing to $\tilde{X}$ such that $\tilde{K}_n$ is included in the interior of $\tilde{K}_{n+1}$ and put $K_n = q^{-1}(\tilde{K}_n \cap \tilde{F}_n)$. Then $\{K_n\}$ are increasing compact subsets of $X - N_1$ and we get obviously the tightness of $\mathbf{M}_1$:

$$P_x(\lim_{n\to\infty} \sigma_{X-K_n} < \zeta) = 0 \quad \forall x \in X - N_1.$$

□

The state space of $\mathbf{M}_1$ can be enlarged to X if necessary by an trivial extension (namely, by making each point of N_1 to be a trap.) [MR 92, Th. 3.5] has given another construction of an m-tight special standard process associated with a quasi-regular Dirichlet space.

Remark 5.3 According to [MR 92, Th. 5.1, Th.6.4] however, two standard processes associated with the same quasi-regular Dirichlet space admit a common properly exceptional set in the standard sense such that their restrictions to the complement of this set share a common transition function. Therefore, in dealing with a standard process associated with a quasi-regular Dirichlet space, we may assume without loss of generality that it is an image of a Hunt process by a quasi-homeomorphism. This viewpoint is very convenient in utilizing the transfer method.

Thus, in the rest of this section, we continue to work with a quasi-regular Dirichlet space $(X, m, \mathcal{E}, \mathcal{F})$ and an associated standard process $\mathbf{M}_1 = (\Omega, \{\mathcal{F}_t\}, X_t, \zeta, P_x)$ on $X - N_1$ defined by (30)$\sim$(33). We can transfer Lemma 5.1 as follows:

Lemma 5.2 *The following conditions for a set $N \subset X$ are equivalent each other:*

(i) *N is $\mathcal{E}_1$-exceptional.*

(ii) *N is $\mathbf{M}_1$-exceptional.*

(iii) *N is contained in a properly exceptional set in the standard sense.*

Remark 5.4 Since the state space of $\mathbf{M}_1$ is $X - N_1$, any $\mathbf{M}_1$-properly exceptional set in the standard sense is required to contain the set N_1 (see Remark 5.2).

Proof. For simplicity, let N be a Borel subset of $X - N_1$ and put $\tilde{N} = q(N)(\subset \tilde{X} - \tilde{N}_1)$. Of course, N is $\mathcal{E}$-exceptional iff $\tilde{N}$ is $\tilde{\mathcal{E}}$-exceptional. By our construction of $\mathbf{M}_1$, we see that the $\mathbf{M}_1$-exceptionality of N (resp. $\mathbf{M}_1$-proper exceptionality in the standard sense of $N \cup N_1$) is equivalent to the $\tilde{\mathbf{M}}_1$-exceptionality of $\tilde{N}$ (resp.$\tilde{\mathbf{M}}_1$-proper exceptionality of $\tilde{N} \cup \tilde{N}_1$). But, in view of Lemma 5.1 and Remark 5.1, three conditions of the present lemma are equivalent for $\tilde{N}$, the regular Dirichlet form $\tilde{\mathcal{E}}$ and the Hunt process $\tilde{\mathbf{M}}_1$. □

Remark 5.5 In relation to the ω-set involved in a properly exceptional set in the standard sense, we make the following remark : for a Borel set $A \subset X$, the ω-set

$$\Gamma = \{\omega \in \Omega : X_t(\omega),\ X_{t-}(\omega) \in A\ \forall t \in [0, \zeta(\omega))\}$$

is $\mathcal{F}_\infty$-measurable, because, for $B = X - A$, and each $T > 0$, the ω-set

$$\{\omega : X_{t-} \in B \cup \{\Delta\}\ \exists t \in [0, T \wedge \zeta(\omega))\}$$

is contained in

$$\{\omega : X_{t-} \text{ exists and is in } B \cup \{\Delta\}\ \exists t \in [0, T]\},$$

and by [BG 68, Prop.10.20] we can see that $\Omega \setminus \Gamma \in \mathcal{F}_\infty$.

The notion of the additive functional $A_t(\omega)$ of the present standard process $\mathbf{M}_1$ on $X - N_1$ is defined exactly in the same way as for an Hunt process except that we now adopt a properly exceptional set in the standard sense (instead of a properly exceptional set) as an exceptional set N $(N_1 \subset N \subset X)$ of the additive functional A. The equivalence of two AF's $A^{(1)},\ A^{(2)}$ of

$\mathbf{M}_1$ is defined in the same way as for an Hunt process. On account of Lemma 5.2 and Remark 5.4, we can then find a common defining set $\Lambda \in \mathcal{F}_\infty$ and a common exceptional set $N(\supset N_1)$ such that

$$A_t^{(1)}(\omega) = A_t^{(2)}(\omega), \ \forall t \geq 0, \ \forall \omega \in \Lambda.$$

We can easily observe that, if $A_t(\omega)$ is an AF of $\mathbf{M}_1$ with a defining set Λ and an exceptional set $N(\supset N_1)$, then $A_t(\omega)$ is an AF of $\tilde{\mathbf{M}}_1$ with some defining set contained in Λ and some exceptional set containing $q(N)(\supset \tilde{N}_1)$. Conversely any AF of $\tilde{\mathbf{M}}_1$ with a defining set Λ and an exceptional set $\tilde{N}(\supset \tilde{N}_1)$ can be viewed as an AF of $\mathbf{M}_1$ with the same defining set and the exceptional set $q^{-1}(\tilde{(N)})$. Two AF's are equivalent with respect to $\mathbf{M}_1$ iff so they are with respect to $\tilde{\mathbf{M}}_1$.

Various classes of AF's of $\mathbf{M}_1$ are defined in the same way as for an Hunt process. In particlur, we have the classes $\mathbf{A}_c^+$ of PCAF's, $\overset{\circ}{\mathcal{M}}$ of margingale AF's of finite energy and $\mathcal{N}_c$ of continuous AF's of zero energy for the process $\mathbf{M}_1$. For $u \in \mathcal{F}$ and its $\mathcal{E}$-quasicontinuous version u^*, we put

$$\tilde{u}^*(\tilde{x}) = u^*(q^{-1}(\tilde{x})) \quad \tilde{x} \in \tilde{X} - \tilde{N}_1.$$

Then $\tilde{u}^*$ is an $\tilde{\mathcal{E}}$-quasicontinuous version of $\tilde{u} = \Phi u$ and

$$u^*(X_t(\omega)) - u^*(X_0(\omega)) = \tilde{u}^*(\tilde{X}_t(\omega)) - \tilde{u}^*(\tilde{X}_0(\omega)) \quad \omega \in \Omega.$$

Hence $A^{[u]}$ (the left hand side) is (up to the equivalence) a finite cadlag AF and uniquely expressible as (3) for $\mathbf{M}_1$ because so is $A^{[\tilde{u}]}$ (the right hand side) for $\tilde{\mathbf{M}}_1$.

For the present quasi-regular Dirichlet form $\mathcal{E}$, the class S of smooth measures and the notion of the energy measure $\mu_{[u]}$ of $u \in \mathcal{F}_b$ are defined also in the same way as for a regular Dirichlet form. q preserves the notion of the smoothness of positive measures. μ is the Revuz measure of a PCAF A of $\mathbf{M}_1$ if $q\mu$ is the Revuz measure of A as a PCAF of $\tilde{\mathbf{M}}_1$. The energy measure of $u \in \mathcal{F}_b$ characterized by the equation (28) can be constructed as the image by q^{-1} of the energy measure of $\Phi u \in \tilde{\mathcal{F}}_b$ with respect to the regular Dirichlet form $\tilde{\mathcal{E}}$.

Summing up what has been mentioned, we get

Theorem 5.2 (i) *The equivalence classes of PCAF's $\mathbf{A}_c^+$ of $\mathbf{M}_1$ and the smooth measures S of $\mathcal{E}$ are in one to one (Revuz) correspondence by (27).*

(ii) *Any $u \in \mathcal{F}$ admits a unique finite smooth measure $\mu_{\langle u \rangle}$ satisfying the equation (28) in the case that $u \in \mathcal{F}_b$.*

(iii) *For any $u \in \mathcal{F}$, the AF (1) admits the decomposition (3) uniquely up to the equivalence where $M^{[u]} \in \overset{\circ}{\mathcal{M}}$, $N^{[u]} \in \mathcal{N}$. $M^{[u]}$ admits its quadratic variation in $\mathbf{A}_c^+$ whose Revuz measure is the energy measure of u.*

Finally we transfer Theorem 5.4.2 of [FOT 94]. Recall that an AF $A_t(\omega)$ is said to be of bounded variation if it is of bounded variation in t on each compact interval of $[0, \zeta(\omega))$ for every fixed ω in its defining set.

Theorem 5.3 *The following conditions are equivalent for $u \in \mathcal{F}$:*

(i) *The AF $A^{[u]}$ defined by* (1) *is a semimartingale in the sense that the CAF $N^{[u]}$ in its decomposition* (3) *is of bounded variation.*

(ii) *There exists a signed smooth measure ν with some attached $\mathcal{E}$-nest $\{E_n\}$ for which the equation* (4) *holds.*

If condition (ii) *holds, then $N^{[u]}$ admits an expression* (5) *for $A^k \in \mathbf{A}_c^+$ with Revuz measure ν^k, $k = 1, 2$, where*

$$\nu = \nu^1 - \nu^2 \tag{35}$$

is the Jordan decomposition of the smooth signed measure ν (hence ν^k, $k = 1, 2$, are automatically smooth). If condition (i) *holds, then $N^{[u]}$ admits an expression* (5) *by some $A^k \in \mathbf{A}_c^+, k = 1, 2$, and condition* (ii) *is fulfilled for the signed smooth measure ν of* (35) *defined by the Revuz measure ν^k of A^k, $k = 1, 2$.*

6 Semimartigale characterizations of AF's

Let $(X, m, \mathcal{E}, \mathcal{F})$ be a quasi-regular Dirichlet space and $\mathbf{M}_1 = (X_t, P_x)$ be an associated standard process specified in Theorem 5.1 as an image of a Hunt process by a quasi-homeomorphism. By Theorem 4.1 and Theorem 5.3, we have

Theorem 6.1 *The following two conditions are equivalent for $u \in \mathcal{F}$:*

(i) *There exists an $\mathcal{E}$-nest $\{E_n\}$ for which the inequality* (2) *is valid for some positive constant C_n.*

(ii) *The AF $A^{[u]}$ defined by* (1) *is a semimartingale in the sense that the CAF $N^{[u]}$ in its decomposition* (3) *is of bounded variation.*

When one of these conditions is satisfied, there exists a signed smooth measure ν with some attached $\mathcal{E}$-nest $\{E_n\}$ for which the equation (4) *holds. Further $N^{[u]}$ admits an expression* (5) *for $A^k \in \mathbf{A}_c^+$ with Revuz measure ν^k, $k = 1, 2$, which are related to ν by* (35).

We now study a simple case that the $\mathcal{E}$-nest appearing in the inequality (2) is trivial. Then (2) is simplified to the condition that, for $u \in \mathcal{F}$, there exists a positive constant C for which the inequality (6) holds for any v in the space $\mathcal{F}_b$. Let $\mathcal{L}$ be a subspace of $\mathcal{F}_b$ satisfying condition ($\mathcal{L}$) described in §1.

Lemma 6.1 *If, for $u \in \mathcal{F}$, the inequality* (6) *holds for any v in the space $\mathcal{L}$, then* (6) *holds for any v in $\mathcal{F}_b$.*

Proof. Take any $v \in \mathcal{F}_b$ and set $M = \|v\|_\infty$. For any $\epsilon > 0$, we can find a real function $\psi(t)$ such that

$$|\psi(t)| \leq M + \epsilon;\ \psi(t) = t,\ -M \leq t \leq M;\ 0 \leq \psi(t) - \psi(s) \leq t - s,$$

and $\psi(\mathcal{L}) \subset \mathcal{L}$. Choose $v_n \in \mathcal{L}$ which are $\mathcal{E}$-convergent to v. Then $\psi(v_n) \in \mathcal{L},\ \psi(v_n)$ is $\mathcal{E}$-convergent to $\psi(v) = v$. From the validity of (6) for functions in $\mathcal{L}$, we get

$$\mathcal{E}(u, \psi(v_n)) \leq \|\psi(v_n)\|_\infty \leq M + \epsilon.$$

It suffices to let $n \to \infty$ and then $\epsilon \downarrow 0$. □

Theorem 6.2 *For $u \in \mathcal{F}$, the following conditions are equivalent:*

(i) *The inequality* (6) *holds for any v in the space $\mathcal{L}$ satisfying condition ($\mathcal{L}$).*

(ii) *There exists a unique finite signed smooth measure ν for which the equation* (4) *is valid for any $v \in \mathcal{F}_b$.*

(iii) *The continuous AF $N_t^{[u]}$ in the decomposition* (3) *of the AF* (1) *is of bounded variation and satisfies the property* (7).

In this case, $N^{[u]}$ admits an expression (5) *by PCAF's $A^k,\ k = 1, 2$, whose Revuz measures $\nu^k,\ k = 1, 2$, are finite smooth measures related to the signed measure ν of* (ii) *by* (35).

Proof. The first two conditions are equivalent by virtue of Theorem 4.2 and Lemma 6.1. Assume (ii) and let (35) be the Jordan decomposition of ν. Then $\nu^k,\ k = 1, 2$, are finite smooth measures and, on account of Thorem 5.3, $N^{[u]}$ is expressible by the PCAF's A^k with Revuz measure $\nu^k,\ k = 1, 2$. Denote by $\{N^{[u]}\}_t$ the total variation of $N^{[u]}$ on $[0, t]$. It is known to be an element of $\mathbf{A}_c^+$. Since it is dominated by $A_t^1 + A_t^2$, we get from the Revuz correspondence (27)

$$\lim_{t\downarrow 0} \frac{1}{t} E_m(\{N^{[u]}\}_t) \leq \lim_{t\downarrow 0} \frac{1}{t} E_m(A_t^1 + A_t^2) = \nu^1(X) + \nu^2(X) < \infty,$$

arriving at (iii).

Conversely, assume (iii) and let

$$A_t^1 = \frac{1}{2}(\{N^{[u]}\}_t + N_t^{[u]}),\ A_t^2 = \frac{1}{2}(\{N^{[u]}\}_t - N_t^{[u]}).$$

On account of Theorem 5.3, condition (ii) holds for the signed smooth measure (35) where ν^k is defined to be the Revuz measure of $A^k,\ k = 1, 2$. Then ν is finite, because by condition (7)

$$\nu^1(X) + \nu^2(X) = \lim_{t\downarrow 0} \frac{1}{t} E_m(A_t^1 + A_t^2) = \lim_{t\downarrow 0} \frac{1}{t} E_m(\{N^{[u]}\}_t) < \infty.$$

□

Remark 6.1 Property (7) says that the Revuz measure of the PCAF $\{N^{[u]}\}_t$ has a finite total mass. By [FOT 94, Th.5.3.1], any PCAF A and its Revuz measure μ are related by

$$E_m\left(\int_0^t f(X_s)dA_s\right) = \int_o^t \langle f \cdot p_s 1, \mu\rangle ds, \quad f \in \mathcal{B}.$$

Hence, property (7) implies the P_m-integrability

$$E_m(\{N^{[u]}\}_t) < \infty, \quad t > 0. \tag{36}$$

If the process is conservative in the sense that $p_s 1 = 1,\ s > 0$, then the integrability (36) implies property (7).

In the rest of this section, we assume that $(X, m, \mathcal{E}, \mathcal{F})$ is a regular Dirichlet space and $\mathbf{M} = (X_t, P_x)$ is any associated Hunt process. Of course, the preceding two theorems remain valid under the present assumption. Let us further assume that $\mathcal{E}$ is strongly local: if $u, v \in \mathcal{F}$ are of compact support and v is constant on a neighbourhood of the support of u, then $\mathcal{E}(u, v) = 0$.

Then the associated Hunt process $\mathbf{M}$ can be taken to be a diffusion (namely, of continuous sample paths on $[0, \zeta)$) with no killing inside. A function u is said to be *locally in* $\mathcal{F}$ ($u \in \mathcal{F}_{loc}$ in notation) if for any relatively compact open set G there is a function $w \in \mathcal{F}$ such that $u = w\ m - a.e.$ on G. Let $u \in \mathcal{F}_{loc}$. u admits an $\mathcal{E}$-quasicontinuous version u^*. The energy measure $\mu_{\langle u \rangle}$ of u is still well defined. The AF $A^{[u]}$ defined by (1) admits a decomposition (3) where $M^{[u]} \in \overset{\circ}{\mathcal{M}}_{loc}$ and $N^{[u]} \in \mathcal{N}_{c,loc}$. The decomposition is unique up to the equivalence of local AF's. The quadratic variation $\langle M^{[u]} \rangle \in \mathbf{A}_c^+$ of $M^{[u]}$ has as its Revuz measure the energy measure $\mu_{\langle u \rangle}$ of u. See [FOT 94, §5.5] for details of the above notions, notations and statements.

As for the semimartingale characterization of $N^{[u]} \in \mathcal{N}_{c,loc}$ for $u \in \mathcal{F}_{loc}$, all statements in Theorem 5.3 remain true except that the $\mathcal{E}$-nest $\{E_n\}$ appearing in the condition (ii) there is now required to consist of compact subsets of X ([FOT 94, th.5.5.4]).

The next condition for a subset $\mathcal{C}$ of $C_0(X)$ is taken from [FOT 94] (condition ($\mathcal{C}$.2) of [FOT 94]).

($\mathcal{C}$) $\mathcal{C}$ is a dense subalgebra of $C_0(X)$. For any compact set K and relatively compact open set G with $K \subset G$, $\mathcal{C}$ admits an element u such that $u \geq 0,\ u = 1$ on K and $u = 0$ on $X - G$.

Let $\mathcal{C}$ be a subset of $\mathcal{F} \cap C_0(X)$ satisfying both conditions $\mathcal{L}$ and $\mathcal{C}$. Such a subset is similar to a special standard core in the sense of [FOT 94], and the only difference lies in the definition of a real function $\phi_\epsilon(t)$ appearing in condition $\mathcal{L}$. For any open set $G \subset X$, we put

$$\mathcal{F}_G = \{u \in \mathcal{F} : u^* = 0\ \mathcal{E} - q.e. \text{ on } X - G\},$$

$$\mathcal{C}_G = \{u \in \mathcal{C} : Supp[u] \subset G\}.$$

$\mathcal{C}_G$ is uniformly dense in $C_0(G)$ and $\mathcal{E}_1$-dense in $\mathcal{F}_G$ ([FOT 94, Lem.2.3.4]).

Theorem 6.3 *The next conditions are equivalent for $u \in \mathcal{F}_{loc}$:*

(i) *For any relatively compact open set $G \subset X$, there is a positive constant C_G such that*

$$|\mathcal{E}(u, v)| \leq C_G \|v\|_\infty \quad \forall v \in \mathcal{C}_G. \tag{37}$$

(ii) *There exists a signed Radon measure ν on X charging no set of zero capacity such that*

$$\mathcal{E}(u, v) = -\int_X v(x)\nu(dx) \quad \forall v \in \mathcal{C}. \tag{38}$$

(iii) *$N_t^{[u]}$ is of bounded variation and satisfies property* (8).

In this case, $N^{[u]}$ admits an expression (5) *by PCAF's $A^k,\ k = 1, 2,$ whose Revus measures $\nu^k,\ k = 1, 2,$ are smooth Radon measures related to the signed measure ν of* (ii) *by* (35).

Proof. The implication (ii) $\Rightarrow$ (i) is trivial. Conversely, suppose condition (i) is fulfilled. Then there exists a unique signed Radon measure on X for which the equation (38) holds. We have to show that ν charges no set of zero capacity.

To this end, fix $w \in \mathcal{C}$ and choose a relatively compact open set G containing the support of w. Take $u_1 \in \mathcal{F}$ satisfying $u = u_1$ on G, then

$$\mathcal{E}(u, wv) = \mathcal{E}(u_1, wv), \;\; v \in \mathcal{C}$$

and we have from (i),

$$|\mathcal{E}(u_1, w \cdot v)| \leq C' \|v\|_\infty \quad \forall v \in \mathcal{C}, \tag{39}$$

where $C' = C_G \cdot \|w\|_\infty$. Since $\mathcal{C}$ is an algebra satisfying condition $(\mathcal{L})$, we can extend inequality (39) from $\mathcal{C}$ to $\mathcal{F}_b$. In fact, keeping the notations in the proof of Lemma 6.1, we can show that $w \cdot \psi(v_n)$ is $\mathcal{E}$-weakly convergent to $w \cdot v$.

By virtue of Proposition 4.1, the inequality (39) holding for u_1 and for any $v \in \mathcal{F}_b$ implies that the equation (15) is valid for u_1 and for a finite signed measure ν_w charging no set of zero capacity. A comparison with (38) yields

$$\nu_w = w \cdot \nu$$

and we conclude that ν charges no set of zero capacity because w is an arbitrary element of $\mathcal{C}$. The proof of the equivalece of (i) and (ii) is complete.

Consider relatively compact open sets $\{G_k\}$ such that

$$\bar{G}_k \subset G_{k+1}, \quad \cup_{k=1}^\infty G_k = X.$$

Just as in the proof of [FOT 94, Cor.5.4.1], we can see that condition (ii) is equivalent to the validity of Theorem 5.3 (ii) for $u \in \mathcal{F}_{loc}$, a signed Radon measure ν charging no set of zero capacity and the $\mathcal{E}$-nest $\{\bar{G}_n\}$, which in turn can been seen to be equivalent to the probabilistic condition (iii) in the same way as the corresponding proof of Theorem 6.2, because Theorem 5.3 is applicable to the present situation in view of the remark made in the parpagraph preceding the introducicition of the space $\mathcal{C}$.

□

Remark 6.2 Property (8) says that the Revuz measure of the PCAF $\{N^{[u]}\}_t$ is a Radon measure. (8) implies the integrability

$$E_m\left(\int_0^t I_K(X_s)d\{N^{[u]}\}_s\right) < \infty, \quad \forall K \text{ compact}, \; \forall t > 0. \tag{40}$$

Conversely (40) implies (8) if the process is conservative.

7 Stochastic characterizations of BV functions and Caccioppoli sets

Let R^d be the d-dimensional Euclidean space and m_0 be the Lebesgue measure on it. We consider a non-negative locally integrable function ρ on R^d and the associated *energy form* defined by

$$\mathcal{E}^\rho(u,v) = \frac{1}{2}\int_{R^d} \nabla u(x)\cdot\nabla v(x)\ \rho(x)\ m_0(dx), \quad u,v \in C_0^1(R^d). \tag{41}$$

Throughout this section, let us assume the Hamza type condition on ρ :

(H) $\rho = 0 \quad m-$a.e. on $S(\rho)$,
where

$$S(\rho) = \{x \in R^d : \int_{U(x)} \rho(y)^{-1} m_0(dy) = \infty\ \forall U(x)\}$$

the *singular set* of ρ.

The complement $R(\rho)$ is called the *regular set* of ρ. Under (H), the support of the measure ρdm_0 equals $\overline{R(\rho)}$. Further the form $\mathcal{E}^\rho$ is closable on $L^2(R^d;\rho\cdot m_0)$ and the closure $(\mathcal{E}^\rho,\mathcal{F}^\rho)$ is a strongly local regular Dirichlet form on $L^2(\overline{R(\rho)};\rho\cdot m_0)$ ([RW 85], [F 97b]).
The associated diffusion $\mathbf{M}^\rho = (X_t^\rho, P_x^\rho)$ on $\overline{R(\rho)}$ is called a *distorted Brownian motion.* The reason of this naming is in that, if we apply the decomposition (3) to coordinate functions

$$\psi_i(x) = x_i \in \mathcal{F}^\rho_{loc},\ 1 \le i \le d,$$

then we get the expression of the sample path

$$X_t^\rho - X_0^\rho = B_t + N_t^\rho \tag{42}$$

where B_t is a d-dimensional Brownian motion and

$$N_t^\rho = (N_t^1,\cdots,N_t^d), \quad N_t^i = N_t^{[\psi_i]} \in \mathcal{N}_{c,loc},\ 1 \le i \le d.$$

Condition (H) is fulfilled in the following two important cases:
(I) ρ is non-negative continuous m_0-a.e. on R^d.
(II) $\rho(x) = I_D(x),\ x \in R^d$, for an open set $D \subset R^d$.
In the second case, $\overline{R(\rho)} = \overline{D}$ and the distorted Brownian motion $\mathbf{M}^{I_D}$ reduces to the *modified reflecting Brownian motion* on $\overline{D}$ associated with the strongly local regular Dirichlet form

$$\mathcal{E}^{I_D}(u,v) = \frac{1}{2}\int_D \nabla u(x)\cdot\nabla v(x)\ m_0(dx),\ \mathcal{F}^{I_D} = \hat{H}^1(D)$$

on $L^2(\overline{D}; I_D\cdot m_0)(= L^2(D;m_0))$ studied in [F 97a]. Here $\hat{H}^1(D)$ denotes the closure of the space $C_0^1(R^d)|_D$ in the Sobolev space $H^1(D)$. The term 'modified' is added because $\hat{H}^1(D)$ could be a proper subset of $H^1(D)$.

A function $\rho \in L^1_{loc}(R^d)$ is called BV (denoted by $\rho \in BV_{loc}$) if for any bounded open set $V \subset R^d$, there exists a positive constant C_V such that

$$|\int_V (div\ v)\rho m_0(dx)| \leq C_V \|v\|_\infty \quad \forall v \in C^1_0(V; R^d). \tag{43}$$

Theorem 7.1 *Suppose a non-negative function $\rho \in L^1_{loc}(R^d)$ satisfies the condition* (H). *Then the following conditions are equivalent:*

(i) $\rho \in BV_{loc}$.

(ii) *The distorted Brownian path X^ρ_t is a semimartingale in the sense that each component N^i_t, $(1 \leq i \leq d)$, in the decomposition* (42) *is of bounded variation and additionally it satisfies that*

$$\lim_{t \downarrow 0} \frac{1}{t} E^\rho_{\rho \cdot m_0} \left(\int_0^t I_K(X^\rho_s) |dN^i_s| \right) < \infty \quad 1 \leq i \leq d, \tag{44}$$

for every compact set $K \subset \overline{R(\rho)}$.

Proof. We apply Theorem 6.3 to the strongly local, regular Dirichlet form $(\mathcal{E}^\rho, \mathcal{F}^\rho)$ on $L^2(X; m)$ and an associated diffusion $\mathbf{M}^\rho = (X^\rho_t, P^\rho_x)$ on X, where

$$X = \overline{R(\rho)} \quad m = \rho \cdot m_0.$$

We take

$$\mathcal{C} = C^1_0(R^d)|_X,$$

which obviously has the properties $(\mathcal{L})$ and $(\mathcal{C})$. Taking as u the coodinate function $\psi_i \in \mathcal{F}_{loc}$, the condition (i) of Theorem 6.3 reads as follows: for any relatively compact open set $G \subset R^d$, there is a positive constant C_G such that

$$|\int_{R(\rho)} \partial_i v(x) \rho(x) m_0(dx)| \leq C_G \sup_{x \in R(\rho)} |v(x)|, \quad \forall v \in C^1_0(G).$$

It is easy to see that $\rho \in BV_{loc}$ if and only if the above condition is satisfied for each $i = 1, 2, \cdots, d$. Thus, we get Theorem 7.1 from Theorem 6.3. □

A measurable set $E \subset R^d$ is called *Caccioppoli* if $I_E \in BV_{loc}$. We know ([EG 92]) that E is Caccioppoli if and only if there exists a positive Radon measure σ on ∂E and a σ-measurable vector

$$\mathbf{n}_E : \partial E \to R^d$$

with $|\mathbf{n}_E| = 1$ σ-a.e. such that

$$\int_E div\ v\ m_0(dx) = -\int_{\partial E} v \cdot \mathbf{n}_E\ d\sigma \quad \forall v \in C^1_0(R^d, R^d). \tag{45}$$

By specifying Theorem 7.1 to $\rho = I_D$, we get

Theorem 7.2 *The following conditions are equivalent for an open set $D \subset R^d$:*

(i) D is Caccioppoli.

(ii) The modified reflecting Brownian path $(X_t, P_x) = (X_t^{I_D}, P_x^{I_D})$ on $\overline{D}$ is a semimartingale in the sense that each component N_t^i of the second term $N_t^{I_D}$ in its decomposition (42) *is of bounded variation and satisfies the additional property that*

$$\lim_{t \downarrow 0} \frac{1}{t} E_{I_D m_0} \left(\int_0^t I_K(X_s) |dN_s^i| \right) < \infty$$

In this case, the modified reflecting Brownian motion admits an expession of Skorohod type:

$$X_t - X_0 = B_t + \int_0^t \mathbf{n}(X_s) dL_s \tag{46}$$

for a PCAF L_t with Revuz measure σ.

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ON REGULAR DIRICHLET SUBSPACES OF $H^1(I)$ AND ASSOCIATED LINEAR DIFFUSIONS

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Abstract

We will give a complete characterization of all regular Dirichlet subspaces of $H^1(I)$ for a finite open interval I by a certain family of scale functions. Each associated diffusion will be constructed from a reflecting Brownian motion on a closed interval by a time change and a transformation of the state space.

1. Introduction

Throughout this paper let I be a finite open interval (a, b) or the real line $\mathbb{R}$. Denote by $L^2(I)$ the space of square integrable functions on I and we let

$$H^1(I) = \{u \in L^2(I) \colon u \text{ is absolutely continuous and } u' \in L^2(I)\}.$$

$$\mathbf{D}(u, v) = \int_I u' \cdot v' \, dx \quad u, v \in H^1(I).$$

$(H^1(I), (1/2)\mathbf{D})$ can be considered as a regular local recurrent Dirichlet form on $L^2(\bar{I})$, where $\bar{I}$ denotes $[a, b]$ (resp. $\mathbb{R}$) for $I = (a, b)$ (resp. $I = \mathbb{R}$). The associated diffusion process on $\bar{I}$ is the reflecting Brownian motion (resp. the Brownian motion).

We call $(\mathcal{F}, \mathcal{E})$ a *Dirichlet subspace of* $(H^1(I), (1/2)\mathbf{D})$ if

$$\mathcal{F} \subset H^1(I), \ \mathcal{E}(u, v) = \frac{1}{2}\mathbf{D}(u, v), \quad u, v \in \mathcal{F}, \tag{1.1}$$

and $(\mathcal{F}, \mathcal{E})$ is a Dirichlet form on $L^2(I)$. It is called *regular on* $L^2(\bar{I})$ $(= L^2(I))$ if $\mathcal{F} \cap C_0(\bar{I})$ is dense both in $\mathcal{F}$ and $C_0(\bar{I})$, where $C_0(\bar{I})$ denotes the space of continuous functions on $\bar{I}$ with compact support. It is called *recurrent* if its extended Dirichlet space $\mathcal{F}_e$ contains the constant function 1. When I is finite, any regular Dirichlet subspace of $(H^1(I), (1/2)\mathbf{D})$ is automatically recurrent.

In this paper, we shall prove that the Sobolev space $(H^1(I), (1/2)\mathbf{D})$ admits as its

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regular Dirichlet subspaces the following family of spaces $(\mathcal{F}^{(\mathrm{s})}, \mathcal{E}^{(\mathrm{s})})_{\mathrm{s}\in\mathbf{S}}$:

$$\mathcal{F}^{(\mathrm{s})} := \left\{ u \in L^2(I) : u \text{ is absolutely continuous} \right. \\ \left. \text{with respect to } d\mathrm{s}(x),\ \int_I \left(\frac{du}{d\mathrm{s}}(x)\right)^2 d\mathrm{s}(x) < \infty \right\} \tag{1.2}$$

$$\mathcal{E}^{(\mathrm{s})}(u, v) := \frac{1}{2}\int_I \frac{du}{d\mathrm{s}}\frac{dv}{d\mathrm{s}}\,d\mathrm{s}, \quad u, v \in \mathcal{F}^{(\mathrm{s})}, \tag{1.3}$$

for s belonging to the space of functions

$$\mathbf{S} = \{\mathrm{s} : \mathrm{s}(x) \text{ is absolutely continuous, strictly increasing in } x \in I \\ \text{and } \mathrm{s}'(x) = 0 \textit{ or } 1 \text{ for a.e. } x \in I,\ \ \mathrm{s}(\eta) = 0\}, \tag{1.4}$$

where η denotes either a or 0 according as I is (a, b) or $\mathbb{R}$.

We shall further consider the subfamily

$$\hat{\mathbf{S}} = \begin{cases} \mathbf{S} & \text{when } I = (a, b), \\ \{\mathrm{s} \in \mathbf{S} : \mathrm{s}(\pm\infty) = \pm\infty\} & \text{when } I = \mathbb{R}, \end{cases} \tag{1.5}$$

of $\mathbf{S}$ and prove that all recurrent regular Dirichlet subspaces of $(H^1(I), (1/2)\mathbf{D})$ are exhausted by the family of spaces $(\mathcal{F}^{(\mathrm{s})}, \mathcal{E}^{(\mathrm{s})})_{\mathrm{s}\in\hat{\mathbf{S}}}$.

For $\mathrm{s} \in \mathbf{S}$, we let $E_{\mathrm{s}} = \{x \in I : \mathrm{s}'(x) = 0\}$ and denote by $|\cdot|$ the Lebesgue measure. Denote by φ the linear function $\varphi(x) = x,\ x \in I$. Clearly, $\varphi \in \mathcal{F}^{(\mathrm{s})}$ ($\mathcal{F}^{(\mathrm{s})}_{\mathrm{loc}}$ when $I = \mathbb{R}$) if and only if $|E_{\mathrm{s}}| = 0$, or equivalently, the inverse function of s is absolutely continuous. In this case, $\mathrm{s}(x)$ equals either $\varphi(x) - a$ or $\varphi(x)$ according as I is (a, b) or $\mathbb{R}$, and $\mathcal{F}^{(\mathrm{s})} = H^1(I)$ of course. A typical example of an element $\mathrm{s} \in \mathbf{S}$ for $I = (0, 2)$ with $|E_{\mathrm{s}}| > 0$ is provided by

$$\mathrm{s} := \mathrm{t}^{-1},\ \ \mathrm{t}(x) := c(x) + x, \quad x \in (0, 1), \tag{1.6}$$

where c is the standard Cantor function on (0, 1).

In this connection, we would like to mention that the second and the third authors have considered in [3] a slightly more general regular Dirichlet form than $(H^1(I), (1/2)\mathbf{D})$ for $I = (0, 1)$ and studied its regular Dirichlet subspace. Unfortunately, there is a flaw in the proof of Theorem 2 in [3]. As is corrected in [4], it should be replaced by the following weaker assertion for which the proof given in [3] works: *Let* $\check{\mathcal{F}}$ *be a subspace of* $\mathcal{F}$ *such that* $(\check{\mathcal{F}}, \mathcal{E})$ *is a regular Dirichlet space on* $L^2(\bar{I}, \rho\,dx)$. *Assume that a scale function* s *of the diffusion process on* $\bar{I}$ *associated with* $(\check{\mathcal{F}}, \mathcal{E})$ *admits an absolutely continuous inverse* t. *Then* $\check{\mathcal{F}} = \mathcal{F}$.

The organization of the present paper is as follows. The next two sections are devoted to the proof of the above mentioned assertions. In particular, we shall show in §2 that any recurrent regular Dirichlet subspace of $(H^1(I), (1/2)\mathbf{D})$ has a scale function belonging to the class $\hat{\mathbf{S}}$.

In §3, we shall construct a recurrent diffusion process on $[a, b]$ (resp. $\mathbb{R}$) associated with the space $(\mathcal{F}^{(\mathtt{s})}, \mathcal{E}^{(\mathtt{s})})$ for $\mathtt{s} \in \hat{\mathbf{S}}$ from the reflecting Brownian motion on a closed interval (resp. the Brownian motion on $\mathbb{R}$) by a time change and a state space transformation. Since the infinitesimal generator of this diffusion is $(d/2dx)(d/d\mathtt{s})$ in Feller's canonical form, such a construction is well known in principle (cf. [6]), but we shall formulate it in relation to the transformations of Dirichlet forms in order to ensure the recurrence of the resulting diffusion and Dirichlet form.

In the last section, we shall state some useful descriptions of the space $\mathbf{S}$ and give examples of $\mathtt{s} \in \mathbf{S} \setminus \hat{\mathbf{S}}$ corresponding to transient regular Dirichlet subspaces of $(H^1(\mathbb{R}), (1/2)\mathbf{D})$.

2. Regular Dirichlet subspaces and scale functions

We recall (cf. [2, p.55]) that the extended Dirichlet space $H_e^1(I)$ of $H^1(I)$ is given by

$$H_e^1(I) = \{u : u \text{ is absolutely continuous on } I \text{ and } u' \in L^2(I)\}. \tag{2.1}$$

In particular, $1 \in H_e^1(I)$ and the Dirichlet form $(H^1(I), (1/2)\mathbf{D})$ is recurrent. $H_e^1(I)$ is continuously imbedded into $C(\bar{I})$ and in fact the following elementary inequality holds for any $x, y \in \bar{I}$:

$$|u(y) - u(x)|^2 \leq |y - x|\mathbf{D}(u, u), \quad u \in H_e^1(I). \tag{2.2}$$

When I is finite, $H_e^1(I) = H^1(I)$.

Let $(\mathcal{F}, \mathcal{E})$ be a regular Dirichlet subspace of $(H^1(I), (1/2)\mathbf{D})$. Since $(\mathcal{F}, \mathcal{E})$ is strongly local, there exists a diffusion process $\mathbf{M} = (X_t, P_x)$ on $\bar{I}$ associated with it. Denote by σ_y the hitting time of the one point set $\{y\}$, $y \in \bar{I}$, for $\mathbf{M}$. The next lemma about the existence of the scale function (a strictly increasing continuous function satisfying (2.3)) is well known for a more general one-dimensional diffusion process ([5]) but we give a self contained proof of it based on the inequality (2.2) in the present special situation.

Lemma 2.1. *There exists a strictly increasing function $\mathtt{s}$ on $\bar{I}$ uniquely up to a linear transformation such that*

$$P_x(\sigma_d < \sigma_c) = \frac{\mathtt{s}(x) - \mathtt{s}(c)}{\mathtt{s}(d) - \mathtt{s}(c)}, \quad c \leq x \leq d, \ c, d \in \bar{I}. \tag{2.3}$$

$\mathtt{s}$ is absolutely continuous on I.

Proof. Let J be a connected open subset of $\bar{I}$. We denote by τ_J the leaving time from J of the diffusion $\mathbf{M}$. We also consider the part $\mathbf{M}_J$ of $\mathbf{M}$ on J the diffusion killed upon the leaving time τ_J. $\mathbf{M}_J$ is then associated with the subspace $\mathcal{F}_J$ of $(\mathcal{F}, \mathcal{E})$ defined by

$$\mathcal{F}_J = \{u \in \mathcal{F} : u(x) = 0, \ \ x \in \bar{I} \setminus J\}.$$

(2.2) implies that each singlèton of J has a positive capacity with respect to the Dirichlet form $(\mathcal{F}_J, \mathcal{E})$. Consequently, the connectedness of the state space J is a synonym for its quasi-connectedness for $(\mathcal{F}_J, \mathcal{E})$ and hence $(\mathcal{F}_J, \mathcal{E})$ is irreducible ([2, p.172]). This implies, by virtue of [2, Theorem 4.6.6], that

$$P_x(\sigma_y < \tau_J) > 0 \quad \forall x, y \in J. \tag{2.4}$$

For any $c, d \in \bar{I}$, $-\infty < c < d < \infty$, we make the following choice of the intervals $J \subset \bar{I}$: when $\bar{I} = [a, b]$ (resp. $\bar{I} = \mathbb{R}$), we take $[a, d)$ and $(c, b]$ (resp. $(-\infty, d)$ and (c, ∞)). We then get from (2.4)

$$P_x(\sigma_c < \sigma_d) > 0, \ \ P_x(\sigma_d < \sigma_c) > 0, \quad \forall x \in (c, d).$$

We also note here that

$$P_c(\sigma_c < \sigma_d) = 1 \quad P_d(\sigma_d < \sigma_c) = 1 \tag{2.5}$$

because the positivity of the capacity of a point implies the $\mathbf{M}$-regularity of the point for itself.

On the other hand, for the finite open interval $J = (c, d) \subset I$, the space $(\mathcal{F}_J, \mathcal{E})$ admits a 0-order potential operator G^0 by virtue of (2.2) again: for any $f \in L^2(J)$,

$$G^0 f \in \mathcal{F}_J, \ \ \mathcal{E}_J(G^0 f, v) = \int_J f v \, dx, \quad \forall v \in \mathcal{F}_J.$$

Therefore

$$E_x(\sigma_c \wedge \sigma_d) = G^0 1_J(x) < \infty, \quad x \in (c, d),$$

and

$$P_x(\sigma_c < \sigma_d) + P_x(\sigma_d < \sigma_c) = 1, \quad x \in (c, d).$$

In particular, the function $p_{c,d}(x) = P_x(\sigma_d < \sigma_c)$, $x \in \bar{I}$, is not only strictly positive but also strictly increasing in $x \in (c, d)$ because the sample path continuity and the strong Markov property of $\mathbf{M}$ implies

$$p_{c,d}(x) = p_{c,y}(x) p_{c,d}(y) < p_{c,d}(y), \quad c < x < y < d. \tag{2.6}$$

In the same way, we have, for $c' \le c < d \le d'$, $c', d' \in \bar{I}$, that

$$\begin{aligned} p_{c',d'}(x) &= p_{c,d}(x)p_{c',d'}(d) + (1 - p_{c,d}(x))p_{c',d'}(c) \\ &= (p_{c',d'}(d) - p_{c',d'}(c))p_{c,d}(x) + p_{c',d'}(c) \quad c \le x \le d. \end{aligned} \tag{2.7}$$

When $I = (a, b)$, we let

$$\mathrm{s}(x) \doteq p_{a,b}(x) \quad x \in \bar{I}.$$

Then s is strictly increasing and its property (2.3) follows from (2.7) with $c' = a$, $d' = b$. When $I = \mathbb{R}$, we put, for any $c < d$ such that $c \le x \le d$ and $c < 0$, $1 < d$,

$$\mathrm{s}(x) = \alpha p_{c,d}(x) + \beta,$$

and determines constants α, β by

$$\mathrm{s}(0) = 0, \quad \mathrm{s}(1) = 1.$$

Then, $\mathrm{s}(x)$ is independent of such a choice of (c, d) because, for any interval $(c', d') \supset (c, d)$, $p_{c,d}$ is a linear function of $p_{c',d'}$ on $[c, d]$ in view of (2.4). Further s satisfies (2.3) because $p_{c,d}(c) = 0$, $p_{c,d}(d) = 1$.

Finally, in order to show the absolute continuity of s, we take any finite interval $(c, d) \subset I$. It suffices to prove that the function $p(x) = p_{c,d}(x)$, $x \in I$, is absolutely continuous since s is a linear function of p on (c, d).

When $I = (a, b)$, $p(x)$ is known to be the 0-order equilibrium potential of $\{d\}$ with respect to the Dirichlet space

$$\mathcal{F}_{(c,b]} = \{u \in \mathcal{F} : u(x) = 0, \ \forall x \le c\},$$

and $p(x)$ is characterized by

$$p \in \mathcal{F}_{(c,b]}, \ p(d) = 1, \ \mathcal{E}(p, v) \ge 0, \quad \forall v \in \mathcal{F}_{(c,b]}, \ v(d) \ge 0. \tag{2.8}$$

In particular, p is absolutely continuous.

When $I = \mathbb{R}$, we consider the space

$$\mathcal{F}_{(c,\infty)} = \{u \in \mathcal{F} : u(x) = 0, \ \forall x \le c\}.$$

By virtue of (2.2), we see that the Dirichlet space $(\mathcal{F}_{(c,\infty)}, \mathcal{E})$ is transient and the function $p(x)$ is the associated 0-order equilibrium potential of $\{d\}$ characterized by

$$p \in \mathcal{F}_{(c,\infty),e}, \quad p(d) = 1, \quad \mathcal{E}(p, v) \ge 0, \quad \forall v \in \mathcal{F}_{(c,\infty),e}, \ v(d) \ge 0, \tag{2.9}$$

where $\mathcal{F}_{(c,\infty),e} (\subset H^1_e(\mathbb{R}))$ is the extended Dirichlet space of $\mathcal{F}_{(c,\infty)}$. Hence p is absolutely continuous. □

We call the function $\mathtt{s}$ in Lemma 2.1 the *scale function* associated with the regular Dirichlet subspace $(\mathcal{F}, \mathcal{E})$ of $(H^1(I), (1/2)\mathbf{D})$.

We continue to consider a finite open interval $J = (c, d) \subset I$ and the corresponding function $p(x) = p_{c,d}(x)$ as in the proof of Lemma 2.1. By virtue of (2.2), the space $(\mathcal{F}_J, \mathcal{E})$ admits the reproducing kernel $g^0(x, y),\ x, y \in J$ characterized by

$$g^0(\cdot, y) \in \mathcal{F}_J,\ \ \mathcal{E}(g^0(\cdot, y), v) = v(y), \quad \forall v \in \mathcal{F}_J.$$

Lemma 2.2. *There exists a constant $C > 0$, such that, for any $x, y \in J$,*

$$g^0(x, y) = \begin{cases} Cp(x)(1 - p(y)), & x \leq y; \\ C(1 - p(x))p(y), & x \geq y. \end{cases}$$

Proof. We consider the function

$$p_y^0(x) := P_x(\sigma_c \wedge \sigma_d > \sigma_y), \quad x, y \in J, \tag{2.10}$$

$p_y^0(\,\cdot\,)$ is the 0-order equilibrium potential of $\{y\}$ with respect to $(\mathcal{F}_J, \mathcal{E})$ characterized by

$$p_y^0 \in \mathcal{F}_J,\ \ p_y^0(y) = 1,\ \ \mathcal{E}(p_y^0, v) \geq 0, \quad \forall v \in \mathcal{F}_J,\ \ v(y) \geq 0. \tag{2.11}$$

The above two characterizations lead us to

$$p_y^0(x) = \frac{g^0(x, y)}{g^0(y, y)} \quad x, y \in J.$$

On the other hand, we have $p_y^0(x) = p_{c,y}(x),\ c < x \leq y$, and we get from (2.7)

$$p_y^0(x) = \begin{cases} \dfrac{p(x)}{p(y)}, & x \leq y \\ \dfrac{1 - p(x)}{1 - p(y)}, & x \geq y, \end{cases}$$

for $x, y \in J$. The desired expression of $g^0(x, y)$ follows from the above two identities. □

Lemma 2.3. *Any function in $\mathcal{F}$ is absolutely continuous with respect to $d\mathtt{s}$.*

Proof. For any finite interval $J = (c, d) \subset I$, let G^0 be the 0-order potential operator associated with $(\mathcal{F}_J, \mathcal{E})$ as was considered in the proof of Lemma 2.1. Then it follows from Lemma 2.2 that, for $f \in L^2(J),\ x \in J$,

$$G^0 f(x) = \int_J g^0(x, y) f(y)\, dy$$

$$= C(1-p(x))\int_c^x p(y)f(y)\,dy + Cp(x)\int_x^d (1-p(y))f(y)\,dy$$
$$= Cp(x)\int_c^d (1-p(y))f(y)\,dy - C\int_c^x\int_c^y f(z)\,dz\,dp(y),$$

which means that $G^0 f$ is absolutely continuous with respect to p, namely, it can be expressed as $\int_c^x \varphi(y)p'(y)\,dy$ by some function $\varphi \in L^1(J;dp)$.

Since $G^0(L^2(J))$ is dense in $\mathcal{F}_J$, there exist, for any $u \in \mathcal{F}_J$, $f_n \in L^2(J)$ such that $u_n = G^0 f_n = \int_0^{\cdot} \varphi_n(y)p'(y)\,dy$ is $\mathcal{E}$ convergent to u. Hence, for any $B \subset J$ on which $p'(x) = 0$ a.e.,

$$\int_B u'(x)^2\,dx = \int_B (u'(x) - \varphi_n(x)p'(x))^2\,dx \le \mathcal{E}(u-u_n, u-u_n) \to 0, \quad n \to \infty,$$

which implies $u'(x) = 0$ a.e. on B, namely, u is absolutely continuous with respect to dp.

Finally, any $u \in \mathcal{F}$ can be expressed as

(2.12) $$u(x) = u(c) + (u(d)-u(c))p(x) + [(u(x)-u(c)) - (u(d)-u(c))p(x)], \quad x \in J,$$

the last term being a member of $\mathcal{F}_J$. Therefore u is absolutely continuous on J with respect to dp and hence with respect to $d\mathtt{s}$. □

Suppose that $(\mathcal{F}, \mathcal{E})$ is recurrent. Then, by [2, Theorem 4.6.6], the property (2.4) for $J = \bar{I}$ is strengthened to

(2.13) $$P_x(\sigma_y < \infty) = 1 \quad \forall x, y \in \bar{I}.$$

Note that, when $I = (a, b)$, $(\mathcal{F}, \mathcal{E})$ is automatically recurrent because, owing to the regularity, $\mathcal{F}$ contains a continuous function v greater than 1 on $[a, b]$ and hence the constant function $1 \wedge v$ as well.

For the scale function $\mathtt{s}$ associated with $(\mathcal{F}, \mathcal{E})$, we let

(2.14) $$E_{\mathtt{s}} = \left\{x \in I:\ \limsup_{h\to 0}\frac{\mathtt{s}(x+h)-\mathtt{s}(x)}{h} = 0\right\}.$$

Lemma 2.4. *Suppose $(\mathcal{F}, \mathcal{E})$ is recurrent.*
(i) *$\mathtt{s}'$ is constant a.e. on $I \setminus E_{\mathtt{s}}$.*
(ii) $\mathtt{s}(\pm\infty) = \pm\infty$.

Proof. (i) Again we fix an arbitrary interval $(c, d) \subset I$ and denote by $p(x)$, $x \in I$, the function $p_{c,d}(x)$ in the proof of Lemma 2.1. We know that p is absolutely continuous on I, strictly increasing on (c, d) and $p((c, d)) = (0, 1)$. Denote by q the inverse function of $p|_{(c,d)}$.

We have then

$$|p(A)| = \int_A p'(x)\,dx, \quad \text{for any Borel set } A \subset (c,d). \tag{2.15}$$

This is clear for a disjoint union of finite number of subintervals of (c,d), and the monotone class lemma (cf. [1]) then applies.

We next let

$$E = \left\{ x \in (c,d) \colon \limsup_{h \to 0} \frac{p(x+h) - p(x)}{h} = 0 \right\}$$

and $F = p(E)$. Then $|F| = 0$ by (2.15). (2.15) further means that, if $A \subset (c,d) \setminus E$ and $|A| > 0$, then $|p(A)| > 0$. Hence q is absolutely continuous on $(0,1) \setminus F$.

On the other hand, for any $\varphi \in C_0^{(1)}((0,1))$, $\varphi(p) \in \mathcal{F}_{(c,b]}$ (resp. $\mathcal{F}_{(c,\infty),e}$) when $I = (a,b)$ (resp. $\mathbb{R}$.) Further $\varphi(p(x)) = 0$, $x \geq d$, $x \in I$, because $p(x) = 1$, $x \geq d$, $x \in I$, on account of (2.13) and (2.5) . Hence, in view of (2.8) and (2.9),

$$\int_0^1 p'(q(x))\varphi'(x)\,dx = \int_c^d p'(x)\varphi'(p(x))p'(x)\,dx = 2\mathcal{E}(p, \varphi(p)) = 0.$$

It follows that $p'(q(x))$ is constant a.e. on $(0,1)$. Therefore p' is constant a.e. on $(c,d) \setminus E$. Since $\mathtt{s}$ is a linear function of p on (c,d), $\mathtt{s}'$ is constant a.e. on $(c,d) \setminus E_{\mathtt{s}}$ as was to be proved.

(ii) Since the recurrence assumption implies the conservativeness of the process $\mathbf{M}$ ([2]), it is easy to see that

$$P_x\left(\lim_{y \to \pm\infty} \sigma_y = \infty \right) = 1 \quad x \in \bar{I},$$

and we can get $\mathtt{s}(-\infty) = -\infty$ by noting (2.11) and letting $c \to -\infty$ in (2.3). Similarly we get $\mathtt{s}(\infty) = \infty$. □

We are now in a position to state a main theorem of this paper. Let $\mathbf{S}$ be the class of functions $\mathtt{s}$ defined by (1.4) and $\hat{\mathbf{S}}$ be its subclass defined by (1.5). For $\mathtt{s} \in \mathbf{S}$, we introduce the space $(\mathcal{F}^{(\mathtt{s})}, \mathcal{E}^{(\mathtt{s})})$ by (1.2) and (1.3).

Theorem 2.1. (i) *For any $\mathtt{s} \in \mathbf{S}$, the space $(\mathcal{F}^{(\mathtt{s})}, \mathcal{E}^{(\mathtt{s})})$ is a regular Dirichlet subspace of $(H^1(I), (1/2)\mathbf{D})$. The scale function associated with $(\mathcal{F}^{(\mathtt{s})}, \mathcal{E}^{(\mathtt{s})})$ equals $\mathtt{s}$ up to a linear transform.*

(ii) *Let $(\mathcal{F}, \mathcal{E})$ be a regular recurrent Dirichlet subspace of $(H^1(I), (1/2)\mathbf{D})$ and $\mathtt{s}$ be the associated scale function. Then, by making a linear modification of $\mathtt{s}$ if necessary, $\mathtt{s}$ belongs to the class $\hat{\mathbf{S}}$ and*

$$\mathcal{F} = \mathcal{F}^{(\mathtt{s})}, \quad \mathcal{E}(u,v) = \mathcal{E}^{(\mathtt{s})}(u,v), \quad u, v \in \mathcal{F}.$$

REMARK. The converse to (ii) (the recurrence of the space $(\mathcal{F}^{(\mathrm{s})}, \mathcal{E}^{(\mathrm{s})})$ for $\mathrm{s} \in \hat{\mathbf{S}}$) will be shown in the next section.

Proof. (i) Suppose $\mathrm{s} \in \mathbf{S}$ and $u, v \in \mathcal{F}^{(\mathrm{s})}$. Then u, v are absolutely continuous with respect to dx and

$$\begin{aligned} \frac{1}{2}\int_I \frac{du}{dx}\frac{dv}{dx}\,dx &= \frac{1}{2}\int_I \frac{du}{d\mathrm{s}}\frac{dv}{d\mathrm{s}}\mathrm{s}'(x)^2\,dx \\ &= \frac{1}{2}\int_I \frac{du}{d\mathrm{s}}\frac{dv}{d\mathrm{s}}\,d\mathrm{s}, \qquad u, v \in \mathcal{F}^{(\mathrm{s})}. \end{aligned} \tag{2.16}$$

Hence $\mathcal{F}^{(\mathrm{s})} \subset H^1(I)$ and $\mathcal{E}^{(\mathrm{s})}(u, v) = (1/2)\mathbf{D}(u, v)$, $u, v \in \mathcal{F}^{(\mathrm{s})}$.

Since $u(d) - u(c) = \int_c^d (du/d\mathrm{s})\,d\mathrm{s}$, we see that

$$|u(d) - u(c)|^2 \le 2|d - c|\mathcal{E}^{(\mathrm{s})}(u, u) \quad (c, d) \subset I, \;\; u \in \mathcal{F}^{(\mathrm{s})},$$

and any $\mathcal{E}_1^{(\mathrm{s})}$-Cauchy sequence is uniformly convergent on any compact interval of I. Hence $(\mathcal{F}^{(\mathrm{s})}, \mathcal{E}^{(\mathrm{s})})$ is a closed symmetric form on $L^2(I)$. Clearly it is Markovian.

The regularity is also verifiable. Indeed, when I is a finite interval, $\mathcal{F}^{(\mathrm{s})}$ contains s and constant functions and hence an algebra generated by them, which separates points of $\bar{I}$. Consequently $\mathcal{F}^{(\mathrm{s})}$ is dense in $C(\bar{I})$ by the Weierstrass theorem. Since the above inequality implies that $\mathcal{F}^{(\mathrm{s})} \subset C(\bar{I})$, we see that $(\mathcal{F}^{(\mathrm{s})}, \mathcal{E}^{(\mathrm{s})})$ is a regular Dirichlet form on $L^2(\bar{I})$.

When $I = \mathbb{R}$, we consider the space

$$\mathcal{C} = \{\varphi(\mathrm{s}) : \varphi \in C_0^1(\mathbb{R})\}.$$

Then $\mathcal{C} \subset \mathcal{F}^{(\mathrm{s})}$. Since $\mathcal{C}$ is an algebra separating points of I, it is dense in $C_0(\mathbb{R})$. Suppose $u \in \mathcal{F}^{(\mathrm{s})}$ is $\mathcal{E}_1$-orthogonal to $\mathcal{C}$: $\mathcal{E}_1(u, v) = 0$ $\forall v \in \mathcal{C}$. Then u is a solution of the equation

$$\frac{1}{2}\frac{d}{dx}\frac{du}{d\mathrm{s}} = u.$$

It is known that the solutions of this equation form a 2-dimensional vector space spanned by a positive increasing function $u^{(1)}$ and a positive decreaing function $u^{(2)}$ ([6]). Obviously, neither $u^{(1)}$ nor $u^{(2)}$ is in $L^2(\mathbb{R})$ and u must vanish. Hence $\mathcal{C}$ is dense in $\mathcal{F}^{(\mathrm{s})}$. Therefore $(\mathcal{F}^{(\mathrm{s})}, \mathcal{E}^{(\mathrm{s})})$ is a regular Dirichlet subspace of $(H^1(I), (1/2)\mathbf{D})$.

In order to prove the second assertion in (i), we consider any finite interval $J = (c, d) \subset \mathbb{R}$, take any $d_1 \in J$ and put

$$r(x) = \left(\frac{\mathrm{s}(x) - \mathrm{s}(c)}{\mathrm{s}(d_1) - \mathrm{s}(c)}\right)^+ \wedge \left(\frac{\mathrm{s}(d) - \mathrm{s}(x)}{\mathrm{s}(d) - \mathrm{s}(d_1)}\right)^+, \qquad x \in \mathbb{R}.$$

We readily see that $r \in \mathcal{F}_J^{(\mathrm{s})}$, $r(d_1) = 1$ and, for any $v \in \mathcal{F}_J^{(\mathrm{s})}$,

$$\begin{aligned}\mathcal{E}^{(\mathrm{s})}(r, v) &= \frac{1}{2(\mathrm{s}(d_1) - \mathrm{s}(c))} \int_c^{d_1} \frac{dv}{d\mathrm{s}} d\mathrm{s} - \frac{1}{2(\mathrm{s}(d) - \mathrm{s}(d_1))} \int_{d_1}^{d} \frac{dv}{d\mathrm{s}} d\mathrm{s} \\ &= \frac{1}{2(\mathrm{s}(d_1) - \mathrm{s}(c))} v(d_1) + \frac{1}{2(\mathrm{s}(d) - \mathrm{s}(d_1))} v(d_1).\end{aligned}$$

Hence r satisfies the condition (2.11) for $(\mathcal{F}^{(\mathrm{s})}, \mathcal{E}^{(\mathrm{s})})$ and $r(x)$ coincides with the function $p_{d_1}^0(x)$ defined by (2.10) on J for the diffusion (X_t, P_x) associated with $(\mathcal{F}^{(\mathrm{s})}, \mathcal{E}^{(\mathrm{s})})$ and in particular

$$r(x) = P_x(\sigma_{d_1} < \sigma_c) \quad x \in (c, d_1).$$

Since

$$r(x) = \frac{\mathrm{s}(x) - \mathrm{s}(c)}{\mathrm{s}(d_1) - \mathrm{s}(c)} \quad c < x < d_1,$$

we have shown that s is a scale function for the space $(\mathcal{F}^{(\mathrm{s})}, \mathcal{E}^{(\mathrm{s})})$.

(ii) The scale function s associated with a given regular recurrent Dirichlet subspace $(\mathcal{F}, \mathcal{E})$ of $(H^1(I), (1/2)\mathbf{D})$ belongs to $\hat{\mathbf{S}}$ (after an appropriate linear transform) by virtue of Lemma 2.1 and Lemma 2.4. We further see from Lemma 2.3 and identity (2.16) for $u, v \in \mathcal{F}$ that $\mathcal{F} \subset \mathcal{F}^{(\mathrm{s})}$ and $\mathcal{E}(u, v) = \mathcal{E}^{(\mathrm{s})}(u, v)$, $u, v \in \mathcal{F}$.

Take an interval $J = (c, d) \subset I$. Consider any function $u \in \mathcal{F}^{(\mathrm{s})}$ with $u(x) = 0$ for $x \notin J$ and assume that u is $\mathcal{E}^{(\mathrm{s})}$-orthogonal to the space $\mathcal{F}_J$:

$$\mathcal{E}^{(\mathrm{s})}(u, v) = 0, \quad \forall v \in \mathcal{F}_J.$$

By the function $p = p_{c,d}$ for $(\mathcal{F}, \mathcal{E})$ as in the proof of Lemma 2.1, we may write

$$\mathrm{s}(x) = c_0 p(x) + c_1, \;\; u(x) = \int_c^x \varphi(\xi)\, dp(\xi), \quad c \le x \le d.$$

Choosing as v the Green function $g^{0,y}(x) = g^0(x, y) \in \mathcal{F}_J$ of Lemma 2.2 for each fixed $y \in J$, we are led to

$$\begin{aligned}\mathcal{E}^{(\mathrm{s})}(u, g^{0,y}) &= \int_0^d \frac{du}{ds} \frac{dg^{0,y}}{ds} ds \\ &= Cc_0^{-1} \int_c^y \varphi(x)(1 - p(y))\, dp(x) - Cc_0^{-1} \int_y^d \varphi(x) p(y)\, dp(x) \\ &= Cc_0^{-1} \int_c^y \varphi(x)\, dp(x) - Cc_0^{-1} p(y) \int_c^d \varphi(x)\, dp(x) = Cc_0^{-1} u(y),\end{aligned}$$

and $u = 0$. Hence any function in $\mathcal{F}^{(\mathrm{s})}$ with compact support belongs to the space $\mathcal{F}$. Since we have seen in (i) that $(\mathcal{F}^{(\mathrm{s})}, \mathcal{E}^{(\mathrm{s})})$ is regular, we have the desired inclusion $\mathcal{F}^{(\mathrm{s})} \subset \mathcal{F}$. □

3. Constructions by time change and state space transform

If the scale function of the diffusion associated with the regular Dirichlet subspace is $\mathtt{s}$, then we know intuitively that, after a state space transformation $\mathtt{s}: I \to \mathtt{s}(I)$, the diffusion becomes another diffusion with hitting distributions identical with that of Brownian motion and that this new diffusion differs from the Brownian motion by a time change. This suggests a way of constructing the original diffusion and Dirichlet subspace from the Brownian motion and Sobolev space.

In this section, we construct a recurrent diffusion process $\tilde{X}$ associated with the Dirichlet form (1.2), (1.3) on $L^2(I)$ for $\mathtt{s} \in \hat{\mathbf{S}}$ from the reflecting Brownian motion on $\mathtt{s}(\bar{I})$ when I is finite and the Brownian motion on $\mathbb{R}$ when $I = \mathbb{R}$ by a time change and a transformation of the state space. We also notice that $\tilde{X}$ is the one-dimensional diffusion on $\bar{I}$ with infinitesimal generator $(1/2)(d/dx)(d/d\mathtt{s})$ in Feller's sense ([5]).

We prepare a lemma.

Lemma 3.1. *Let (E, m) be a σ-finite measure space, $X = (X_t, P_x)$ be an m-symmetric Markov process on E and $(\mathcal{F}, \mathcal{E})$ be the associated Dirichlet space on $L^2(E; m)$. Let γ be a one-to-one measurable transformation from E onto a space $\tilde{E}$ and $\tilde{m}$ be the image measure; $\tilde{m}(B) = m(\gamma^{-1}(B))$. We put*

$$\tilde{X}_t = \gamma(X_t), \quad \tilde{P}_x = P_{\gamma^{-1}x}, \quad x \in \tilde{E}.$$

Then $\tilde{X} = (\tilde{X}_t, \tilde{P}_x)$ is an $\tilde{m}$-symmetric Markov process on $\tilde{E}$ and the associated Dirichlet space $(\tilde{\mathcal{F}}, \tilde{\mathcal{E}})$ on $L^2(\tilde{E}, \tilde{m})$ satisfies

$$\tilde{\mathcal{F}} = \{u \in L^2(\tilde{E}; \tilde{m}) : u \circ \gamma \in \mathcal{F}\}$$
$$\tilde{\mathcal{E}}(u, v) = \mathcal{E}(u \circ \gamma, v \circ \gamma) \quad u, v \in \tilde{\mathcal{F}}.$$

Proof. It was proved in [1, p. 325] that $\tilde{X}$ is a Markov process on $\tilde{E}$ with transition function

$$\tilde{p}_t f(y) = p_t(f \circ \gamma)(\gamma^{-1}(y)) \quad y \in \tilde{E}, \ f \in \mathcal{B}^+,$$

where p_t is the transition function of X.

The $\tilde{m}$-symmetry of $\tilde{p}_t$ and the above relation of the Dirichlet spaces follow from

$$\begin{aligned} \int_{\tilde{E}} \tilde{p}_t f \cdot g d\tilde{m} &= \int_{\tilde{E}} p_t(f \circ \gamma)(\gamma^{-1}(y))(g \circ \gamma)(\gamma^{-1}(y))\, dm(\gamma^{-1} y) \\ &= \int_E p_t(f \circ \gamma)(g \circ \gamma)\, dm, \end{aligned}$$

and

$$\frac{1}{t} \int_{\tilde{E}} (f - \tilde{p}_t f) \cdot g\, d\tilde{m} = \frac{1}{t} \int_E (f \circ \gamma - p_t(f \circ \gamma)) g \circ \gamma\, dm.$$

That completes the proof. $\square$

The process $\tilde{X} = (\tilde{X}_t, \tilde{P}_x)_{x\in\tilde{E}}$ in the above lemma is called the process obtained from $X = (X_t, P_x)_{x\in E}$ by the transformation γ of the state space from E to $\tilde{E}$.

Take any $\mathtt{s}$ from the class $\hat{\mathbf{S}}$ defined by (1.5) and let $\mathtt{t}$ be its inverse function. Clearly

$$J = \mathtt{s}(I) = \begin{cases} (0, b - a - |E_{\mathtt{s}}|), & I = (a, b), \\ \mathbb{R}, & I = \mathbb{R}. \end{cases}$$

Let $(B_t, P_x)_{x\in\bar{J}}$ be the reflecting Brownian motion on $\bar{J}$ when $I = (a, b)$ and the Brownian motion on $\mathbb{R}$ when $I = \mathbb{R}$. It is associated with the regular local recurrent Dirichlet form $(\mathcal{H}^1(J), (1/2)\mathbf{D})$ on $L^2(\bar{J})$. The transition function of (B_t, P_x) is absolutely continuous with respect to dx. Each one point set has a positive 1-capacity with respect to this Dirichlet form. Hence the quasi-support of a positive Radon measure on $\bar{J}$ coincides with its topological support.

Let A_t be the PCAF (positive continuous additive functional) in the strict sense (B_t, P_x) with Revuz measure $d\mathtt{t}$. Since the support of $d\mathtt{t}$ is $\bar{J}$, the fine support of A_t is also $\bar{J}$ and A_t is strictly increasing in t a.s. Let τ_t be the inverse of A_t and denote by X the time change of B_t by (τ_t):

$$X_t = B_{\tau_t}. \tag{3.1}$$

Theorem 3.1. (i) *Let*

$$\tilde{X}_t = \mathtt{t}\left(B_{\tau_t}\right), \quad t \geq 0, \qquad \tilde{P}_x = P_{\mathtt{s}(x)}, \quad x \in \bar{I}. \tag{3.2}$$

Then $(\tilde{X}_t, \tilde{P}_x)_{x\in\bar{I}}$ is a diffusion process on $\bar{I}$ associated with the regular Dirichlet subspace $(\mathcal{F}^{(\mathtt{s})}, \mathcal{E}^{(\mathtt{s})})$ on $L^2(\bar{I})$ of $(H^1(I), (1/2)\mathbf{D})$.
(ii) *$(\mathcal{F}^{(\mathtt{s})}, \mathcal{E}^{(\mathtt{s})})$ is recurrent.*

Proof. (i) By virtue of (6.2.22) in [2], the time changed process $(X_t, P_x)_{x\in\bar{J}}$ is $d\mathtt{t}$-symmetric and its Dirichlet space $(\mathcal{F}^J, \mathcal{E}^J)$ on $L^2(\bar{J}; d\mathtt{t})$ is given by

$$\mathcal{F}^J = H^1_e(J) \cap L^2(\bar{J}, d\mathtt{t}), \ \ \mathcal{E}^J(u, v) = \frac{1}{2}\mathbf{D}(u, v), \quad u, v \in \mathcal{F}^J,$$

for the extended Dirichlet space $H^1_e(J)$ defined by (2.1) for $H^1(J)$.

Since $\tilde{X} = (\tilde{X}_t, \tilde{P}_x)$ is obtained from the time changed process (X_t, P_x) of (3.1) by means of the transformation $\mathtt{t}$ of the state space from $\bar{J}$ onto $\bar{I}$, we see by Lemma 3.1 that $\tilde{X}$ is symmetric with respect to the image measure by $\mathtt{t}$ of $d\mathtt{t}$, which is obviously the Lebesgue measure dx on $\bar{I}$, and the associated Dirichlet space $(\tilde{\mathcal{F}}, \tilde{\mathcal{E}})$ on $L^2(\bar{I}) =$

$L^2(I)$ is given by

$$\tilde{\mathcal{F}} = \{u \in L^2(I) : u \circ \mathtt{t} \in \mathcal{F}_J\} = \{u \in L^2(I) : u \circ \mathtt{t} \in H_e^1(J)\}, \tag{3.3}$$

$$\tilde{\mathcal{E}}(u, v) = \frac{1}{2}\mathbf{D}(u \circ \mathtt{t}, v \circ \mathtt{t}), \quad u, v \in \tilde{\mathcal{F}}. \tag{3.4}$$

We claim that

$$\tilde{\mathcal{F}} = \mathcal{F}^{(\mathtt{s})}, \ \tilde{\mathcal{E}}(u, v) = \mathcal{E}^{(\mathtt{s})}(u, v), \quad u, v \in \tilde{\mathcal{F}}. \tag{3.5}$$

By (3.3), $u \in \tilde{\mathcal{F}}$ if and only if $u \in L^2(I)$ and there exists a function $\phi \in L^2(J)$ such that

$$u(\mathtt{t}(x)) = \int_0^x \phi(y)\,dy + C, \quad x \in J,$$

for some constant C. In this case,

$$u(x) = \int_0^{\mathtt{s}(x)} \phi(y)\,dy + C = \int_a^x \phi(\mathtt{s}(y))\,d\mathtt{s}(y) + C, \ \ x \in I$$

and

$$\frac{1}{2}\int_I \left(\frac{du}{d\mathtt{s}}\right)^2 d\mathtt{s} = \frac{1}{2}\int_I \phi(\mathtt{s}(x))^2\,d\mathtt{s}(x) = \frac{1}{2}\int_J \phi(x)^2\,dx,$$

and hence $\tilde{\mathcal{F}} \subset \mathcal{F}^{(\mathtt{s})}$ and $\tilde{\mathcal{E}} = \mathcal{E}^{(\mathtt{s})}$ on $\tilde{\mathcal{F}} \times \tilde{\mathcal{F}}$. Converse inclusion can be shown in the same way.

(ii) We have only to show this for $I = \mathbb{R}$. By virtue of [2, (6.2.23)], the extended Dirichlet space of $(\mathcal{F}^{\mathbb{R}}, \mathcal{E}^{\mathbb{R}})$ coincides with $(H_e^1(\mathbb{R}), (1/2)\mathbf{D})$ and hence contains constant functions. Since the Dirichlet space $(\tilde{\mathcal{F}}, \tilde{\mathcal{E}})$ is obtained by (3.3) and (3.4), its extended Dirichlet space also contains constant functions. □

From the proof, it also follows that ,for $\mathtt{s} \in \hat{\mathbf{S}}$, $u \in \mathcal{F}^{(\mathtt{s})}$ if and only if $u \circ \mathtt{t} \in H^1(J)$. Equivalently $\mathcal{F}^{(\mathtt{s})} = \{u \circ \mathtt{s} : u \in H^1(J)\}$.

4. Some descriptions of the class S

We can give more tractable descriptions of the class $\mathbf{S}$ of scale functions defined by (1.4).

Let $\mathbf{T}$ be the totality of function $\mathtt{t}$ defined on some open interval $J \subset \mathbb{R}$ expressed as

$$\mathtt{t}(x) = c(x) + x, \quad x \in J, \tag{4.1}$$

for a non-decreasing singular continuous function $c(x)$ on J.

Let $\mathbf{E}$ be the totality of measurable subset E of I satisfying that, for any $x, y \in I$, $x < y$, $|(I \setminus E) \cap (x, y)| > 0$, i.e., the complement of E has a positive measure on any non-empty open subinterval. Two sets in $\mathbf{E}$ are regarded to be equivalent if they differ by a zero-measure set.

The following theorem illustrates the structure of $\mathbf{S}$ and shows that any regular recurrent Dirichlet subspace of $(H^1(I), \mathbf{D})$ may be obtained in the same way as done in the example in §1.

Theorem 4.1. *Let $\mathtt{s}$ be a strictly increasing function on I.*
(1) *$\mathtt{s} \in \mathbf{S}$ if and only if its inverse function belongs to $\mathbf{T}$.*
(2) *$\mathtt{s} \in \mathbf{S}$ if and only if there exists a set $E \in \mathbf{E}$ such that*

$$\mathtt{s}(x) = \int_\eta^x 1_{E^c}(y)\,dy, \quad x \in I, \tag{4.2}$$

where η denotes a when $I = (a, b)$ and 0 when $I = \mathbb{R}$. The set E is uniquely determined by $\mathtt{s}$ up to the equivalence.

Proof. (1) For $\mathtt{s} \in \mathbf{S}$, we let $\mathtt{t}(x) = \mathtt{s}^{-1}(x)$, $x \in J = \mathtt{s}(I)$. In view of the first part of the proof of Lemma 2.4 (i), we see that $\mathtt{t}'(x) = 1$ a.e. $x \in J$, and accordingly

$$\mathtt{t}(x) = c(x) + x, \quad x \in J,$$

for some nondecreasing singular continuous function $c(x)$. Hence $\mathtt{t} \in \mathbf{T}$.

Conversely if $\mathtt{t} \in \mathbf{T}$, then $\mathtt{t}(x) = c(x) + x$ is a strictly increasing continuous function with $\mathtt{t}' = 1$ a.e. on J. Further, for any $x, y \in J$, $x < y$, $(y - x) \le \mathtt{t}(y) - \mathtt{t}(x)$. It follows that $\mathtt{s}(x) = \mathtt{t}^{-1}(x)$, $x \in I = \mathtt{t}(J)$, is absolutely continuous. Clearly $\mathtt{s}$ is differentiable at $\mathtt{t}(x)$ if and only if $\mathtt{t}$ has a non-zero derivative at $x \in J$ and hence

$$\mathtt{s}'(\mathtt{t}(x)) = \frac{1}{\mathtt{t}'(x)} = 1, \quad \text{a.e. } x \in J,$$

which implies that $\mathtt{s}' = 1$ a.e. on $I \setminus E_{\mathtt{s}}$ in the same way as in the second part of the proof Lemma 2.4 (i).

As for (2), for any $\mathtt{s} \in \mathbf{S}$, $E_{\mathtt{s}} \in \mathbf{E}$ and conversely for $E \in \mathbf{E}$, it is easy to check that $\mathtt{s} \in \mathbf{S}$ as defined in (4.2). □

By this theorem, we can readily conceive functions in $\mathbf{S} \setminus \hat{\mathbf{S}}$ when $I = \mathbb{R}$. For example, for any non-decreasing singular continuous function $c(x)$ on $\mathbb{R}$ with $c(\pm\infty) = \pm\infty$, we put

$$\mathtt{t}(x) = c\left(\frac{x}{1 - |x|}\right) + x, \quad x \in (-1, 1) \tag{4.3}$$

and let $\mathtt{s}$ be the inverse function of $\mathtt{t}$.

Another example is provided by

$$\mathtt{s}(x) = \int_0^x 1_G(y)\,dy, \quad x \in \mathbb{R}, \quad \text{for } G = \bigcup_{r_n \in Q} \left(r_n - \frac{1}{2^{n+1}}, r_n + \frac{1}{2^{n+1}} \right), \tag{4.4}$$

where $Q = \{r_n\}$ is the set of all rational numbers.

In both cases, $\mathtt{s}(-\infty)$ and $\mathtt{s}(\infty)$ are finite and the corresponding spaces $(\mathcal{F}^{(\mathtt{s})}, \mathcal{E}^{(\mathtt{s})})$ are transient Dirichlet subspaces of $(H^1(\mathbb{R}), (1/2)\mathbf{D})$ by Theorem 2.1.

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Entrance Law, Exit System and Lévy System of Time Changed Processes

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Abstract

Let $(X, \widehat{X})$ be a pair of Borel standard processes on a Lusin space E that are in weak duality with respect to some σ-finite measure m that has full support on E. Let F be a finely closed subset of E. In this paper, we obtain the characterization of a Lévy system of the time changed process of X by a positive continuous additive functional (PCAF in abbreviation) of X having support F, under the assumption that every m-semipolar set of X is m-polar for X. The characterization of the Lévy system is in terms of Feller measures, which are intrinsic quantities for the part process of X killed upon leaving $E \setminus F$. Along the way, various relations between the entrance law, exit system, Feller measures and the distribution of the starting and ending point of excursions of X away from F are studied. We also show that the time changed process of X is a special standard process having a weak dual and that the μ-semipolar set of Y is μ-polar for Y, where μ is the Revuz measure for the PCAF used in the time change.

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Contents

1 Introduction

Given a Markov process X on a state space E and a subset F of E, we may associate the minimal process X^0 on $E_0 = E \setminus F$ and the time changed process Y on F; X^0 and Y are obtained from X by killing upon leaving E_0 and by the time substitute with the inverse of the local time on F respectively. There is yet another associated process that has attracted the interests of researchers for many years: the excursions of X away from the set F.

Intuitively, the joint distribution of the starting and ending points of excursions should contribute to the jumping measure of Y, as has been verified by explicit computations for the reflecting Brownian motion on a smooth Euclidean domain by P. Hsu [23]. In §5 of this paper, we shall show that this is indeed the case in a great generality. One can also naturally guess that the entrance law governing the excursions ought to be determined uniquely by the dual of the minimal process X^0. When F is just a one point set, this is confirmed recently by M. Fukushima and H. Tanaka [17] for a symmetric diffusion X. In §3 of the present paper, we shall establish this identification for a more general Markov process X and a general finely closed set F.

The stated results in §3 and §5 of the present paper will lead us in §5 to the characterization of the jumping and killing measures of Y by means of the Feller measures, which are the intrinsic quantities for X^0 and its dual process. The Feller measure was introduced by W. Feller [6] in his study of boundary theory for Markov chains. Such a characterization has been obtained previously by Y. LeJan [28] for a Hunt process associated with a non-symmetric Dirichlet form under some restrictive condition and quite recently by our joint paper [4] for the most general symmetric Markov processes. The following is a more detailed introduction of the present paper.

Let $X = \{X_t, \mathbf{P}_x\}$ be a standard process on a Lusin space E that has a weak dual standard process $\widehat{X} = \{\widehat{X}_t, \widehat{\mathbf{P}}_x\}$ with respect to a σ-finite measure m having full support on E. We assume for X that

(A.1) every m-semipolar set is m-polar.

Fix a subset F of E satisfying

(A.2) F is q.e. finely closed,

(A.3) $\mathbf{P}_x(\sigma_F < \infty) > 0$ for m_0-a.e. $x \in E_0$,

where $E_0 := E \setminus F$, $m_0 := m|_{E_0}$ and $\sigma_F := \inf\{t > 0 : X_t \in F\}$, the hitting time of F by X.

There are two important stochastic objects relevant to the set F: Maisonneuve's exit system for the homogeneous random set

$$M(\omega) := \{t \in [0, \zeta(\omega)) : X_t(\omega) \in F \text{ or } X_{t-}(\omega) \in F\} \cup \{\zeta(\omega)\},$$

and the trace process Y on F obtained from X by a time change with respect to a positive continuous additive functional (PCAF in abbreviation) having support F. The aim of the present paper is to describe some basic characteristics in these two objects in terms of the specific quantities related to the minimal processes X^0 and $\widehat{X}^0$, which are the subprocesses of X and $\widehat{X}$, respectively, killed upon leaving E_0.

By the specific quantities, we mean an X^0-entrance law $\{\mu_t^f,\ t \geq 0\}$ on E_0 that is characterized by

$$\widehat{\mathbf{H}}f \cdot m_0 = \int_0^\infty \mu_t^f dt \qquad \text{for every } f \in \mathcal{B}^+(F),$$

the Feller measure U and the supplementary Feller measure V that are defined by

$$U(f,g) := L^{(0)}(\widehat{\mathbf{H}}f \cdot m_0, \mathbf{H}g) \quad \text{and} \quad V(f) := L^{(0)}(\widehat{\mathbf{H}}f \cdot m_0, 1 - \mathbf{H}1) \qquad \text{for } f, g \in \mathcal{B}^+(F).$$

Here for $x \in E_0$,

$$\mathbf{H}f(x) := \mathbf{E}_x\left(f(X_{\sigma_F}); \sigma_F < \infty\right) \quad \text{and} \quad \widehat{\mathbf{H}}f(x) := \widehat{\mathbf{E}}_x\left(f(\widehat{X}_{\sigma_F}); \sigma_F < \infty\right),$$

and $L^{(0)}$ is the X^0-energy functional of an excessive measure and an excessive function for X^0. We emphasize that those quantities are well computable in many examples. The entrance law $\{\mu_t^f,\ t \geq 0\}$ can be expressed in terms of the joint distribution of the hitting time and hitting place of F of the dual process $\widehat{X}$. When X is the d-dimensional Brownian motion and F is a compact smooth hypersurface, concrete expressions of the Feller measures U and V are derived in [15, Example 2.1] and [4, Example 2.12], respectively.

The exit system $(\mathbf{P}_x^*, K + J)$ defined in §3 for the set $M(\omega)$ will describe the behaviors of the sample path $X_t(\omega)$ for t belonging to the time set $[0, \infty) \setminus M(\omega)$, which is a disjoint union of excursion intervals away from the set F. In particular, $\mathbf{P}_x^*$ for $x \in F$ may be considered as a σ-finite measure on the space of paths continuously entering from x so that

$$Q_t^*(x, B) = \mathbf{E}_x^*\left(I_B(X_t); t < \sigma_F\right)$$

is an X^0-entrance law governing the excursions. In Theorem 3.3, we shall establish an identity linking $Q_t^*(\cdot, B)$, $B \in \mathcal{B}(E_0)$, to the above mentioned X^0-entrance law $\mu_t^f(B)$.

Theorem 3.3 can be regarded as an extension of a part of a recent paper [17], where F is a one-point set. R. Getoor (through a private communication) has shown a similar formula to Theorem

3.3 for a general right process X and for any excessive measure m of it with F being a one-point set.

Theorem 3.3, which relates the X^0-entrance law μ_t^f to the exit system and the Lévy system of X, is a key of the present paper in the sense that all subsequent theorems will be deduced from it. In Theorem 3.4, the Feller measures U and V are represented as joint distributions of the starting and ending points of excursions in terms of the exit system. In Theorem 4.1, the Feller-Neveu measure $\Theta^{f,g}(du)$ is represented by a joint distribution of the starting point, ending point and the length of excursions. The Feller measure and the Feller-Neveu measure were first introduced by W. Feller [6] and J. Neveu [32], respectively, for a Markov process on a denumerable state space with a finite number of ideal boundary points. When F is just a one point set, $\Theta(du)$ is nothing but the Lévy measure of the inverse local time at the point ([17]).

As a consequence of Theorem 3.4, the Feller measures are identified in Corollary 3.5 with (generalized) Revuz measures of certain homogeneous random measures involving the starting and ending points of excursions. On the other hand, the latter quantities will be identified in Theorem 5.5 of §5 with the jumping and killing measures $\check{J}$, $\check{\kappa}$ of the time changed process Y on F obtained from X by a PCAF having support F. Combining Corollary 3.5 with Theorem 5.5, we shall get in Theorem 5.6 the following identifications

$$\check{J} = U + J|_{F\times F} \qquad \text{and} \qquad \check{\kappa} = V + \kappa|_F,$$

where J and κ are the jumping measure and killing measure of X, respectively. To be more precise, we shall consider in §5 the totality S_F of smooth measures whose quasi support coincide with F q.e.. We take an arbitrary but fixed $\mu \in S_F$ and consider the PCAF A having Revuz measure μ. Then the time changed process Y of X by the inverse of A can be seen to be a right process on F possessing a weak dual right process with respect to μ and still satisfying condition **(A.1)**. Therefore we can verify that both X and Y are special standard processes and admit their Lévy systems (under the original topologies) for quasi every starting points. The above mentioned jumping and killing measures of them are well defined in terms of their Lévy systems. In particular, the identifications in Theorem 5.6 holds independently of the choice of $\mu \in S_F$.

Theorem 5.6 is a generalization of the corresponding part of §2 of our recent joint paper [4], where a general irreducible m-symmetric Markov process X on a Lusin space E and a quasi closed subset F of E with positive capacity are considered. The method employed in [4] is to identify the jumping and killing measure appearing in the Beurling-Deny representation of the quasi-regular Dirichlet form on $L^2(F;\mu)$ of the time changed process Y by making full use of the stochastic calculus on martingale additive functionals of X in the Dirichlet form setting.

Under some extra condition, LeJan has obtained in §3 of [28] the same results as §2 of [4] for a Hunt process X associated with a non-symmetric sectorial regular Dirichlet form and for a closed set F. Along with [27], nice potential theoretic methods were systematically utilized in [28] under the condition that the Dirichlet space is continuously embedded into $L^2(E;m)$. This condition however excludes many interesting examples such as the reflecting Brownian motion on the unit disk while F is the unit circle. Our Theorem 5.6 also extends the corresponding part of §3 of [28] because the Hunt process associated with a non-symmetric sectorial Dirichlet form is known to satisfy the present condition **(A.1)** (cf. [27], [36], [9]).

When X is an m-symmetric conservative diffusion and F is a closed set, Corollary 3.5 can be readily obtained by a direct computation as was done in §3 of the paper by M. Fukushima, P. He and J. Ying [15]. The proof of Theorem 5.5 will be carried out by extending a time change argument in the proof of [15, Theorem 5.1]. In this sense, the present paper can be viewed as an extension of [15] and of a part of [17] methodologically.

Here we mention a close relevance of the present work to an article of M. Motoo [31], where the Lévy system $(\check{N}, \check{H})$ of the time changed process Y on a closed set F of a Hunt process X was studied under some strong analytic conditions on the resolvent of X^0. The excursions of X away from F were studied using jump times of the inverse local time and also a variant Ψ of the Feller kernel on F was introduced. It was then shown in [31] that

$$\int_0^t \check{N}f(Y_s)d\check{H}_s \geq \int_0^t \Psi f(Y_s)ds, \quad t > 0, \quad f \in \mathcal{B}^+(F),$$

with the equality holds if and only if X admits no jump from F to F. See M. Blumenthal [1] for an interpretation of Motoo's results by making use of an exit system.

There are many literatures on the subject of excursions and exit systems of Markov processes, [8], [19], [21], [23], [24], [25], [29] to name a few. For example, [21], [8] and [25] studied the excursion laws using the exit system, under the strong duality assumption (which assumes the existence of transition density functions) or classical duality assumption (which assumes he existence of potential kernels). In contrast, in the present paper we define Feller measures independently of the exit system, and then relate these measures to the joint distributions of the starting and ending points of the excursions defined via exit system.

In the next section, we collect some basic facts for standard processes in weak duality from articles [14], [22], [30], [12], [34] and develop them in a convenient way for later uses. Subsections §2.2, §2.3 and §2.4 contain some new results proven under the condition **(A.1)**. In §2.2, we show that the above mentioned homogeneous random set $M(\omega)$ is actually a closed subset of $[0, \infty)$. It will be shown in §2.3 that the quasi support of a smooth measure coincides with the support of the associated PCAF, extending the corresponding part of [16] for m-symmetric Markov processes. Subsection §2.4 deals with the existence of the associated Lévy system under the original topology.

Our study originated in the work of J. L. Doob [5], where the Douglas integral representation of the Dirichlet integral of a harmonic function on a unit disk was generalized to a general Green space with Martin boundary using Naim's kernel. In subsequent works by M. Fukushima [13] and H. Kunita [26], Naim's kernel was replaced by a Feller kernel (a density function of a Feller measure). In these works, what was given a priori is a minimal process X^0, and what were searched for were its possible extensions together with their intrinsic boundaries. In a series of papers [15], [4] and the present one, we assume instead that a process on E and a subset F of E are given in advance and the behaviors of X around F are investigated in relation to Feller measures defined by the absorbed process X^0 and its dual process.

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2 Markov processes in weak duality

Let E be a Lusin space (i.e., a space that is homeomorphic to a Borel subset of a compact metric space) and $\mathcal{B}(E)$ be the Borel σ-algebra on E. Let $X = (\Omega, \mathcal{M}, \mathcal{M}_t, X_t, \mathbf{P}_x, x \in E)$ be a standard process on E. Here, a standard process on the Lusin space E is a normal, right continuous strong Markov process which is quasi-left continuous on $(0, \zeta)$, where ζ is the lifetime of the process. The shift operators $\{\theta_t, t \geq 0\}$ satisfy $X_s \circ \theta_t = X_{s+t}$ identically for $s, t \geq 0$. Adjoined to the state space E is an extra point $\partial \notin E$; the process X retires to ∂ at its "lifetime"

$$\zeta := \inf\{t \geq 0 : X_t = \partial\}.$$

Denote $E \cup \{\partial\}$ by E_∂. The transition semigroup $\{P_t, t \geq 0\}$ of the process X is defined by

$$P_t f(x) := \mathbf{E}_x[f(X_t)] = \mathbf{E}_x[f(X_t); t < \zeta].$$

(Here and in the sequel, unless mentioned otherwise, we use the convention that a function defined on E takes the value 0 at the cemetery point ∂.)

A standard process is said to be *Borel* if $P_t f$ is Borel measurable for each bounded Borel function f. The family of all nearly Borel subsets of E with respect to the process X will be denoted by $\mathcal{B}^n(E)$. We shall start with a Borel standard process X. But a right process or a standard process obtained from X by some transformations such as restriction to an X-invariant set, a killing, a time change may no longer be Borel; however these processes have the following weaker measurability that

$$\text{the semigroup maps bounded nearly Borel functions into nearly Borel functions.} \tag{2.1}$$

Let m be a σ-finite measure on $(E, \mathcal{B}(E))$ with $\operatorname{supp}[m] = E$. Throughout this paper except for §2.1 **(vi)**,**(vii)** and §2.4, we assume that X is a Borel standard process and there is another Borel standard process

$$\widehat{X} = (\Omega, \widehat{\mathcal{M}}, \widehat{\mathcal{M}}_t, \widehat{X}_t, \widehat{\mathbf{P}}_x, x \in E)$$

that is in weak dual to X with respect to the measure m in the sense that

$$\int_E g(x)\widehat{P}_t f(x) m(dx) = \int_E f(x) P_t g(x) m(dx), \qquad f, g \in \mathcal{B}^+(E). \tag{2.2}$$

Here $\widehat{P}_t$ is the transition semigroup of $\widehat{X}$. For $\alpha \geq 0$, the α-resolvents of X and $\widehat{X}$ will be denoted as G_α and $\widehat{G}_\alpha$, respectively. The point ∂ will play the role of the cemetery for X and $\widehat{X}$. The quantities relative to $\widehat{X}$ will be denoted with a hat $\widehat{}$ and designated by the prefix co-.

We know (cf. [2, I,(9.15)]) that almost surely the left limit of X_t exists in E for $t < \zeta$. So without loss of generality, we assume that $X_t(\omega)$ has left limits in E for every $t \in (0, \zeta(\omega))$ for each $\omega \in \Omega$.

Under the weak duality assumption, the measure m is an excessive measure of X, that is, m is a σ-finite Borel measure on $\mathcal{B}(E)$ such that $mP_t \le m$ for all $t > 0$. Here mP_t denotes the measure μ defined by $\int_E f(x)\mu(dx) = \int_E P_t f(x) m(dx)$ for any Borel function $f \ge 0$ on E. Since X is a standard process, we have $\lim_{t\to 0} mP_t = m$ setwise.

2.1 Exceptional sets and fine topology

In this subsection, we list some known basic statements about exceptional sets and fine topology related to X and $\widehat{X}$ which have been presented in [14, §2] and in [22, §6].

The hitting time of $B \subset E_\partial$ is defined by $\sigma_B = \inf\{t > 0, X_t \in B\}$ with the convention that $\inf \emptyset = \infty$.

A subset $B \subset E$ is said to be *m-polar* if there exists a nearly Borel set $\widetilde{B}$ containing B such that

$$\mathbf{P}_m(\sigma_{\widetilde{B}} < \infty) = 0.$$

The set $\widetilde{B}$ above can in fact be chosen to be a Borel subset of E. It is known that any m-polar set is m-negligible. In the sequel, a statement is said to hold *quasi-everywhere* (*q.e.* in abbreviation) if it holds except on an m-polar set.

For two subsets B_1, B_2 of E, we write $B_1 \subset B_2$ q.e. if $B_1 \setminus B_2$ is m-polar. Thus $B_1 = B_2$ q.e. if their symmetric difference is m-polar, and in this case we call them *q.e. equivalent*. A subset of E is called *q.e. finely open* (respectively, *q.e. finely closed*) if it is q.e. equivalent to a nearly Borel finely open (respectively, closed) set. A subset of E is called *m-semipolar* if it is q.e. equivalent to a semipolar set. An m-semipolar set is therefore a union of a semipolar set and an m-polar set.

A function u defined q.e. on E is called *finely continuous q.e.* if there exists an m-polar set $N \in \mathcal{B}^n$ such that $E \setminus N$ is finely open and u is nearly Borel measurable and finely continuous on $E \setminus N$.

For $F \in \mathcal{B}^n$, we put for $f \in \mathcal{B}^+(E)$, $x \in E$,

$$P_t^0 f(x) = \mathbf{E}_x\left(f(X_t); t < \sigma_F\right) \qquad \text{for } t > 0,$$

$$\mathbf{H}f(x) = \mathbf{E}_x\left(f(X_{\sigma_F}); \sigma_F < \infty\right) \quad \text{and} \quad \mathbf{H}_\alpha f(x) = \mathbf{E}_x\left(e^{-\alpha\sigma_F} f(X_{\sigma_F})\right) \qquad \text{for } \alpha > 0.$$

In the following, we will use (u, v) to denote the inner product of u, v in $L^2(E, m)$, that is, $(u, v) := \int_E u(x)v(x)m(dx)$. For a set A, we use 1_A to denote its indicator function, that is,

$$1_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

The following facts are known.

(i) When F is Borel, the following hold for $f, g \in \mathcal{B}^+(E)$:

$$(\widehat{P}_t^0 f,\, g) = (f,\, P_t^0 g) \qquad \text{for } t > 0, \tag{2.3}$$

$$(\widehat{\mathbf{H}}_\alpha \widehat{G}_\alpha f,\, g) = (f,\, \mathbf{H}_\alpha G_\alpha g) \qquad \text{for every } \alpha > 0. \tag{2.4}$$

(ii) m-polarity and m-co-polarity are equivalent.

Clearly the identities (2.3) and (2.4) extend to $F \in \mathcal{B}^n$. We note here that, although $F \in \mathcal{B}^n$ is not necessarily co-nearly Borel measurable, there are Borel sets B_1, B_2 such that $B_1 \subset F \subset B_2$ and $B_2 \setminus B_1$ is m-polar and hence m-co-polar, so that the left hand sides of the above identities make sense.

(iii) If u is finely continuous q.e. on E and $u \geq 0$ m-a.e. on a finely open set $G \in \mathcal{B}^n$, then $u \geq 0$ q.e. on G.

A set $E_1 \in \mathcal{B}^n$ is called X-*invariant* if

$$\mathbf{P}_x\,(X_t \in E_1 \text{ for every } t \in [0,\zeta) \text{ and } X_{t-} \in E_1 \text{ for every } t \in (0,\zeta)) = 1 \quad \text{for every } x \in E_1.$$

The restriction of X to an X-invariant set E_1 is a standard process on E_1. We say that a set N is *properly exceptional* if $N \in \mathcal{B}^n$, $m(N) = 0$ and $E \setminus N$ is X-invariant. A properly exceptional set is m-*inessential* in the sense of [22] but the converse is not true.

(iv) A set N is m-polar if and only if N is contained in a properly exceptional set $\tilde{N}$. The set $\tilde{N}$ can be taken to be Borel.

A function u is q.e. finely continuous on E if and only if there exists a Borel properly exceptional set N such that u is Borel measurable and finely continuous on $E \setminus N$.

The assertions **(i)**, **(ii)** and **(iii)** were proved in [14]. The first assertion of **(iv)** was proved in [14] and in [22, (6.12)] with $\tilde{N}$ being taken to be an m-inessential set, but the proof of [16, Theorem 4.1.1] works to get the above stronger one by making use of [22, (15.7)].

(v) The following two conditions are equivalent:

$$\text{every } m\text{-semipolar set is } m\text{-polar.} \tag{2.5}$$

$$\text{A function is q.e. finely continuous if and only if it is q.e. co-finely continuous.} \tag{2.6}$$

In [14], condition (2.6) was proved to be equivalent to the condition that every semipolar set is m-polar, which is obviously equivalent to (2.5).

Finally, for later use in §5 for a time changed process, we add to our list two statements shown in [22] for a pair of right processes $(X,\, \widehat{X})$ in weak duality with respect to m (the standardness is not needed here).

(vi) For $B \in \mathcal{B}^n(E)$, B is m-semipolar if and only if

$$\mathbf{P}_m(X_t \in B \text{ for uncoutably many } t) = 0.$$

If B is m-semipolar, then

$$\mathbf{P}_x(X_t \in B \text{ for uncoutably many } t) = 0 \qquad \text{for q.e } x \in E.$$

The second assertion in the above is immediate from [2, (II.3.4)].

(vii) m-semipolarity and m-co-semipolarity are equivalent.

2.2 Closedness of a homogeneous random set

From now on, X and $\widehat{X}$ are two Borel standard processes X, $\widehat{X}$ in weak duality with respect to m with an additional assumption that for the process X,

(A.1) every m-semipolar set is m-polar.

Remark 2.1 Since m-polar is m-semipolar and m-co-polar is m-co-semipolar, by **(vii)**, the assumption **(A.1)** amounts to saying that m-polarity, m-semipolarity, m-co-polarity and m-co-semipolarity are all the same.

In this subsection, we formulate a lemma that is important in the next section. A point $x \in E$ is called *regular* for a set $F \in \mathcal{B}^n$ if $\mathbf{P}_x(\sigma_F = 0) = 1$. The set of all regular points for F is denoted by F^r.

We consider the random subset $M(\omega)$ of $[0, \zeta)$ defined by

$$M(\omega) := \{t \in [0, \zeta(\omega)) : X_t(\omega) \in F \text{ or } X_{t-}(\omega) \in F\} \cup \{\zeta(\omega)\}, \qquad \omega \in \Omega, \tag{2.7}$$

with the convention that $X_{0-}(\omega) = X_0(\omega)$.

Lemma 2.2 *Let $F \in \mathcal{B}^n$ be q.e. finely closed. Then the random set $M(\omega)$ defined by* (2.7) *is a closed subset of* $[0, \infty)$ $\mathbf{P}_x$*-a.s. for* q.e. $x \in E$.

Proof. Since $F^r \subset F$ q.e. by assumption and $F \setminus F^r$ is semipolar and hence m-polar by (A.1), we can choose a properly exceptional set N containing the symmetric difference of F and F^r by (vi). We then have for $F_1 = F \setminus N$

$$F_1^r = F_1 \qquad \text{and} \qquad \mathbf{P}_x(\sigma_F = \sigma_{F_1}) = 1 \quad \text{for } x \in E \setminus N.$$

Moreover, for $x \in E \setminus N$, $\mathbf{P}_x$-a.s.,

$$M(\omega) = M_1(\omega) := \{t \in [0, \zeta(\omega)) : X_t(\omega) \in F_1 \text{ or } X_{t-}(\omega) \in F_1\} \cup \{\zeta(\omega)\}.$$

In view of the right continuity of the 1-excessive function $f_1(x) = \mathbf{E}_x[e^{-\sigma_F}]$ along the path X_t, it is clear that the set $M(\omega)$ is righthand closed $\mathbf{P}_x$-a.s. for $x \in E \setminus N$.

To show the left hand side closedness, we proceed as follows. Under weak duality condition, it is shown in Mitro [30, §6] (which is an improved version of a theorem of Weil [38] that is proved originally under the strong duality assumption) that for every $x \in E$, $\mathbf{P}_x$-a.s.,

$$t \mapsto f(X_{t-}) \text{ is left-continuous for every co-finely continuous } f \text{ on } (0, \zeta).$$

Define $f_n(x) = \mathbf{E}_x\left[e^{-n\sigma_F}\right]$ for $x \in E$. Note that f_n is n-excessive for process X and hence it is finely continuous. According to the assumption **(A.1)** and **(v)** of §2.1, f_n is q.e. co-finely continuous on E. Thus, by **(iv)**, we have for q.e. $x \in E$, $\mathbf{P}_x$-a.s.,

$$t \mapsto f_n(X_{t-}) \text{ is left-continuous and } t \mapsto f_n(X_t) \text{ is right continuous having left limit.} \tag{2.8}$$

In particular, we have for q.e. $x \in E$, $\mathbf{P}_x$-a.s.

$$\lim_{s \uparrow t} f_n(X_s) = \lim_{s \uparrow t} f_n(X_{s-}) = f(X_{t-}) \qquad \text{for every } t \in (0, \zeta).$$

Since $f_n(x)$ decreases to $1_{F^r}(x) = 1_F(x)$ as $n \uparrow \infty$ for $x \in E \backslash N$, hence by (2.8) for q.e. $x \in E \backslash N$,$\mathbf{P}_x$-a.s. and for $t < \zeta$,

$$\begin{aligned}
\limsup_{s \uparrow t} 1_F(X_{s-}) &\leq \lim_{n \to \infty} \limsup_{s \uparrow t} f_n(X_{s-}) \\
&= \lim_{n \to \infty} f_n(X_{t-}) \\
&= 1_F(X_{t-})
\end{aligned}$$

and similarly,

$$\limsup_{s \uparrow t} 1_F(X_s) \leq 1_F(X_{t-}),$$

for q.e. $x \in E \setminus E$. Hence for q.e. $x \in E$, $\mathbf{P}_x$-a.s., if $t_n \in M(\omega)$ and $t_n \uparrow t$ with $t < \zeta$, then $X_{t-} \in F$ and so $t \in M(\omega)$. This shows that $M(\omega) \cap [0, \zeta(\omega))$ is a closed subset of $[0, \zeta(\omega))$ and so $M(\omega)$ is closed. □

2.3 Smooth measures and positive continuous additive functionals

We continue to consider a pair of Borel standard processes $(X, \widehat{X})$ that are in weak duality with respect to m under the assumption **(A.1)**.

Let μ be a Borel measure on E charging no m-polar set. A set $F \subset E$ is said to be a *quasi-support* of μ if the next two conditions are satisfied:

$$F \text{ is q.e. finely closed and } \mu(E \setminus F) = 0. \tag{2.9}$$

$$\text{if } \tilde{F} \text{ is another set with property (2.9), then } F \subset \tilde{F} \text{ q.e.} \tag{2.10}$$

The quasi support of μ is unique up to q.e. equivalence. Since a closed set is finely closed, the quasi-support of μ, if exists, is contained in the topological support of μ q.e. Analogously to [16, Theorem 4.6.2], we have the following criterion of the quasi support.

We denote by $\mathcal{D}$ the space of all non-negative functions u on E such that $u = f_1 - f_2$ for some bounded 1-excessive functions $f_1,\ f_2$ on E with respect to X.

Proposition 2.3 *The following conditions are equivalent to each other for any Borel measure μ on E charging no m-polar set and for any q.e. finely closed set $F \subset E$:*
(i) *F is a quasi support of μ.*
(ii) *For every function u in $\mathcal{D}$, $u = 0$ μ-a.e. on E if and only if $u = 0$ q.e. on F .*
(iii) *For any q.e. finely continuous function u, $u = 0$ μ-a.e. on E if and only if $u = 0$ q.e. on F.*

Proof. (i)$\to$(iii): Suppose condition (i) holds. Then "if" part of (iii) is trivially ture. Let u be a q.e. finely continuous function vanishes μ-a.e. on E. Then, $\widetilde{F} := \{x \in E : u(x) = 0\}$ is q.e finely closed having $\mu(\widetilde{F}) = 0$ and hence $F \subset \tilde{F}$ q.e.. Therefore $u = 0$ q.e. on F, proving the "only if" part of (iii).

(iii)$\to$(ii): This is true trivially.

(ii)$\to$(i): Suppose condition (ii) holds so that, Define

$$\mathcal{N}_\mu := \left\{ u \in \mathcal{D} : \int_E |u| d\mu = 0 \right\} \qquad \text{and} \qquad \mathcal{D}_{F^c} = \{u \in \mathcal{D} : u = 0 \text{ q.e. on } F\}.$$

Then the condition (ii) can be rephrased as $\mathcal{N}_\mu = \mathcal{D}_{F^c}$. Assume this holds and define

$$v_F(x) := \mathbf{E}_x \left[\int_0^{\sigma_F} e^{-s} 1_E(X_s) ds \right] = 1 - \mathbf{E}_x \left[e^{-\sigma_F \wedge \zeta} \right] \qquad \text{for } x \in E.$$

Note that $v_F(x) = G_1 1(x) - \mathbf{H}_1 G_1 1(x)$ is in $\mathcal{D}$ and $v_F = 0$ q.e. on F; in other words, $v_F \in \mathcal{D}_{F^c}$. So $v_F \in \mathcal{N}_\mu$ and hence $\int_E v_F(x)\mu(dx) = 0$.Since F^c is q.e. finely open, by the definition of v_F, $v_F > 0$ q.e. on F^c. Therefore $\mu(E \setminus F) = 0$. Consider another q.e. finely closed set F_1 with $\mu(E \setminus F_1) = 0$. Since $v_{F_1} = 0$ q.e. on F_1, v_{F_1} belongs to $\mathcal{N}_\mu = \mathcal{D}_{F^c}$. Since $v_{F_1} > 0$ q.e. on F_1^c, it follows that $F \subset F_1$ q.e. and hence F is a quasi support of μ, proving (i). □

Corollary 2.4 *Any q.e. finely closed set F admits a bounded Borel measure charging no m-polar set whose quasi support equals F q.e.*

Proof. Let F be a q.e. finely closed set. Take a strictly positive bounded m-integrable function f on E and put

$$\mu(B) = \int_E f(x)\mathbf{H}_1 1_B(x) m(dx) = \mathbf{E}_{f\cdot m}\left(e^{-\sigma_F} 1_B(X_{\sigma_F})\right), \qquad B \in \mathcal{B}(E).$$

Clearly μ is a finite measure charging no m-polar set with $\mu(E \setminus F) = 0$. If $u \in \mathcal{D}$ vanishes μ-a.e. then

$$(f, \mathbf{H}_1 u) = \int_E u(y)\mu(dy) = 0,$$

and so $\mathbf{H}_1 u = 0$ m-a.e. on E. Since $\mathbf{H}_1 u \in \mathcal{B}$ and finely continuous, $\mathbf{H}_1 u = 0$ q.e. on E by **(iii)**. In particular, $u = \mathbf{H}_1 u = 0$ q.e. on F. Hence μ satisfies condition (ii) of Corollary 2.3. □

For a Borel measure charging no m-polar set, its *co-quasi support* is well defined by replacing "q.e. finely closed" in (2.8) by "q.e. co-finely closed".

Corollary 2.5 *Let μ be a Borel measure on E charging no m-polar set. Then the quasi support of μ and the co-quasi support of μ is q.e. equivalent.*

Proof. This follows from **(v)** in §2.1 and Proposition 2.3 (iii). □

We call a functional $A = \{A_t(\omega),\ t \geq 0,\ \omega \in \Omega\}$, a *positive continuous additive functional* (PCAF in abbreviation) of X if the following conditions are satisfied:

(1) A is adapted to the minimum completed admissible filtration $\{\mathcal{F}_t\}$;

(2) there exist a set $\Lambda \in \mathcal{F}_\infty$ and a properly exceptional set $N \subset E$ such that

$$\mathbf{P}_x(\Lambda) = 1 \ \text{ for every } x \in E \setminus N, \qquad \text{and} \qquad \theta_t \Lambda \subset \Lambda \ \text{ for } t > 0,$$

and, moreover, for each $\omega \in \Omega$, $t \mapsto A_{\cdot}(\omega)$ is non-negative finite continuous on $[0, \zeta(\omega))$, $A_0(\omega) = 0$, $A_t(\omega) = A_{\zeta(\omega)}(\omega)$ for every $t \geq \zeta(\omega)$, and for every $t, s > 0$,

$$A_{t+s}(\omega) = A_s(\omega) + A_t(\theta_s \omega).$$

The sets Λ and N are called the *defining set* and the *exceptional set* of A, respectively. A PCAF for which $N = \emptyset$ is called a *PCAF in the strict sense.* In short, a PCAF is a PCAF in the strict sense of the restricted standard process $X|_{E \setminus N}$ for some properly exceptional set N. Two PCAF's A and $\widetilde{A}$ are regarded to be *equivalent* if $\mathbf{P}_x(A_t = \widetilde{A}_t) = 1$ for q.e. $x \in E$ for each $t > 0$. The latter is equivalent to

$$\mathbf{P}_x(A_t = \widetilde{A}_t \text{ for every } t \geq 0) = 1 \qquad \text{for q.e. } x \in E.$$

We call a positive measure μ on E the Revuz measure of a PCAF A of X with respect to the excessive measure m if

$$\int_E f(x)\mu(dx) = \uparrow \lim_{t \downarrow 0} \frac{1}{t} \mathbf{E}_m \left[\int_0^t f(X_s) dA_s \right] \tag{2.11}$$

for any Borel $f \geq 0$. Here $\uparrow \lim_{t\downarrow 0}$ means the quantity is increasing as $t \downarrow 0$. Obviously the Revuz measure is uniquely determined by the equivalent class of PCAFs and charges no m-polar set.

The family of all PCAF's of X is denoted by $\mathbf{A}_c^+$. For $A \in \mathbf{A}_c^+$, the associated α-potential $U_A^\alpha f$ is defined for $\alpha \geq 0$, $f \in \mathcal{B}(E)$ by

$$U_A^\alpha f(x) = \mathbf{E}_x \left[\int_0^\infty e^{-\alpha t} f(X_t) dA_t \right], \quad x \in E \setminus N, \tag{2.12}$$

where N is an exceptional set for A. The next fundamental *Revuz formula* valid under the present weak duality setting ([22, (9.3)]) will be utilized in §3 and §5:

$$(f, U_A^\alpha g) = \langle \widehat{G}_\alpha f, g \cdot \mu \rangle, \quad f, g \in \mathcal{B}^+(E), \tag{2.13}$$

where μ is the Revuz measure of A.

The *support* of $A \in \mathbf{A}_c^+$ is defined by

$$\check{F} = \{x \in E \setminus N : \mathbf{P}_x(\inf\{t > 0 : A_t(\omega) > 0\} = 0) = 1\}, \tag{2.14}$$

where N is an exceptional set for A. The support $\check{F}$ is nearly Borel measurable and finely closed with respect to the process $X|_{E \setminus N}$; moreover (cf. [2, p215])

$$A_t = \int_0^t 1_{\check{F}}(X_s) dA_s, \quad \mathbf{P}_x\text{-a.s. for } x \in E \setminus N. \tag{2.15}$$

The support $\check{F}$ is uniquely determined up to q.e. for equivalent PCAF's.

Proposition 2.6 *The support $\check{F}$ of $A \in \mathbf{A}_c^+$ is a quasi support of the Revuz measure μ_A of A.*

Proof. $\check{F}$ is q.e. finely closed as was noted above and $\mu_A(E \setminus \check{F}) = 0$ in view of (2.11) and (2.15). Therefore it suffices to verify the condition (iii) of Proposition 2.3: if u is q.e. finely continuous and vanishes μ_A-a.e. on E, then $u = 0$ q.e. on $\check{F}$. But this can be shown in exactly the same way as the proof of [16, Theorem 5.1.5]. □

When $X = \widehat{X}$ and the associated Dirichlet form on $L^2(E; m)$ is regular, the family (of equivalence classes of) $\mathbf{A}_c^+$ was shown in [16] to be in one to one correspondence with the class of smooth measures under (2.10). This has been extended by Fitzsimmons-Getoor [12] to a general right process with respect to its excessive measure m. A Borel measure μ on E is called *smooth* if it charges no m-polar set and there exists an increasing sequence of finely open sets $\{E_n\}$ such that $\mu(E_n) < \infty$ for all n and

$$\mathbf{P}_m(\lim_{n\to\infty} \tau_{E_n} < \zeta) = 0, \tag{2.16}$$

where $\tau_{E_n} := \inf\{t > 0 : X_t \notin E_n\}$. The class of all smooth measures is denoted by S. In [12], the smoothness of a measure μ is defined as above but with a stronger requirement that m charges no m-semipolar set, which is equivalent to the above, however, under the present assumption **(A.1)**.

The next proposition is a special case of [12, Theorem 3.11].

Proposition 2.7 *The equivalence classes of $\mathbf{A}_c^+$ and S are in one to one correspondence by the Revuz relation* (2.11).

By Proposition 2.6 and Proposition 2.7, we conclude that any smooth measure $\mu \in S$ admits a quasi support which equals the support of the corresponding PCAF A.

An increasing sequence of q.e. finely open subset $\{E_n, n \geq 1\}$ is called an X*-nest* if condition (2.16) is satisfied. An $\widehat{X}$-nest can be defined analogously. An X-nest $\{E_n, n \geq 1\}$ appearing in the above definition of the smooth measure μ will be said to be *associated with* μ.

Proposition 2.8 (i) $\{E_n, n \geq 1]$ *is an* X*-nest if and only if it is an* $\widehat{X}$*-nest,*

(ii) *A Borel measure* μ *on* E *is a smooth measure for* X *if and only if it is a smooth measure for* $\widehat{X}$.

(iii) *Let* $\mu \in S$ *and let* A, $\widehat{A}$ *be the PCAFs of* X, $\widehat{X}$, *respectively, having Revuz measure* μ. *Then for every* $\alpha > 0$, *there exists an* X*-nest* $\{E_n\}$ *associated with* μ *such that the* α*-potential and the co-*α*-potential* $U^\alpha_{A_n}1$, $\widehat{U}^\alpha_{\widehat{A}_n}1$ *of* $A_n := 1_{E_n} \cdot A$, $\widehat{A}_n := 1_{E_n} \cdot \widehat{A}$ *are bounded on* E *for each* n.

Here the PCAFs $(1_{E_n} \cdot A)_t := \int_0^t 1_{E_n}(X_s) dA_s$ *and* $(1_{E_n} \cdot \widehat{A})_t := \int_0^t 1_{E_n}(\widehat{X}_s) d\widehat{A}_s$.

Proof. (i) Under the condition **(A.1)**, it is known that a set A is q.e. finely open for X if and only if it is q.e. finely open for $\widehat{X}$. Take f and g two strictly positive bounded functions on E so that $f, g \in L^1(E, m)$. Then $\widehat{G}_1 f$ and $G_1 g$ are strictly positive on E q.e.. Let $\{E_n, n \geq 1]$ be an X-nest. Define $F_n := E \setminus E_n$. Then by (2.4),

$$\int_E g(x) \widehat{\mathbf{E}}_x \left[e^{-\widehat{\sigma}_{F_n}} \widehat{G}_1 f(\widehat{X}_{\widehat{\sigma}_{F_n}}) \right] m(dx) = \int_E f(x) \mathbf{E}_x \left[e^{-\sigma_{F_n}} G_1 g(X_{\sigma_{F_n}}) \right] m(dx) \qquad \text{for every } n \geq 1.$$

It follows from $\{E_n, n \geq 1\}$ being an X-nest that

$$\lim_{n \to \infty} \int_E g(x) \widehat{\mathbf{E}}_x \left[e^{-\widehat{\sigma}_{F_n}} \widehat{G}_1 f(\widehat{X}_{\widehat{\sigma}_{F_n}}) \right] m(dx) = 0.$$

Define $T := \lim_{n \to \infty} \widehat{\sigma}_{F_n}$. By the quasi-left continuity of $\widehat{X}$ on $[0, \widehat{\zeta})$, we have from the above display and the Fatou lemma by letting $n \to \infty$ that

$$\int_E g(x) \widehat{\mathbf{E}}_x \left[e^{-T} \widehat{G}_1 f(\widehat{X}_T); T < \widehat{\zeta} \right] m(dx) = 0$$

It follows that $\mathbf{P}_m(T < \widehat{\zeta}) = 0$ and so $\{E_n, n \geq 1\}$ is an $\widehat{X}$-nest. Interchange the role of X and $\widehat{X}$, we see that every $\widehat{X}$-nest is an X-nest.

(ii) follows immediately from (i) and the definition of smooth measure.

(iii) First assume that $\mu(E) < \infty$. Then by (2.13), $U^\alpha_A 1(x)$, $\widehat{U}^\alpha_{\widehat{A}} 1(x)$ are finite m-a.e. and hence q.e. on E. Define, for $n \geq 1$,

$$E_n := \{x \in E : U^\alpha_A 1(x) < n, \ \widehat{U}^\alpha_{\widehat{A}} 1(x) < n\}.$$

Clearly, $\{E_n, n \geq 1\}$ are q.e. finely open sets increasing to E q.e. and consequently is an X-nest. Furthermore, we have for each $n \geq 1$,

$$U^\alpha_{A_n} 1(x) = \mathbf{E}_x \left[e^{-\alpha \sigma_{E_n}} U^\alpha_{A_n} 1(X_{\sigma_{E_n}}) \right] \leq n \qquad \text{for q.e. } x \in E.$$

Similarly, $\widehat{U}^\alpha_{\widehat{A}_n} 1$ is dominated by n.

For a general smooth measure with an associated X-nest $\{G_\ell\}$, the measure $\mu_n = 1_{G_\ell} \cdot \mu$ is finite for each ℓ, and admits an X-nest $\{E_n^{(\ell)}\}$ possessing the above property for $A_\ell = 1_{G_\ell} \cdot A$, $\widehat{A}_\ell = 1_{G_\ell} \cdot \widehat{A}$. Then $E_n = \cup_{\ell=1}^n (G_\ell \cap E_n^{(\ell)})$, $n = 1, 2, \cdots$, is a desired X-nest associated with μ. □

2.4 Special standard process and Lévy system

We continue to consider a pair of standard processes X, $\widehat{X}$ in weak duality with respect to m under the assumption **(A.1)**. But in this subsection, we do not assume that X and $\widehat{X}$ are Borel and we allow their weaker measurability (2.1). We call X *μ-special standard* if, for any sequence of stopping times $T_n \uparrow T$, $\mathcal{F}_T^\mu = \sigma(\cup_{n=1}^\infty \mathcal{F}_{T_n}^\mu)$, where $\{\mathcal{F}_t, t \geq 0\}$ denotes the filtration generated by X. X is said to be *special standard* if it is μ-special standard for any initial law μ.

Lemma 2.9 *Under the assumption* **(A.1)**, *X is m-special standard. There exists a properly exceptional set N such that the restricted process $X|_{X\setminus N}$ is special standard. The same is true for $\widehat{X}$.*

Remark 2.10 This lemma is a consequence of Theorem 16.19 and Theorem 16.21 and remarks preceding them in [22]. We remark here that this lemma holds more generally for right processes X and $\widehat{X}$ possessing the left limits in E up to the life times which are in weak duality with respect to m and the first part of this lemma holds if every m-semipolar set is *m-co-polar* while the second part of this lemma holds if every m-co-semipolar set is *m-polar*. This remark will be applied in §5 to a time changed processes of X and $\widehat{X}$.

Any right process is known to admit a Lévy system under the Ray topology ([34, **(73.1)**]). But we like to have a Lévy system under the original topology.

Lemma 2.11 *Suppose that X is a special standard process on E with lifetime ζ. For this lemma only, we use X_{t-}^0 and X_{t-} to denote the left limit under the original topology on E and under the Ray topology, respectively. Then for any $x \in E$, $\mathbf{P}_x$-a.s. on $\{\zeta < \infty\}$, $X_{\zeta-}^0$ exists in E if and only if $X_{\zeta-} \in E$. In this case, these two limits are the same.*

Proof. It is known that there is an m-polar set N such that $E \setminus N$ is X-invariant and that $X\big|_{E\setminus N}$ is special standard process on $E \setminus N$. Hence without loss of generality, we may and do assume that $N = \emptyset$.

By [34, Remark **(46.3)**], we know that if $X_{\zeta-} \in E$ exists in the Ray topology, then $X_{\zeta-}^0$ exists in E and coincides with $X_{\zeta-}$.

Note that (see [34, **(47.6)**]) under the Ray topology, X is a Hunt process. By [34, Theorem **(46.2)**], the event $\{X_{t-}^0$ does not exit in $E\}$ is predictably meager. So in particular, we have a.s. on $\{X_{\zeta-}^0$ does not exit in $E\}$ that $X_{\zeta-} = X_\zeta = \partial$.

So it remains to show that $\mathbf{P}_x(A) = 0$, where

$$A := \{\zeta < \infty \text{ and } X_{\zeta-}^0 \text{ exists in } E \text{ but differs from } X_{\zeta-}\}.$$

Define

$$\zeta_p := \begin{cases} \zeta, & \text{if } X_{\zeta-} = \partial \text{ and } \zeta < \infty, \\ \infty, & \text{otherwise.} \end{cases}$$

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which is predictable by [34, **(44.5)**]. So there is an increasing sequence of stopping times $\{T_n, n \geq 1\}$ such that $T_n < \zeta_p$ and $\lim_{n\to\infty} T_n = \zeta_p$. According to [34, Theorem **(46.2)**], the event

$$\{X^0_{t-} \text{ exists in } E \text{ but differs from } X_{t-}\}$$

is also predictably meager. So $A \subset [[\zeta_p]]$. Define $h(x) := \mathbf{P}_x(A)$. Then h is excessive on E. Define

$$M_t = \mathbf{E}_x\left[1_A \middle| \mathcal{F}_t\right],$$

which is a bounded martingale. From the Markov property of X, we have $M_t = h(X_t)$ for $t < \zeta_p$. Since X is special standard, the filtration $\{\mathcal{F}_t, t \geq 0\}$ is quasi-left continuous by [34, **(47.6)**]. As ζ_p is predictable, M_t is continuous at $t = \zeta_p$. On the other hand, since X is a Hunt process under the Ray topology and h is finely continuous of X,

$$\lim_{n\to\infty} M_{T_n} = \lim_{n\to\infty} h(X_{T_n}) = h(X_{\zeta_p -}) = 0.$$

This implies that $h(x) = \mathbf{E}_x M_0 = 0$. So the lemma is established. □

By virtue of [34, **(73.1)**, **(47.10)**] and Lemma 2.11, we can conclude as follows:

Lemma 2.12 *Any special standard process X on E has a Lévy system (N, H). That is, $N(x, dy)$ is a kernel on $(E_\partial, \mathcal{B}(E_\partial))$ and H is a PCAF H of X in the strict sense with bounded 1-potential such that for any nonnegative Borel function f on $E \times E_\partial$ that vanishes on the diagonal and is extended to be zero elsewhere,*

$$\mathbf{E}_x\left(\sum_{s\leq t} f(X_{s-}, X_s)\right) = \mathbf{E}_x\left(\int_0^t \int_{E_\partial} f(X_s, y) N(X_s, dy) dH_s\right) \tag{2.17}$$

for every $x \in E$ and $t \geq 0$, where $X_{\zeta-}$ is defined by

$$X_{\zeta-} := \begin{cases} \lim_{t\uparrow\zeta} X_t, & \text{if the limit } \lim_{t\uparrow\zeta} X_t \text{ exists in } E, \\ \partial, & \text{otherwise.} \end{cases} \tag{2.18}$$

3 Entrance law, exit system and Feller measures

Let E, m be as in §2 and consider two Borel standard processes $X = (X_t, \zeta, \mathbf{P}_x)$ and $\widehat{X} = (\widehat{X}_t, \widehat{\zeta}, \widehat{\mathbf{P}}_x)$ on E in weak duality with respect to the measure m. We assume for X the condition **(A.1)** as in the last three subsections.

We fix a set $F \in \mathcal{B}^n$ and put $E_0 := E \setminus F$, $\quad m_0 := m|_{E_0}$, $\quad (u, v)_0 := \int_{E_0} u(x)v(x)m_0(dx)$.

We then assume that

(A.2) F is q.e. finely closed,

(A.3) $\mathbf{P}_x(\sigma_F < \infty) > 0$, for m_0-a.e. $x \in E_0$.

Let $X^0 = (X^0_t, \zeta^0, \mathbf{P}_x)_{x \in E^0}$ be the subprocess of X killed upon leaving F, that is, with $\zeta^0 = \zeta \wedge \sigma_F$,

$$X^0_t := \begin{cases} X_t & \text{if } t \in [0, \zeta^0), \\ \partial, & \text{if } t \geq \zeta^0, \end{cases}$$

where ∂ is an extra point added to E_0. Denote by P^0_t, G^0_α the transition function and the resolvent of X^0. In particular, $P^0_t f(x) = \mathbf{E}_x\left(f(X_t); t < \sigma_F\right)$, $x \in E_0$, $f \in \mathcal{B}(E_0)$, and we have from (2.3)

$$(\widehat{P}^0_t f, g)_0 = (f, P^0_t g)_0 \quad \text{and} \quad (\widehat{G}^0_\alpha f, g)_0 = (f, G^0_\alpha g)_0 \qquad \text{for } f, g \in \mathcal{B}^+(E_0). \tag{3.1}$$

The assumptions **(A.1)** and **(A.2)** imply that F and F^r are q.e. equivalent as was observed in the proof of Lemma 2.2.

The assumption **(A.3)** together with (3.1) implies as in [16, Lemma 1.6.5] that $G^0_{0+} f(x) < \infty$ for some strictly positive $f \in \mathcal{B}_b(E_0)$ m-a.e. on E_0 and hence q.e. on E_0 by **(iii)** in §2.1.

Furthermore, by Lemma 2.2, the random set $M(\omega)$ defined for F by (2.7) is a closed subset of $[0, \infty)$, and moreover, if $t_n \in M(\omega)$ increase to $t < \zeta$, then $X_{t-}(\omega) \in F$, $\mathbf{P}_x$-a.s. for q.e. $x \in E$.

Therefore, by **(iv)** in §2.1, there exists a properly exceptional set $N \subset E$ such that, denoting $E \setminus N$, $F \setminus N$, $E_0 \setminus N$ and the restriction $X|_{E \setminus N}$ by E, F, E and X, respectively, again, we can assume from the beginning that the following properties hold.

(I) $F = F^r$.

(II) X^0 is transient:

there is a strictly positive $f \in \mathcal{B}_b(E_0)$ on E_0 such that $G^0_{0+} f(x) < \infty$ for every $x \in E_0$.

(III) $M(\omega)$ is closed, and furthermore, if $t_n \in M(\omega)$ increase to $t < \zeta$, then $X_{t-}(\omega) \in F$, $\mathbf{P}_x$-a.s. for every $x \in E$.

As in §2, we consider, for $f \in \mathcal{B}^+(E)$, $\alpha > 0$ and $x \in E$,

$$\mathbf{H} f(x) = \mathbf{E}_x\left[f(X_{\sigma_F}); \sigma_F < \infty\right] \qquad \text{and} \qquad \mathbf{H}_\alpha f(x) = \mathbf{E}_x\left[e^{-\alpha\sigma_F}(X_{\sigma_F})\right],$$

together with the corresponding notations $\widehat{\mathbf{H}}$, $\widehat{\mathbf{H}}_\alpha$ to $\widehat{X}$.

Since

$$\widehat{P}^0_t \widehat{\mathbf{H}} f(x) = \widehat{\mathbf{E}}_x\left[f(\widehat{X}_{\sigma_F}); t < \sigma_F\right] \leq \widehat{\mathbf{H}} f(x), \quad x \in E_0, \tag{3.2}$$

we see from (3.1) that the measure $\widehat{\mathbf{H}} f \cdot m_0$ is X^0-excessive for any $f \in \mathcal{B}^+_b(E)$.

In accordance with [20, Prop. 3.6] (see also [15, Lemma 2.1]) , the *energy functional* $L^{(0)}(\eta, u)$ of an X^0-excessive measure η on E_0 and an X^0-excessive function u on E_0 (with respect to X^0) is well defined by

$$L^{(0)}(\eta, u) := \uparrow \lim_{t \downarrow 0} \frac{1}{t} \langle \eta - \eta P^0_t, u \rangle.$$

We can then define a bimeasure U on $E \times E$ by

$$U(f \otimes g) := L^{(0)}(\widehat{\mathbf{H}}f \cdot m_0, \mathbf{H}g) =\uparrow \lim_{t \downarrow 0} \frac{1}{t}(\widehat{\mathbf{H}}f, \mathbf{H}g - P_t^0 \mathbf{H}g) \qquad \text{for } f, g \in \mathcal{B}_b^+(E). \tag{3.3}$$

We call U the *Feller measure.* Actually U charges on $\widehat{F} \times F$, where $\widehat{F}$ denotes the co-fine closure of F. We also define the *supplementary Feller measure* V by

$$V(f) = L(\widehat{\mathbf{H}}f \cdot m_0, q) \qquad \text{for } f \in \mathcal{B}_b^+(E), \tag{3.4}$$

where $q(x) := 1 - \mathbf{H}1(x) = \mathbf{P}_x(\sigma_F = \infty)$. The measure V charges on $\widehat{F}$.

We call a family of σ-finite measures $\{\nu_t,\ T > 0\}$ on E_0 an X^0-*entrance law* if

$$\nu_t p_s^0 = \nu_{t+s} \qquad \text{for } t, s > 0.$$

The next lemma is an extension of Lemma 2.2 of [17].

Lemma 3.1 (i) *For any $f \in \mathcal{B}_b^+(E)$, there exists a unique X^0-entrance law μ_t^f such that*

$$\widehat{\mathbf{H}}f \cdot m_0 = \int_0^\infty \mu_t^f \, dt. \tag{3.5}$$

.

(ii) *Denote by $\mu_\alpha^{\#f}$ the Laplace transform of μ_t^f. Then, for any $v \in \mathcal{B}^+(E_0)$,*

$$\int_t^\infty \langle \mu_s^f, v \rangle ds = (\widehat{\mathbf{H}}f, P_t^0 v)_0 \qquad \textit{for every } t > 0, \tag{3.6}$$

and

$$\langle \mu_\alpha^{\#f}, v \rangle = (\widehat{\mathbf{H}}_\alpha f, v)_0. \tag{3.7}$$

(iii) *For any $f \in \mathcal{B}_b^+(E)$ and X^0-excessive function v on E_0, $\langle \mu_t^f, v \rangle$ is right continuous, decreasing in $t > 0$ and*

$$L^{(0)}(\widehat{\mathbf{H}}f \cdot m_0, v) =\uparrow \lim_{t \downarrow 0} \langle \mu_t^f, v \rangle.$$

In particular,

$$U(f, g) =\uparrow \lim_{t \downarrow 0} \langle \mu_t^f, \mathbf{H}g \rangle \qquad \textit{and} \qquad V(f) =\uparrow \lim_{t \downarrow 0} \langle \mu_t^f, q \rangle. \tag{3.8}$$

Proof. The proof of this lemma is analogous to the proof of Lemma 2.2 of [17]. But we spell out its proof for completeness.
(i). By (3.1) and (3.2),

$$\lim_{t \to \infty} \langle (\widehat{\mathbf{H}}f \cdot m_0) P_t^0,\, v \rangle = \lim_{t \to \infty} (\widehat{P}_t^0 \widehat{\mathbf{H}}f, v)_0 = 0 \qquad \text{for } v \in L^1(E_0, m_0),$$

namely $\widehat{\mathbf{H}}f \cdot m_0$ is purely excessive. Since X^0 is transient (see **(II)**), the assertion follows from [20, Th. 5.21] (see also [7]).

(ii) For $v \in L^1(E_0; m_0)$, we have

$$\int_t^\infty \langle \mu_t^f, v\rangle dt = \int_0^\infty \langle \mu_{t+s}^f, v\rangle dt = \int_0^\infty \langle \mu_s^f, P_t^0 v\rangle ds = (\widehat{\mathbf{H}}f \cdot m_0, P_t^0 v\rangle = (\widehat{\mathbf{H}}f, P_t^0 v)_0,$$

and

$$\langle \mu_t^f, v\rangle = -\frac{d}{dt}(\widehat{\mathbf{H}}f, P_t^0 v)_0, \qquad \text{a.e. } t.$$

Hence

$$\begin{aligned}\langle \mu_\alpha^{\#f}, v\rangle &= -\int_0^\infty e^{-\alpha t}\frac{d}{dt}(\widehat{\mathbf{H}}f, P_t^0 v)_0 dt \\ &= \Big[-e^{-\alpha t}(\widehat{\mathbf{H}}f, P_t^0 v)_0\Big]_0^\infty - \alpha\int_0^\infty e^{-\alpha t}(\widehat{\mathbf{H}}f, P_t^0 v)_0 dt \\ &= (\widehat{\mathbf{H}}f, v)_0 - \alpha(\widehat{\mathbf{H}}f, G_\alpha^0 v)_0\end{aligned}$$

$= (\widehat{\mathbf{H}}f - \alpha\widehat{G}_\alpha^0\widehat{\mathbf{H}}f, v)_0 = (\widehat{\mathbf{H}}_\alpha f, v)_0$.

(iii) If v is X^0-excessive, then $\langle \mu_{t+s}, v\rangle = \langle \mu_t, P_t^0 v\rangle \uparrow \langle \mu_t, v\rangle$ as $s \downarrow 0$.

For $v \in L^1(E_0; m_0)$, we get from (i) and (ii) that

$$\langle \widehat{\mathbf{H}}f \cdot m_0 - (\widehat{\mathbf{H}}f \cdot m_0)P_t^0, v\rangle = \int_0^t \langle \mu_s^f, v\rangle ds,$$

which extends to any X^0-excessive v and leads us to the desired identity. □

Recall the random set $M(\omega)$ defined by (2.7) for F. It is a closed subset of $[0,\infty)$ almost surely by **(III)** of §3. Let I denote the left endpoints for each components of the relatively open set $M(\omega)^c$ in $[0,\infty)$. Note that $\zeta(\omega) \in I(\omega)$ if $\zeta(\omega) < \infty$. $M(\omega)^c \cap [0,\zeta)$ consists of those "excursion intervals" away from F of the sample path $X(\omega)$. Clearly M is homogeneous on $(0,\infty)$, i.e., for every $s \geq 0$,

$$(M - s) \cap (0,\infty) = (M \circ \theta_s) \cap (0,\infty),$$

so is I.

Define

$$R(\omega) = \inf\{s > 0 : s \in M(\omega)\}.$$

Since X and $\widehat{X}$ is a pair of standard processes in weak duality, we have by [22, Proposition 15.7] that for q.e. $x \in E$, $\mathbf{P}_x$-almost surely

$$\begin{aligned}\sigma_F &:= \inf\{t > 0 : X_t \in F\} \\ &= \inf\{t > 0 : X_t \in F \text{ or } X_{t-} \in F\},\end{aligned} \tag{3.9}$$

and so

$$R = \sigma_F \wedge \zeta \qquad \mathbf{P}_x\text{-a.s. for q.e. } x \in E.$$

By enlarging the properly exceptional set $N \subset E$ appearing in the paragraph preceding **(I)-(III)**, we may assume that (3.9) holds $\mathbf{P}_x$-a.s. for every $x \in E$. This, combined with the property **(I)**, implies the identity

$$F = \{x \in E : \mathbf{P}_x(R = 0) = 1\}.$$

Let $(\mathbf{P}_x^*, K + J)$ be an *exit system* in the sense of Maisonneuve [29, (4.11)]. That is, K is a PCAF of X carried on F, $dJ_t = \sum_{s \in I : X_s \in E \setminus F} \varepsilon_s(dt)$, and $\mathbf{P}^*$ is a kernel from E to Ω such that

$$\mathbf{E}_x \left[\sum_{s \in I \cap [0,\zeta)} Z_s \cdot \Lambda \circ \theta_s \right] = \mathbf{E}_x \left[\int_0^\infty Z_s \cdot \mathbf{P}_{X_s}^*(\Lambda) d(K_s + J_s) \right], \quad x \in E, \tag{3.10}$$

for any positive predictable process Z_s and positive r.v. Λ. Here $\varepsilon_s(dt)$ is the unit atomic measure on $\mathbb{R}$ concentrated at the point $\{s\}$. Note that, for $x \in E_0$, $\mathbf{P}_x^*$ is defined to be $\mathbf{P}_x$.

Remark 3.2 In Maisonneuve [29], the exit system is constructed for a conservative strong Markov process X that satisfies the condition that its excessive functions are nearly Borel and right continuous along the sample paths of X. In our setting, the Borel standard process X may have finite lifetime. However, the results of Maisonneuve [29] are applicable here since we can regard the cemetery point ∂ as the absorbing state of X. □

Denote by $Q_t^*(x, \cdot)$, $x \in F$, the *entrance law with respect to* $\mathbf{P}_x^*$ defined by [29, (6.2)]:

$$Q_t^* g(x) := \mathbf{E}_x^* \left[g(X_t); t < R \right] \qquad \text{for } t > 0,\ x \in E,\ g \in \mathcal{B}^+(E). \tag{3.11}$$

Note that $Q_t^* g(x) = P_t^0 g(x)$ for $x \in E_0$.

On account of Lemma 2.9 and Lemma 2.12, X is an m-special standard process and has a Lévy system (N, H) in the following sense: $N(x, dy)$ is a kernel on $(E_\partial, \mathcal{B}(E_\partial))$ and H is a PCAF H of X with bounded 1-potential such that for any nonnegative Borel function f on $E \times E_\partial$ that vanishes on the diagonal and is extended to be zero elsewhere,

$$\mathbf{E}_x \left(\sum_{s \le t} f(X_{s-}, X_s) \right) = \mathbf{E}_x \left(\int_0^t \int_{E_\partial} f(X_s, y) N(X_s, dy) dH_s \right)$$

for q.e. $x \in E$ and $t \ge 0$, where $X_{\zeta-}$ is defined by (2.18). The Revuz measure of H with respect to the excessive measure m will be denoted as μ_H.

Theorem 3.3 *For any Borel subset $B \subset E_0$ and $f \in \mathcal{B}_b(F)$,*

$$\mu_t^f(B) = \int_F f(x) Q_t^*(x, B) \mu_K(dx) + \int_{F \times (E \setminus F)} f(x) P_t^0 1_B(y) N(x, dy) \mu_H(dx), \tag{3.12}$$

where μ_K is the Revuz measures of PCAF*'s K with respect to the measure m.*

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Proof. Put

$$\mathbf{Q}^*_\alpha g(x) = \int_0^\infty e^{-\alpha t} Q^*_t g(x) dt, \qquad g \in \mathcal{B}_b(E).$$

By virtue of the identity (3.8), we have for any $v \in \mathcal{B}_b(E)$ vanishing on F and for every $x \in E$,

$$\begin{aligned}
\mathbf{H}_\alpha G_\alpha v(x) &= G_\alpha v(x) - G^0_\alpha v(x) \\
&= \mathbf{E}_x \left(\int_R^\infty e^{-\alpha t} v(X_t) 1_{M^c}(t) dt \right) \\
&= \mathbf{E}_x \left(\sum_{s \in I} \int_s^{s+R\circ\theta_s} e^{-\alpha t} v(X_t) dt \right) \\
&= \mathbf{E}_x \left(\sum_{s \in I} e^{-\alpha s} \int_0^R e^{-\alpha t} v(X_t) dt \circ \theta_s \right) \\
&= \mathbf{E}_x \left(\int_0^\infty e^{-\alpha s} \mathbf{E}^*_{X_s} \left[\int_0^R e^{-\alpha t} v(X_t) dt \right] d(K_s + J_s) \right).
\end{aligned}$$

Therefore we have for any $f \in \mathcal{B}^+_b(E)$,

$$(f, \mathbf{H}_\alpha G_\alpha v) = \mathbf{E}_{f\cdot m} \left[\int_0^\infty e^{-\alpha s} \mathbf{Q}^*_\alpha v(X_s) d(K_s + J_s) \right]. \tag{3.13}$$

On the other hand, owing to the fundamental Revuz formula (2.13), we obtain

$$\mathbf{E}_{f\cdot m} \left[\int_0^\infty e^{-\alpha s} \mathbf{Q}^*_\alpha v(X_s) dK_s \right] = \langle \mathbf{Q}^*_\alpha v \cdot \mu_K, \widehat{G}_\alpha f \rangle. \tag{3.14}$$

Furthermore, since $\mathbf{Q}^*_\alpha(x, \cdot) = G^0_\alpha(x, \cdot)$ for $x \in E_0$, we have

$$\begin{aligned}
& \mathbf{E}_{f\cdot m} \left[\int_0^\infty e^{-\alpha s} \mathbf{Q}^*_\alpha v(X_s) dJ_s \right] \\
=\ & \mathbf{E}_{f\cdot m} \left[\sum_{s \in I, X_s \in E\setminus F} e^{-\alpha s} G^0_\alpha v(X_s) \right] \\
=\ & \mathbf{E}_{f\cdot m} \left[\sum_s e^{-\alpha s} 1_F(X_{s-}) 1_{E\setminus F}(X_s) G^0_\alpha v(X_s) \right] \\
=\ & \mathbf{E}_{f\cdot m} \left[\int_0^\infty e^{-\alpha s} 1_F(X_s) \int_{E\setminus F} N(X_s, dz) G^0_\alpha v(z) dH_s \right] \\
=\ & \langle 1_F \cdot \int_{E\setminus F} N(\cdot, dz) G^0_\alpha v(z) \cdot \mu_H,\ \widehat{G}_\alpha f \rangle.
\end{aligned} \tag{3.15}$$

Here in the last equality we used (2.13) again.

Applying the duality relation (2.4), we get from (3.13),(3.14) and (3.15)

$$(\widehat{\mathbf{H}}_\alpha \widehat{G}_\alpha f, v) = \langle \mathbf{Q}^*_\alpha v \cdot \mu_K + 1_F(\cdot) \int_{E\setminus F} N(\cdot, dz) G^0_\alpha v(z) \cdot \mu_H,\ \widehat{G}_\alpha f \rangle.$$

Since this identity holds for an arbitrary $f \in \mathcal{B}_b(E)$, we obtain for any $\beta > 0$, $f \in C_b^+(E) \cap \mathcal{F}$ and $v \in L^1(E_0, m)$,

$$(\widehat{\mathbf{H}}_\alpha \widehat{G}_\beta f, v) = \langle \mathbf{Q}_\alpha^* v \cdot \mu_K + 1_F(\cdot) \int_{E\setminus F} N(\cdot, dz) G_\alpha^0 v(z) \cdot \mu_H, \, \widehat{G}_\beta f \rangle. \tag{3.16}$$

Multiplying β on both side of (3.16) with $f = 1$ and then letting $\beta \to \infty$, we have by monotone convergence theorem,

$$\langle \mathbf{Q}_\alpha^* v \cdot \mu_K + 1_F \cdot \int_{E\setminus F} N(\cdot, dz) G_\alpha^0 v(z) \cdot \mu_H, \, 1 \rangle = \langle \widehat{\mathbf{H}}_\alpha 1, v \rangle \le \int_{E_0} v(x) m(dx) < \infty.$$

Now multiplying both side of (3.16) β and letting $\beta \to \infty$ in the above equation, we have by bounded convergence theorem,

$$(\widehat{\mathbf{H}}_\alpha f, \, v) = \langle \mathbf{Q}_\alpha^* v \cdot \mu_K + 1_F(\cdot) \int_{E\setminus F} N(\cdot, dz) G_\alpha^0 v(z) \cdot \mu_H, \, f \rangle.$$

This combined with (3.7) proves the desired identity (3.12) since the above display is nothing but the Laplace transform of (3.12). □

Theorem 3.3 allows us to make the connection between Feller measures U and V with the exit system $(\mathbf{P}_x^*, K + J)$.

Theorem 3.4 *The Feller measure U charges on $F \times F$ and*

$$U(dx, dy) = \mu_K(dx) \mathbf{P}_x^*(X_{\sigma_F} \in dy) + \mu_H(dx)\big|_F \int_{E\setminus F} N(x, dz) \mathbf{P}_z(X_{\sigma_F} \in dy). \tag{3.17}$$

The supplementary Feller measure V charges on F and

$$V(dx) = \mu_K(dx) \mathbf{P}_x^*(\sigma_F = \infty) + \mu_H(dx)\big|_F \int_{E\setminus F} N(x, dz) \mathbf{P}_z(\sigma_F = \infty). \tag{3.18}$$

Proof. It follows from (3.8), (3.12) and the definition of $Q_t^*(x, dy)$ that for any $f, g \in \mathcal{B}^+(E)$,

$$\begin{aligned}
&\int_{F\times F} f(x) g(y) U(dx, dy) \\
=\;& \lim_{t\downarrow 0} \langle \mu_t^f, \, \mathbf{H} g \rangle \\
=\;& \lim_{t\downarrow 0} \left[\int_{F\times(E\setminus F)} f(x) \mathbf{H} g(y) Q_t^*(x, dy) \mu_K(dx) + \int_{F\times(E\setminus F)} f(x) P_t^0 \mathbf{H} g(y) N(x, dy) \mu_H(dx) \right] \\
=\;& \lim_{t\downarrow 0} \int_F f(x) \mathbf{E}_x^* \left[\mathbf{H} g(X_t); t < R \right] \mu_K(dx) + \int_{F\times(E\setminus F)} f(x) \mathbf{H} g(y) N(x, dy) \mu_H(dx) \\
=\;& \lim_{t\downarrow 0} \int_F f(x) \mathbf{E}_x^* \left[g(X_{\sigma_F}) \circ \theta_t; t < R \right] \mu_K(dx) + \int_{F\times(E\setminus F)} f(x) \mathbf{H} g(y) N(x, dy) \mu_H(dx) \\
=\;& \lim_{t\downarrow 0} \int_F f(x) \mathbf{E}_x^* \left[g(X_{\sigma_F}); t < R \right] \mu_K(dx) + \int_{F\times(E\setminus F)} f(x) \mathbf{H} g(y) N(x, dy) \mu_H(dx) \\
=\;& \int_F f(x) \mathbf{E}_x^* \left[g(X_{\sigma_F}) \right] \mu_K(dx) + \int_{F\times(E\setminus F)} f(x) \mathbf{E}_y \left[g(X_{\sigma_F}) \right] N(x, dy) \mu_H(dx).
\end{aligned}$$

In the third to the last equality, we used the strong Markov property for the excursion measure $\mathbf{P}^*$ (see Theorem 5.1 of [29]). Identity (3.17) now follows.

The proof for (3.18) is similar to that for (3.17). For completeness, we spell out the details. Note that for $x \in E_0$, $q(x) = 1 - \mathbf{H}1(x) = \mathbf{P}_x(\sigma_F = \infty)$. Thus by (3.8), for any $f \in \mathcal{B}^+(E)$,

$$\begin{aligned}
& \int_F f(x)V(dx) \\
= \;& \lim_{t\downarrow 0}\langle \mu_t^f, q\rangle \\
= \;& \lim_{t\downarrow 0}\left[\int_{F\times(E\setminus F)} f(x)q(y)Q_t^*(x,dy)\mu_K(dx) + \int_{F\times(E\setminus F)} f(x)P_t^0 q(y)N(x,dy)\mu_H(dx)\right] \\
= \;& \lim_{t\downarrow 0}\int_F f(x)\mathbf{E}_x^*\left[q(X_t); t<R\right]\mu_K(dx) + \int_{F\times(E\setminus F)} f(x)q(y)N(x,dy)\mu_H(dx) \\
= \;& \lim_{t\downarrow 0}\int_F f(x)\mathbf{E}_x^*\left[g(X_{\sigma_F})\circ\theta_t; t<R\right]\mu_K(dx) + \int_{F\times(E\setminus F)} f(x)\mathbf{H}g(y)N(x,dy)\mu_H(dx) \\
= \;& \lim_{t\downarrow 0}\int_F f(x)\mathbf{E}_x^*\left[1_{\{\sigma_F=\infty\}}\circ\theta_t; t<R\right]\mu_K(dx) + \int_{F\times(E\setminus F)} f(x)q(y)N(x,dy)\mu_H(dx) \\
= \;& \int_F f(x)\mathbf{P}_x^*(\sigma_F=\infty)\mu_K(dx) + \int_{F\times(E\setminus F)} f(x)\mathbf{P}_y(\sigma_F=\infty)N(x,dy)\mu_H(dx).
\end{aligned}$$

This establishes (3.18). Since μ_K charges on F, we conclude from the expressions (3.17) and (3.18) that the measures U and V charge on $F\times F$ and F, respectively. □

The next corollary relates the Feller measures to the distributions of end places of excursions:

Corollary 3.5 *For every $\Psi \in \mathcal{B}^+(E\times E\setminus d)$ and $f\in\mathcal{B}^+(E)$ that are extended to be zero off $E\times E\setminus d$ and E, respectively, we have*

$$\int_{F\times F}\Psi(x,y)U(dx,dy) = \lim_{t\downarrow 0}\frac{1}{t}\mathbf{E}_m\left[\sum_{s\in I\cap[0,\zeta),\, s\le t}\Psi(X_{s-}, X_{\sigma_F}\circ\theta_s)\right] \tag{3.19}$$

and

$$\int_F f(\xi)V(d\xi) = \lim_{t\downarrow 0}\frac{1}{t}\mathbf{E}_m\left[\sum_{s\in I\cap[0,\zeta),\, s\le t} f(X_{s-})1_{\{\sigma_F=\infty\}}\circ\theta_s\right]. \tag{3.20}$$

Proof. Take any positive measurable functions f, g on E and put

$$Z_s = f(X_{s-})1_{(0,t]}(s) \qquad \Lambda = g(X_{\sigma_F}).$$

By the formula (3.10), we then have

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$$
\begin{aligned}
&\lim_{t\downarrow 0}\frac{1}{t}\mathbf{E}_m\left[\sum_{s\in I\cap[0,\zeta),\,s\le t} f(X_{s-})g(X_{\sigma_F}\circ\theta_s)\right]\\
=\;&\lim_{t\downarrow 0}\frac{1}{t}\mathbf{E}_m\left[\int_0^t f(X_{s-})\mathbf{E}^*_{X_s}(g(X_{\sigma_F}))dK_s\right]+\lim_{t\downarrow 0}\frac{1}{t}\mathbf{E}_m\left[\sum_{s\in I,X_s\in E\setminus F,s\le t} f(X_{s-})\mathbf{E}_{X_s}(g(X_{\sigma_F}))\right]\\
=\;&\int_E f(x)\mathbf{E}^*_x(g(X_{\sigma_F}))\mu_K(dx)+\lim_{t\downarrow 0}\frac{1}{t}\mathbf{E}_m\left[\sum_{s\le t}(1_Ff)(X_{s-})1_{E\setminus F}(X_s)\mathbf{H}g(X_s)\right]\\
=\;&\int_E f(x)\mathbf{E}^*_x(g(X_{\sigma_F}))\mu_K(dx)+\lim_{t\downarrow 0}\frac{1}{t}\mathbf{E}_m\left[\int_0^t (1_Ff)(X_s)\int_{E\setminus F}\mathbf{H}g(z)N(X_s,dz)dH_s\right]\\
=\;&\int_{F\times F} f(x)g(y)\mathbf{P}^*_x(X_{\sigma_F}\in dy)\mu_K(dx)+\int_{F\times(E\setminus F)} f(x)\mathbf{H}g(z)N(x,dz)\mu_H(dx),
\end{aligned}
$$

which, together with (3.17), leads us to (3.19) for $\Psi = f\otimes g$. A monotone class argument establishes (3.19) for general $\Psi \ge 0$ on $F\times F$.

Again by (3.10),

$$
\begin{aligned}
&\lim_{t\downarrow 0}\frac{1}{t}\mathbf{E}_m\left[\sum_{s\in I\cap[0,\zeta),\,s\le t} f(X_{s-})1_{\{\sigma_F=\infty\}}\circ\theta_s\right]\\
=\;&\lim_{t\downarrow 0}\frac{1}{t}\mathbf{E}_m\left[\int_0^t f(X_{s-})\mathbf{P}^*_{X_s}(\sigma_F=\infty)dK_s\right]+\lim_{t\downarrow 0}\frac{1}{t}\mathbf{E}_m\left[\sum_{s\in I,\,X_s\in E\setminus F,s\le t} f(X_{s-})\mathbf{P}_{X_s}(\sigma_F=\infty)\right]\\
=\;&\int_F f(x)\mathbf{P}^*_x(\sigma_F=\infty)\mu_K(dx)+\lim_{t\downarrow 0}\frac{1}{t}\mathbf{E}_m\left[\int_0^t (1_Ff)(X_s)\int_{E\setminus F}\mathbf{P}_y(\sigma_F=\infty)N(X_s,dy)dH_s\right]\\
=\;&\int_F f(x)\mathbf{P}^*_x(\sigma_F=\infty)\mu_K(dx)+\int_{F\times(E\setminus F)} f(x)\mathbf{P}_y(\sigma_F=\infty)N(x,dy)\mu_H(dy),
\end{aligned}
$$

which, together with (3.18), leads us to (3.20). □

Corollary 3.5 will be used in §5.

4 Feller-Neveu measure

This section is a continuation of the preceding one. For $\alpha > 0$, we define the *α-order Feller measure* U_α by

$$U_\alpha(f,g) = \alpha(\widehat{\mathbf{H}}_\alpha f,\,\mathbf{H}g),\qquad f,g\in\mathcal{B}_b^+(E). \tag{4.1}$$

We then have as in [15, §2]

$$\lim_{\alpha\to\infty} U_\alpha(f,g) = U(f,g). \tag{4.2}$$

The notions U_α and U go back to [6]. The next theorem is a consequence of Theorem 3.3.

Theorem 4.1 *For each fixed* $f, g \in \mathcal{B}_b^+(F)$*, define*

$$\Theta^{f,g}((s,t]) := \langle \mu_s^f, \mathbf{H}g - P_{t-s}^0 \mathbf{H}g \rangle, \quad 0 < s \le t. \tag{4.3}$$

This uniquely extends to a measure $\Theta^{f,g}$ *on* $(0,\infty)$ *defined by*

$$\begin{aligned} \Theta^{f,g}(du) &= \int_F f(x)\mathbf{E}_x^*(g(X_{\sigma_F}); \sigma_F \in du)\mu_K(dx) \\ &\quad + \int_{F\times(E\setminus F)} f(x)\mathbf{E}_y\left[g(X_{\sigma_F}); \sigma_F \in du\right] N(x,dy)\mu_H(dx). \end{aligned} \tag{4.4}$$

There exist Borel sets B_n *increasing to* E *such that* $\Theta^{f,1_{B_n}}(du)$ *is* σ*-finite on* $[0,\infty)$ *for each* n*. Furthermore,*

$$U_\alpha(f,g) = \int_0^\infty (1 - e^{-\alpha u})\Theta^{f,g}(du), \qquad \alpha > 0. \tag{4.5}$$

Remark 4.2 (i) The measure $\Theta^{f,g}$ governs the distribution of the excursion length.

(ii) When $m(E) < \infty$, we see that $\langle \mu_t^f, \mathbf{H}g \rangle$ is finite by (3.7), right continuous, decreasing in t by Lemma 3.1 (iii), and

$$\Theta^{f,g}((s,t]) = \langle \mu_s^f, \mathbf{H}g \rangle - \langle \mu_t^f, \mathbf{H}g \rangle, \qquad 0 < s < t,$$

which therefore extends to a unique σ-finite measure on $(0,\infty)$. In the following proof for the general case, we shall proceed differently. Such kind of measure that is related to $U_\alpha(f,g)$ via (4.5) first appeared in [32]. So we call it the *Feller-Neveu measure.*

(iii) Letting $\alpha \to \infty$ in (4.5), we obtain

$$U(f,g) = \Theta^{f,g}([0,\infty)),$$

which combined with (4.4) gives another proof of the first half of Theorem 3.4.

Proof. Clearly the measure $\Theta^{f,g}(du)$ given by (4.4) is well-defined. By making use of Theorem 3.3, we first show that $\Theta^{f,g}((s,t])$ defined by (4.3) is the charge of the measure $\Theta^{f,g}$ of (4.4) on the interval $(s,t]$;

$$\begin{aligned} \Theta^{f,g}((s,t]) &= \int_F f(x)\mathbf{E}_x^*\left[g(X_{\sigma_F}); s < \sigma_F \le t\right] \mu_K(dx) \\ &\quad + \int_{F\times(E\setminus F)} f(x)\mathbf{E}_y\left[g(X_{\sigma_F}); s < \sigma_F \le t\right] N(x,dy)\mu_H(dx), \end{aligned} \tag{4.6}$$

for $0 < s < t$.

Clearly from the definition (4.3),

$$\Theta^{f,g}((s,t]) = \langle \mu_s^f, \mathbf{E}.(g(X_{\sigma_F}); \sigma_F \le t - s) \rangle,$$

which combined with (3.12) leads us to

$$\begin{aligned}
\Theta^{f,g}((s,t]) &= \langle \mu_s^f, \mathbf{E}.[g(X_{\sigma_F}); \sigma_F \le t-s]\rangle \\
&= \int_F f(x)\mathbf{E}_y\left[g(X_{\sigma_F}); \sigma_F \le t-s\right] Q_s^*(x,dy)\mu_K(dx) \\
&\quad + \int_{F\times(E\setminus F)} f(x)\mathbf{E}_y^0\left[\mathbf{E}_{X_s}\left(g(X_{\sigma_F}); \sigma_F \le t-s\right)\right] N(x,dy)\mu_H(dx) \\
&= \int_F f(x)\mathbf{E}_x^*\left[\mathbf{E}_{X_s}\left[g(X_{\sigma_F}); \sigma_F \le t-s\right]; s < R\right]\mu_K(dx) \\
&\quad + \int_{F\times(E\setminus F)} f(x)\mathbf{E}_y\left[g(X_{\sigma_F}); s < \sigma_F \le t\right] N(x,dy)\mu_H(dx) \\
&= \int_F f(x)\mathbf{E}_x^*\left[g(X_{\sigma_F}); s < \sigma_F \le t\right]\mu_K(dx) \\
&\quad + \int_{F\times(E\setminus F)} f(x)\mathbf{E}_y\left[g(X_{\sigma_F}); s < \sigma_F \le t\right] N(x,dy)\mu_H(dx).
\end{aligned}$$

A comparison of the right hand sides of (3.12) and (4.6) gives

$$\Theta^{f,g}((s,t]) \le \langle \mu_s^f, g\rangle, \qquad 0 < s \le t.$$

Since μ_s^f is σ-finite, $\Theta^{f,g}$ extends to a unique measure on $[0,\infty)$ satisfying the equation (4.4) with a stated σ-finiteness property.

We finally prove the relation (4.5). By letting $t \to \infty$ in (4.3), we get from (3.2) the identity

$$\Theta^{f,g}((s,\infty)) := \langle \mu_s^f, \mathbf{H}g\rangle, \quad 0 < s.$$

Without loss of generality, we may assume that $\Theta^{f,g}(du)$ is σ-finite on $[0,\infty)$. Then by the Fubini theorem and (3.7),

$$\begin{aligned}
\int_0^\infty (1-e^{-\alpha u})\Theta^{f,g}(du) &= \alpha\int_0^\infty \left(\int_0^u e^{-\alpha s}ds\right)\Theta^{f,g}(du) \\
&= \alpha\int_0^\infty e^{-\alpha s}\left(\int_{s+}^\infty \Theta^{f,g}(du)\right)ds \\
&= \alpha\int_0^\infty e^{-\alpha s}\Theta^{f,g}(s,\infty)ds \\
&= \alpha\int_0^\infty e^{-\alpha s}\langle \mu_s^f, \mathbf{H}g\rangle ds \\
&= \alpha\langle \widehat{\mathbf{H}}_\alpha f, \mathbf{H}g\rangle = U_\alpha(f,g).
\end{aligned}$$

□ .

5 Lévy system of time change process and Feller measures

We continue to work with Borel standard processes $X = (X_t, \zeta, \mathbf{P}_x)$ and $\widehat{X} = (\widehat{X}_t, \widehat{\zeta}, \widehat{\mathbf{P}}_x)$ on E in weak duality with respect to m satisfying condition **(A.1)** and a set $F \in \mathcal{B}^n$ satisfying the conditions **(A.2)** and **(A.3)** formulated in §3.

Recall the family S of all smooth measures on E for X introduced in §2.3. Let

$$S_F = \{\mu \in S: \text{ the quasi support of } \mu = F \text{ q.e.}\}. \tag{5.1}$$

S_F is non-empty by Corollary 2.4. We take and fix a $\mu \in S_F$. There exists a PCAF A_t of X with Revuz measure μ by virtue of Proposition 2.7. The support of A_t coincides with F q.e. on account of Proposition 2.6.

In the same way as the proof of [16, Lemma 5.1.11], one can then show that

$$\mathbf{P}_x\left(\sigma_F = \inf\{t > 0 : A_t > 0\}\right) = 1, \qquad \text{q.e. } x \in E.$$

Hence, by restricting ourselves to outside a certain properly exceptional set including that for A, we can assume from the beginning that not only the properties **(I)**, **(II)** and **(III)** of §3 but also

(IV) A is a PCAF of X in the strict sense and

$$\mathbf{P}_x\left(\sigma_F = \inf\{t > 0 : A_t > 0\}\right) = 1 \qquad \text{for every } x \in E.$$

Note that **(I)** and **(IV)** imply that F is just the support of A.

We consider the right continuous inverse τ_t of A defined by

$$\tau_t = \inf\{s \geq 0 : A_s > t\} \qquad \text{with } \inf \emptyset := \infty,$$

and the time changed process $Y = (Y_t, \check{\zeta}, \mathbf{P}_x)_{x \in F}$ defined by

$$Y_t = \begin{cases} X_{\tau_t} & \text{for } 0 \leq t < \check{\zeta} := A_\infty; \\ \partial & \text{for } t \geq \check{\zeta}. \end{cases}$$

It is known (cf. [34, **(65.9)**]) that Y is a right process with state space F. We denote $F \cup \{\partial\}$ by F_∂.

Consider again the random set defined by (2.7):

$$M(\omega) = \{t \in [0, \zeta(\omega)) : X_t(\omega) \in F \text{ or } X_{t-}(\omega) \in F\} \cup \{\zeta(\omega)\},$$

which is closed by **(III)** in §3 almost surely. Let I denote the set of left endpoints of those components of the relatively open set $M(\omega)^c$ in $[0, \infty)$.

We define, for $t > \sigma_F$,

$$L(t) := \sup [0, t] \cap M$$

and, for $t \geq 0$,

$$R(t) := \inf(t,\infty) \cap M = \inf\{s > t : \ s \in M\} \tag{5.2}$$

with the convention that $\inf \emptyset = \infty$. When $t > \sigma_F$, we call $(L(t), R(t))$ the excursion straddling on t. Clearly $t \mapsto R(t)$ is right continuous and increasing. In view of (3.9), we can see that for every $x \in \mathbf{E}$, $\mathbf{P}_x$-a.s.,

$$R(t) = (\sigma_F \circ \theta_t + t) \wedge \zeta \qquad \text{for every } t \in [0, \zeta), \tag{5.3}$$

while $R(t) = \infty$ for $t \geq \zeta$. So $\mathbf{P}_x$-a.s.,

$$R(t) \circ \theta_s + s = R(t+s) \qquad \text{for every } t, s, \geq 0.$$

We also note that

(a) for $t \in M$, $t = R(t-)$, and

(b) for $t > \sigma_F$, $R(t-) < R(t)$ if and only if $t \in I$; and in this case $t = R(t-) = L(t)$.

On account of **(IV)**, for each fixed $t > 0$ and $x \in E$, $\mathbf{P}_x$-a.s.,

$$\tau_{A_t} = \inf\{s : \ A_s > A_t\} = \inf\{s > t : \ A_{s-t} \circ \theta_t > 0\} = \sigma_F \circ \theta_t + t,$$

and so, by (5.3), $R_t = \tau_{A_t} \wedge \zeta$ if $t < \zeta$ and $R_t = \tau_{A_t}$ if $t \geq \zeta$. By the right continuity of $t \mapsto \tau_{A_t}$ and $t \mapsto R(t)$, we conclude from above by first applying to rational $t > 0$ that for every $x \in E$, $\mathbf{P}_x$-a.s.,

$$R(t) = \begin{cases} \tau_{A_t} \wedge \zeta & \text{if } t < \zeta, \\ \tau_{A_t} & \text{if } t \geq \zeta, \end{cases} \qquad \text{for every } t \geq 0. \tag{5.4}$$

This means that $t \mapsto A_t$ is constant on each connected component of $M(\omega)^c$ and particularly on each excursion interval of sample paths leaving F. Since A_t is continuous in t and $Y_t \in F$, $0 \leq t < \check{\zeta}$, the next lemma follows immediately from (5.4) and property **(III).**

Lemma 5.1 *The following properties hold $\mathbf{P}_x$ -a.s. for every $x \in E$:*

(i) $Y_{A_t} = X_{R(t)}$ *for every* $t > 0$,

(ii) *the left limit* Y_{t-} *exists and in* F *for any* $t \in (0, \check{\zeta})$ *and* $Y_{A_t -} = X_{R(t-)-}$.

We now prove that the time changed process Y is actually μ-special standard. We prepare a lemma.

Lemma 5.2 *The right process Y has a right process $\widehat{Y}$ in weak duality with respect to the measure μ.*

Proof. On account of Remark 2.1, the process X also satisfies the condition **(A.1)**. Moreover, by Proposition 2.8, $\mu \in S_F$ is also a smooth measure for $\widehat{X}$. Hence a PCAF $\widehat{A}$ of $\widehat{X}$ exists with μ as Revuz measure by Proposition 2.7. Since the co-quasi support of μ is q.e. equivalent to F by Corollary 2.5, the support $\widehat{F}$ of $\widehat{A}$ with respect to $\widehat{X}$ equals F q.e. by virtue of Proposition 2.6 applied to $\widehat{X}$. Let $\widehat{Y}$ be the time changed process of $\widehat{X}$ with respect to $\widehat{A}$. $\widehat{Y}$ is a right process whose state space equals $\widehat{F}$. We shall show that Y and $\widehat{Y}$ are in weak duality with respect to the measure μ.

For $\alpha > 0$, we introduce the α-energy functional of α-co-excessive function u and α-excessive function v with respect to X by

$$L^{\alpha}(u, v) = \lim_{\beta \to \infty} \beta(u, v - \beta G_{\alpha+\beta} v),$$

which, by the weak duality, also equals

$$\lim_{\beta \to \infty} \beta(u - \beta \widehat{G}_{\alpha+\beta} u, v).$$

Recall the α-potential operator U_A^{α} associated with A defined by (2.12). The α-co-potential operator $\widehat{U}_{\widehat{A}}^{\alpha}$ associated with $\widehat{A}$ is defined analogously. Due to the fundamental Revuz formula (2.13) and the equation

$$U_A^{\alpha} g - U_A^{\alpha+\beta} g - \beta G_{\alpha+\beta} U_A^{\alpha} g = 0,$$

we have

$$L^{\alpha}(\widehat{U}_{\widehat{A}}^{\alpha} f, U_A^{\alpha} g) = \langle \widehat{U}_{\widehat{A}}^{\alpha} f, g \rangle_{\mu} \qquad \text{for } f, g \in \mathcal{B}^{+}(E), \tag{5.5}$$

where $\langle f, g \rangle_{\mu}$ denotes the integral $\int f(x) g(x) \mu(dx)$. Similarly we see that the left hand side of (5.5) is equal to $\langle f, U_A^{\alpha} g \rangle_{\mu}$ and consequently

$$\langle \widehat{U}_{\widehat{A}}^{\alpha} f, g \rangle_{\mu} = \langle f, U_A^{\alpha} g \rangle_{\mu} \qquad \text{for } f, g \in \mathcal{B}^{+}(F). \tag{5.6}$$

Next we put for $p > 0$

$$U_{p,A}^{\alpha} g(x) = \mathbf{E}_x \left[\int_0^{\infty} e^{-\alpha t - p A_t} g(X_t) dA_t \right],$$

and observe the equation

$$U_{p,A}^{\alpha} g - U_A^{\alpha} g + p U_A^{\alpha} U_{p,A}^{\alpha} g = 0.$$

An analogous quantity and equation can be associated with $\widehat{A}$.

For a moment, we assume that $\mu(F) < \infty$ and $U_A^{\alpha} 1$, $\widehat{U}_{\widehat{A}}^{\alpha} 1$ are bounded on E. Note that under this assumption, by linearity, identity (5.6) holds for any $f, g \in \mathcal{B}_b(F)$. Now taking $f, g \in \mathcal{B}_b^{+}(F)$ and replacing f and g in the equation (5.6) by

$$f - p \widehat{U}_{p,\hat{A}}^{\alpha} f \qquad \text{and} \qquad g - p U_{p,A}^{\alpha} g$$

respectively, we arrive at

$$\langle \widehat{U}_{p,\hat{A}}^{\alpha} f, g \rangle_{\mu} = \langle f, U_{p,A}^{\alpha} g \rangle_{\mu} \qquad \text{for } f, g \in \mathcal{B}_b^{+}(E). \tag{5.7}$$

Now for a general $\mu \in S_F$, we can take an associated X-nest $\{E_n\}$ satisfying condition (iii) of Proposition 2.8. We then let

$$\mu_n = 1_{E_n} \cdot \mu, \quad A_t^n = \int_0^t 1_{E_n}(X_s)dA_s, \quad \widehat{A}_t^n = \int_0^t 1_{E_n}(\widehat{X}_s)d\widehat{A}_s$$

to get the equation (5.7) holding for $\mu_n, A^n, \widehat{A}^n$. Replacing f, g in the resulting equation by $1_{E_\ell} \cdot f, 1_{E_\ell} \cdot g$, respectively, and letting first $n \to \infty$ and then $\ell \to \infty$, we see the validity of (5.7) for the general μ. Finally, by letting $\alpha \downarrow 0$, we obtain the duality relation of the p-resolvents of Y and $\widehat{Y}$ with respect to μ. □

It follows from Lemma 5.2 that the measure μ is Y-excessive. In the sequel, the μ-polar sets and μ-semipolar sets for process Y will be called μ^Y-polar and μ^Y-semipolar, respectively, in order to distinguish them from m-polar set and m-semipolar set for X.

Lemma 5.3 (i) *A subset $B \subset F$ is m-polar if and only if it is μ^Y-polar; a subset $B \subset F$ is m-co-polar if and only if it is μ^Y-co-polar.*

(ii) *Every μ^Y-semipolar set is μ^Y-polar, and every μ^Y-co-semipolar set is μ^Y-co-polar*

(iii) *The μ^Y-polarity, μ^Y-semipolarity, μ^Y-co-polarity and μ^Y-co-semipolarity are all the same.*

Proof. (i) This part in fact has been (implicitly) established in Fitzsimmons [10, Proposition 4]. Although the statement in [10] assumes that X is symmetric, its proof does not rely on the symmetry of X and uses only the property that X is a standard process and the condition that m-semipolar is m-polar. Fitzsimmons' argument uses Hunt's balayage theorem. For reader's convenience, we present below a slightly different proof, without using Hunt's deep result.

If $B \subset F$ is m-polar, by **(iv)** of §2.1, it is contained in a Borel properly X-exceptional set N. Since $E \setminus N$ is X-invariant and Y is a time-change of X living on F, $F \setminus N$ is Y-invariant. Since the smooth measure μ does not charge on m-polar set, we have $\mu(N) = 0$. Thus $N \cap F$ is a μ^Y-polar set by **(iv)** of §2.1, and so is B. This shows that any m-polar subset of F is a μ^Y-polar set.

Conversely, were there a nearly Borel (of X) set $B \subset F$ that is μ^Ypolar but not m-polar, there would exist a compact subset K of B such that K is not m-polar by [34, p57]. Define $\sigma_K := \inf\{t > 0 : X_t \in K\}$ and $f(x) := \mathbf{P}_x(\sigma_K < \infty)$. Then f is an excessive function of X.

In view of **(I)** of §3, (5.3) and (5.4), we then have, $\mathbf{P}_x$-a.s. on $\{\sigma_K < \infty\}$, for $x \in E$,

$$\sigma_K = R(\sigma_K) = \tau_{A_{\sigma_K}} \qquad \text{and} \qquad Y_{A_{\sigma_K}} = X_{\sigma_K} \in K.$$

So if we define $\sigma_K^Y := \inf\{t > 0 : Y_t \in K\}$, then

$$f(x) = \mathbf{P}_x(\sigma_K < \infty) = \mathbf{P}_x^Y(\sigma_K^Y < \infty), \quad x \in F.$$

Here, to distinguish it from $\mathbf{P}_x$ for the process X, $\mathbf{P}_x^Y$ denotes the law of the process Y starting from x. Since K is μ_Y-polar, $\int_F f(x)\mu(dx) = 0$. On the other hand, by the Revuz identity (2.11),

$$\uparrow \lim_{t \downarrow 0} \frac{1}{t} \mathbf{E}_m \left[\int_0^t f(X_t) dA_t \right] = \int_F f(x)\mu(dx) = 0,$$

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and consequently,

$$0 = \mathbf{E}_m \left[\int_0^\infty f(X_t) dA_t \right] \geq \mathbf{E}_m \left[\Lambda(\omega); \sigma_K < \infty \right], \tag{5.8}$$

where $\Lambda(\omega)$ is the random variable defined by

$$\Lambda(\omega) := \mathbf{E}_{X_{\sigma_K(\omega)}(\omega)} \left[\int_0^\infty f(X_s) dA_s \right].$$

Denote by K^r the set of all regular points of K for X. Then $X_{\sigma_K} \in K^r$ $\mathbf{P}_m$-a.s. on $\{\sigma_K < \infty\}$ because $K \setminus K^r$ is semipolar for process X and hence m-polar by **(A.1)** in §2.2. Since $f > 0$ on K^r, the set $\{f > 0\}$ is finely open, $K^r \subset F^r = F$ by **(I)** in §3 and F is the support of the PCAF A by **(IV)** in §5, we see that $\Lambda > 0$ $\mathbf{P}_m$-a.s. on $\{\sigma_K < \infty\}$. This contradicts to the inequality (5.8) and our assumption that K is not m-polar. Thus we have shown that every μ^Y-polar subset of F must be m-polar.

Applying the above argument to the dual processes $\widehat{X}$ and $\widehat{Y}$, we have every μ^Y-co-polar subset of F is m-co-polar. This completes the proof of (i).

(ii) Assume that $B \subset F$ is an μ^Y-semipolar set for process Y. Then by **(vi)** of §2.1,

$$\mathbf{P}_y(Y_t \in B \text{ for uncountably many } t) = 0$$

for Y-q.e. $y \in F$ and hence by (i) for q.e. $y \in F$. On the other hand, if we define the time sets

$$R(X) := \{t : X_t \in B\} \qquad \text{and} \qquad R(Y) := \{t : Y_t \in B\},$$

then it follows from [2, V(3.8)] that for $x \in E$, $\mathbf{P}_x$-a.s.,

$$R(Y) \subset R(X) \qquad \text{and} \qquad R(X) \setminus R(Y) \text{ is at most countable.}$$

Therefore, by the strong Markov property of X,

$$\begin{aligned} \mathbf{P}_m(X_t \in B \text{ for uncountably many } t) &= \mathbf{E}_m \left[\mathbf{P}_{X_{\sigma_F}}(X_t \in B \text{ for uncountably many } t) \right] \\ &= \mathbf{E}_m \left[\mathbf{P}_{X_{\sigma_F}}(Y_t \in B \text{ for uncountably many } t) \right] \\ &= 0. \end{aligned}$$

This implies that B is m-semipolar by virtue of **(vi)** of §2.1. So B is m-polar, and hence, by (i), is μ^Y-polar.

Applying the above argument to the dual process $\widehat{Y}$, we see that every μ^Y-co-semipolar set is μ^Y-co-polar.

(iii) By **(ii)** of §2.1, a set $B \subset F$ is m-polar if and only if it is m-co-polar. Thus we have that, by (i), a subset of F is μ^Y-polar if and only if it is μ^Y-co-polar. This together with (ii) establishes (iii) of this lemma. □

Proposition 5.4 *The process Y is a μ-special standard process on F having another μ-special standard process $\widehat{Y}$ on F in weak dual with respect to the measure μ. Moreover, the semigroup of Y map bounded nearly Borel measurable functions (with respect to Y) on F into bounded nearly Borel measurable functions (with respect to Y) on F, and the semigroup of $\widehat{Y}$ map bounded nearly Borel measurable functions (with respect to $\widehat{Y}$) on F into bounded nearly Borel measurable functions (with respect to $\widehat{Y}$) on F.*

Proof. On account of Remark 2.10, the first assertion follows from Lemma 5.1, Lemma 5.2 and Lemma 5.3(iii). The second assertion follows from the fact that Y and $\widehat{Y}$ are a time change of the Borel standard processes X and $\widehat{X}$, respectively. □

Combining Proposition 5.4 with Lemma 2.12, we can conclude that the time changed process Y on F admits a Lévy system $(\check{N}.\check{H})$. That is, $\check{N}(x,dy)$ is a kernel on $(F_\partial, \mathcal{B}(F_\partial))$ and $\check{H}$ is a PCAF of Y with bounded 1-potential such that for any nonnegative Borel function f on $F \times F_\partial$ that vanishes on the diagonal and is extended to be zero elsewhere,

$$\mathbf{E}_x\left(\sum_{s\le t} f(Y_{s-},Y_s)\right) = \mathbf{E}_x\left(\int_0^t \int_{F_\partial\setminus N} f(X_s,y)\check{N}(X_s,dy)d\check{H}_s\right) \tag{5.9}$$

for q.e. (or equivalently μ^Y-q.e.) $x \in F$ and $t \ge 0$. Here, $\check{\zeta}$ is the lifetime of Y and $Y_{\check{\zeta}-}$ is defined by

$$Y_{\check{\zeta}-} := \begin{cases} \lim_{t\uparrow\check{\zeta}} Y_t, & \text{if the limit } \lim_{t\uparrow\check{\zeta}} Y_t \text{ exists in } F, \\ \partial, & \text{otherwise.} \end{cases} \tag{5.10}$$

The Revuz measure of $\check{H}$ with respect to the Y-excessive measure μ will be denoted as $\check{\mu}_{\check{H}}$. Define

$$\check{J}(dx,dy) := \check{N}(x,dy)\check{\mu}_{\check{H}}(dx) \qquad \text{and} \qquad \check{\kappa} := \check{N}(x,\{\partial\})\check{\mu}_{\check{H}}(dx). \tag{5.11}$$

We call $\check{J}$ and $\check{\kappa}$ the *jumping measure* and the *killing measure* of Y, respectively.

By (5.9), we have then the formula for the jumping and killing measures:

$$\int_{F\times F\setminus d} \Psi(x,y)\check{J}(dx,dy) =\uparrow \lim_{t\downarrow 0}\frac{1}{t}\mathbf{E}_\mu\left[\sum_{0<s\le t}\Psi(Y_{s-},Y_s)\right], \tag{5.12}$$

for any $\Psi \in \mathcal{B}^+(F\times F)$ that vanishes along the diagonal and is extended to zero elsewhere, and any $f \in \mathcal{B}^+(F)$ that is extended to be zero off F,

$$\int_F f(x)\check{\kappa}(dx) =\uparrow \lim_{t\downarrow 0}\frac{1}{t}\mathbf{E}_\mu\left[f(Y_{\check{\zeta}-});\, \check{\zeta}\le t\right], \tag{5.13}$$

We note that, since $\Lambda(\omega,dt) = \Psi(Y_{s-},Y_s)\epsilon_s(dt)$ is a homogeneous random measure for Y, the jumping measure $\check{J}$ is well-defined by (5.12) for *any* right process Y with left limits up to the life time and for any Y-excessive measure μ.

The following theorem relates the jumping and killing measures of Y to the excursions for process of X away from F.

Theorem 5.5 *For any $\Psi \in \mathcal{B}^+(F \times F)$ vanishing along d and $f \in \mathcal{B}^+(F)$,*

$$\int_{F\times F} \Psi(x,y)\check{J}(dx,dy) =\uparrow \lim_{s\downarrow 0} \frac{1}{s}\mathbf{E}_m \left[\sum_{t\leq s,\, t\in M,\, R(t)<\infty} \Psi(X_{t-}, X_{R(t)}) \right] \tag{5.14}$$

and

$$\int_F f(x)\check{\kappa}(dx) =\uparrow \lim_{s\downarrow 0} \frac{1}{s}\mathbf{E}_m \left[\sum_{t\leq s,\, t\in I} 1_{\{\sigma_F=\infty\}} \circ \theta_t\, f(X_{t-}) \right]. \tag{5.15}$$

Proof. We note that for $t \in M$, $t = R(t-)$ and so (5.14) can be rewritten as

$$\int_{F\times F} \Psi(x,y)\check{J}(dx,dy) =\uparrow \lim_{s\downarrow 0} \frac{1}{s}\mathbf{E}_m \left[\sum_{t\leq s,\, t\in M,\, R(t)<\infty} \Psi(X_{R(t-)-}, X_{R(t)}) \right].$$

The proof of this theorem is similar to the proofs of [15, Theorem 5.1] and [4, Theorem 4.2] but we present a proof for completeness. Let A be the PCAF of X with Revuz measure μ.

We take any Ψ as in the statement of the lemma and extend it to $E_\partial \times E_\partial$ by setting its value outside $F \times F$ to be zero. From the formula (5.12), we have

$$\int_{F\times F} \Psi(\xi,\eta)\check{J}(d\xi,d\eta) =\uparrow \lim_{\alpha\uparrow\infty} \alpha\mathbf{E}_\mu \left[\sum_{0<t<\infty} e^{-\alpha t}\Psi(Y_{t-}, Y_t) \right].$$

We now make a change of variable, replacing t with A_t. In virtue of Lemma 5.1, we then obtain

$$\begin{aligned} \int_{F\times F} \Psi(\xi,\eta)\check{J}(d\xi,d\eta) &=\uparrow \lim_{\alpha\uparrow\infty} \alpha\mathbf{E}_\mu \left[\sum_{t\in M, R(t)<\infty} e^{-\alpha A_t}\Psi\left(X_{R(t-)-}, X_{R(t)}\right) \right] \\ &=\uparrow \lim_{\alpha\uparrow\infty} \int_F \alpha\mathbf{E}^x\left[\Sigma_\alpha\right]\mu(dx), \end{aligned}$$

where

$$\Sigma_\alpha := \sum_{t\in M,\, R(t)<\infty} e^{-\alpha A_t}\Psi(X_{R(t-)-}, X_{R(t)}).$$

It follows from (2.11) and [34, **(32.6)**] that

$$\begin{aligned} \int_{F\times F} \Psi(\xi,\eta)\check{J}(d\xi,d\eta) &= \uparrow \lim_{\alpha\uparrow\infty} \alpha \int_F \mathbf{E}_x[\Sigma_\alpha]\mu(dx) \\ &= \uparrow \lim_{\alpha\uparrow\infty} \alpha \left(\uparrow \lim_{s\downarrow 0} \frac{1}{s}\mathbf{E}_m \left[\int_0^s \mathbf{E}_{X_u}[\Sigma_\alpha] dA_u \right] \right) \\ &= \uparrow \lim_{s\downarrow 0} \frac{1}{s} \left(\sup_{\alpha>0} \alpha\mathbf{E}_m \left[\int_0^s \Sigma_\alpha \circ \theta_u\, dA_u \right] \right). \end{aligned} \tag{5.16}$$

Now

$$
\begin{aligned}
&\alpha \mathbf{E}_m \left[\int_0^s \Sigma_\alpha \circ \theta_u \, dA_u \right] \\
&= \alpha \mathbf{E}_m \left[\int_0^s \sum_{t \in M \circ \theta_u,\, R(t+u) < \infty} e^{-\alpha(A_{t+u} - A_u)} \Psi(X_{R(t+u-)-}, X_{R(t+u)}) dA_u \right] \\
&= \alpha \mathbf{E}_m \left[\int_0^s e^{\alpha A_u} dA_u^\mu \sum_{t > u,\, t \in M,\, R(t) < \infty} e^{-\alpha A_t} \Psi(X_{R(t-)-}, X_{R(t)}) \right] \\
&= \mathbf{E}_m \left[\sum_{t \in M,\, R(t) < \infty} e^{-\alpha A_t} \Psi(X_{R(t-)-}, X_{R(t)}) \int_0^s 1_{\{t > u\}} de^{\alpha A_u} \right] \\
&= \mathbf{E}_m \left[\sum_{t \in M,\, R(t) < \infty} e^{-\alpha A_t} \Psi(X_{R(t-)-}, X_{R(t)}) \cdot (e^{\alpha A_{s \wedge t}} - 1) \right] \\
&= I_{\alpha,s}^- + I_{\alpha,s}^+,
\end{aligned}
$$

where

$$
I_{\alpha,s}^- = \mathbf{E}_m \left[\sum_{t \le s,\, t \in M,\, R(t) < \infty} (1 - e^{-\alpha A_t}) \Psi(X_{R(t-)-}, X_{R(t)}) \right],
$$

$$
I_{\alpha,s}^+ = \mathbf{E}_m \left[(e^{\alpha A_s} - 1) \sum_{t > s,\, t \in M,\, R(t) < \infty} e^{-\alpha A_t} \Psi(X_{R(t-)-}, X_{R(t)}) \right].
$$

It follows from (5.16) that

$$
\int_{F \times F} \Psi(\xi, \eta) \check{J}(d\xi, d\eta) = \lim_{s \downarrow 0} \frac{1}{s} \left[\sup_\alpha (I_{\alpha,s}^- + I_{\alpha,s}^+) \right]. \tag{5.17}
$$

Note that, since Ψ vanishes along the diagonal d, we can insert in the summand of $I_{\alpha,s}^-$ the condition that $\sigma_F < t$, which is equivalent to $A_t > 0$ in view of the property **(IV)** in §5.

From (5.17), we can then conclude that

$$
\int_{F \times F} \Psi(\xi, \eta) \check{J}(d\xi, d\eta) \ \ge \uparrow \lim_{s \downarrow 0} \frac{1}{s} \mathbf{E}_m \left[\sum_{t \le s,\, t \in M,\, R(t) < \infty} \Psi(X_{R(t-)-}, X_{R(t)}) \right], \tag{5.18}
$$

because we see that $I_{\alpha,s}^-$ increases to the right hand side of (5.18) as $\alpha \to \infty$ by taking the above note into account. If the right hand side of (5.18) is infinite, then the desired equality holds true trivially.

Without loss of generality, we may and do assume that the right hand side of (5.18) is finite. In order to obtain the converse inequality to (5.18), observe that we may take a strictly positive

m-integrable function f on E because m is σ-finite. By weak duality $G_1 f$ is then strictly positive on E and m-integrable. Define for $n \ge 1$,

$$E_n = \left\{ x \in E : G_1 f(x) > \frac{1}{n} \right\}, \quad m_n = m|_{E_n}, \quad \text{and} \quad \tau_n = \inf\{t \ge 0 : X_t \notin E_n\}.$$

Then $m_n(E_n) < \infty$ and $\lim_{n\to\infty} \tau_n = \zeta$, $\mathbf{P}_x$-a.s. for every $x \in E$.

Let $X^n := (X_t^n, \tau_n, \mathbf{P}_x)_{x \in E_n}$ be the subprocess of X killed upon leaving E_n; that is,

$$X_t^n = \begin{cases} X_t, & \text{for } t < \tau_n, \\ \partial, & \text{for } t \ge \tau_n. \end{cases}$$

Since X^n is in weak duality under the measure m_n with the subprocess of $\widehat{X}$ killed upon leaving E_n (see §2), m_n is X^n-excessive. We then define, for any $u \ge 0$ and $n \ge 1$,

$$\begin{aligned} \Sigma_{\alpha,u}^n &:= \sum_{u<t<\infty, t\in M, R(t)<\tau_n} e^{-\alpha A_t} \Psi(X_{R(t-)-}^n, X_{R(t)}^n) \\ &= \sum_{u<t<\infty, t\in M, R(t)<\tau_n} e^{-\alpha A_t} \Psi(X_{R(t-)-}, X_{R(t)}). \end{aligned}$$

Note that $\mathbf{E}_{m_n}((e^{\alpha A_s} - 1)\Sigma_{\alpha,s}^n)$ increases to $I_{\alpha,s}^+$ as $n \to \infty$. It can be easily verified that

$$e^{\alpha A_s} \cdot \Sigma_{\alpha,s}^n = \Sigma_{\alpha,0}^n \circ \theta_s^n,$$

where θ_s^n is the shift operator for the process X^n: $X_t^n \circ \theta_s^n = X_{s+t}^n$.

We next take a truncation function $\chi_N(x) = x \wedge N$, $x \in \mathbb{R}$ and set

$$I_{\alpha,s,n,N}^+ = \mathbf{E}_{m_n}\left(\chi_N(e^{\alpha A_s} \cdot \Sigma_{\alpha,s}^n) - \chi_N(\Sigma_{\alpha,s}^n)\right).$$

Since $0 \le \chi_N(b) - \chi_N(a) \uparrow b - a$, $N \uparrow \infty$ for $a < b$, we see that $I_{\alpha,s,n,N}^+$ increases to $I_{\alpha,s}^+$ when we let $N \uparrow \infty$ and then $n \uparrow \infty$.

Since the measure m_n is X^n-excessive and finite, we have

$$\begin{aligned} I_{\alpha,s,n,N}^+ &= \mathbf{E}_{m_n}\left(\mathbf{E}_{X_s^n}(\chi_N(\Sigma_{\alpha,0}^n))\right) - \mathbf{E}_{m_n}(\chi_N(\Sigma_{\alpha,s}^n)) \\ &\le \mathbf{E}_{m_n}(\chi_N(\Sigma_{\alpha,0}^n)) - \mathbf{E}_{m_n}(\chi_N(\Sigma_{\alpha,s}^n)) \\ &= \mathbf{E}_{m_n}(\chi_N(\Sigma_{\alpha,0}^n) - \chi_N(\Sigma_{\alpha,s}^n)) \\ &\le \mathbf{E}_{m_n}(\Sigma_{\alpha,0}^n - \Sigma_{\alpha,s}^n) \\ &\le \mathbf{E}_m\left(\sum_{0<t\le s, t\in M, R(t)<\infty} e^{-\alpha A_s} \Psi(X_{R(t-)-}, X_{R(t)}) \right). \end{aligned}$$

The last expectation in the above is finite under the present assumption, since

$$I_{\alpha,s}^- + I_{\alpha,s,n,N}^+ \le \mathbf{E}_m\left(\sum_{0<t\le s, t\in M, R(t)<\infty} \Psi(X_{R(t-)-}, X_{R(t)}) \right).$$

Therefore we have from (5.17)

$$\begin{aligned}\int_{F\times F}\Psi(\xi,\eta)\check{J}(d\xi,d\eta) &= \lim_{s\downarrow 0}\frac{1}{s}\sup_{\alpha}\left(I^-_{\alpha,s}+\lim_{n\to\infty}\lim_{N\to\infty}I^+_{\alpha,s,n,N}\right)\\ &\le \lim_{s\downarrow 0}\frac{1}{s}\mathbf{E}_m\left(\sum_{0<t\le s, t\in M, R(t)<\infty}\Psi(X_{R(t-)-},X_{R(t)})\right),\end{aligned}$$

which, together with (5.18), completes the proof for (5.14).

Next we show that for any $f\in\mathcal{B}^+(F)$,

$$\int_F f(x)\check{\kappa}(dx) =\uparrow \lim_{s\downarrow 0}\frac{1}{s}\mathbf{E}_m\left[\sum_{t\le s,\, t\in M}1_{\{\sigma_F=\infty\}}\circ\theta_t f(X_{R(t-)-})\right]. \tag{5.19}$$

This is because, since the lifetime $\check{\zeta}$ of Y is A_∞ and $Y_{\check{\zeta}-}=X_{A_\infty-}$, we have by (5.13) and the Revuz identity (2.11),

$$\begin{aligned}\int_F f(x)\check{\kappa}(dx) &= \uparrow\lim_{t\downarrow 0}\frac{1}{t}\mathbf{E}_\mu\left[f(Y_{\check{\zeta}-});\,\check{\zeta}\le t\right]\\ &= \uparrow\lim_{\alpha\uparrow\infty}\alpha\mathbf{E}_\mu\left[e^{-\alpha\check{\zeta}}f(Y_{\check{\zeta}-})\right]\\ &= \uparrow\lim_{\alpha\uparrow\infty}\alpha\mathbf{E}_\mu\left[e^{-\alpha A_\infty}f(X_{A_\infty-})\right]\\ &= \uparrow\lim_{\alpha\uparrow\infty}\alpha\left(\uparrow\lim_{s\downarrow 0}\frac{1}{s}\mathbf{E}_m\left[\int_0^s\left(e^{-\alpha A_\infty}f(X_{A_\infty-})\right)\circ\theta_u dA_u\right]\right)\\ &= \uparrow\lim_{s\downarrow 0}\frac{1}{s}\left(\uparrow\lim_{\alpha\uparrow\infty}\alpha\mathbf{E}_m\left[\int_0^s e^{\alpha A_u}e^{-\alpha A_\infty}f(X_{A_\infty-})dA_u\right]\right)\\ &= \uparrow\lim_{s\downarrow 0}\frac{1}{s}\left(\uparrow\lim_{\alpha\uparrow\infty}\mathbf{E}_m\left[(e^{\alpha A_s}-1)e^{-\alpha A_\infty}f(X_{A_\infty-})\right]\right)\\ &= \uparrow\lim_{s\downarrow 0}\frac{1}{s}\mathbf{E}_m\left[f(X_{A_\infty-});\,A_s=A_\infty\right]\\ &= \uparrow\lim_{s\downarrow 0}\frac{1}{s}\mathbf{E}_m\left[\sum_{t\le s,\,t\in M}1_{\{\sigma_F=\infty\}}\circ\theta_t\, f(X_{t-})\right].\end{aligned}$$

The last inequality is due to the property **(IV)** of §5. This proves (5.19). Note that for $t\in M\setminus I$ with $\sigma_F\circ\theta_t=\infty$, we have $t=R(t-)=R(t)=\infty$ and so (5.15) follows. □

In §3, we have considered the Lévy system (N,H) of X and the Revuz measure μ_H of H. We define the jumping measure J and the killing measure κ of X by

$$J(dx,dy):=\mu_H(dx)N(x,dy) \quad \text{and} \quad \kappa(dx):=N(x,\{\partial\})\mu_H(dx), \tag{5.20}$$

respectively. Combining Corollary 3.5 with Theorem 5.5, we can establish the following identification of the jumping measure $\check{J}$ and the killing measure $\check{\kappa}$ of the time changed process Y.

Theorem 5.6 *We have*

$$\check{J} \ = \ U \ + \ J|_{F\times F}\,, \tag{5.21}$$

and

$$\check{\kappa}(dx) = V(dx) + \kappa(dx)|_F, \tag{5.22}$$

where U *and* V *are the Feller measure and supplement Feller measure defined by* (3.3) *and* (3.4)*, respectively.*

Proof. The sum in the right hand side of the identity in Theorem 5.5 can be divided into two parts: $t \in I$ where $t = R(t-) < R(t) = t + \sigma_F \circ \theta_t$, and $t \in M \setminus I$ where $t = R(t-) = R(t)$. Thus for $\Psi \geq 0$ on $F \times F$ vanishing along the diagonal d and off $F \times F$,

$$\begin{aligned}
&\lim_{s\downarrow 0} \frac{1}{s}\mathbf{E}_m \left[\sum_{t\leq s,\, t\in M,\, R(t)<\infty} \Psi(X_{R(t-)-}, X_{R(t)}) \right] \\
=\ & \lim_{s\downarrow 0} \frac{1}{s}\mathbf{E}_m \left[\sum_{t\leq s,\, t\in I, R(t)<\infty} \Psi(X_{t-}, X_{\sigma_F}\circ\theta_t) \right] + \lim_{s\downarrow 0} \frac{1}{s}\mathbf{E}_m \left[\sum_{t\leq s,\, X_{t-}, X_t\in F} \Psi(X_{t-}, X_t). \right] \\
=\ & \int_{F\times F} \Psi(x,y)U(dx,dy) + \int_{F\times F} \Psi(x,y)J(dx,dy).
\end{aligned}$$

In the last equality, (3.19) and the Lévy system for X are used. This proves (5.21).

By (3.20) and the Lévy system for X, for any $f \geq 0$ that vanishes on $E_\partial \setminus F$,

$$\begin{aligned}
&\lim_{s\downarrow 0} \frac{1}{s}\mathbf{E}_m \left[\sum_{t\leq s,\, t\in I} 1_{\{\sigma_F=\infty\}} \circ \theta_t f(X_{t-}) \right] \\
=\ & \lim_{s\downarrow 0} \frac{1}{s}\mathbf{E}_m \left[\sum_{t\leq s,\, t\in I\cap[0,\zeta)} f(X_{t-}) 1_{\{\sigma_F=\infty\}} \circ \theta_t \right] + \lim_{s\downarrow 0} \frac{1}{s}\mathbf{E}_m \left[1_{\{X_{\zeta-}\in F\}} f(X_{\zeta-});\, \zeta\leq s \right]. \\
=\ & \int_F f(x)V(dx) + \int_F f(x)\kappa(dx).
\end{aligned}$$

In view of (5.15), this establishes the identity $\check{\kappa}(dx) = V(dx) + \kappa(dx)|_F$. □

This theorem extends a recent result of the authors [4] from symmetric Markov process X to non-symmetric Markov process having a weak dual. It in particular shows that the jumping and killing measure of Y are independent of the choice of $\mu \in S_F$. In fact, this independence of $\mu \in S_F$ can also be deduced from Theorem 5.5. One can also apply results from [11] and [18] to see this independence at the sample path level; it is shown in [18] that two PCAFs having same fine support are time-change to each other by a strictly increasing PCAF, while [11, Theorem 6.2] shows that the correspondence between PCAF and its Revuz measure is invariant under time change.

From jumping measure $\check{J}$ and $\check{\kappa}$, one can easily deduce a Lévy system for Y.

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Extending Markov Processes in Weak Duality by Poisson Point Processes of Excursions

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Dedicated to Professor Kiyosi Itô on the occasion of his 90th birthday

Summary. Let a be a non-isolated point of a topological space E. Suppose we are given standard processes X^0 and $\widehat{X}^0$ on $E_0 = E \setminus \{a\}$ in weak duality with respect to a σ-finite measure m on E_0 which are of no killings inside E_0 but approachable to a. We first show that their extensions X and $\widehat{X}$ to E admitting no sojourn at a and keeping the weak duality are uniquely determined by the approaching probabilities of X^0, $\widehat{X}^0$ and m up to a non-negative constant δ_0 representing the killing rate of X at a. We then construct, starting from X^0, such X by piecing together returning excursions around a and a possible non-returning excursion including the instant killing. This extends a recent result by M. Fukushima and H. Tanaka [16] which treats the case where X^0, X are m-symmetric diffusions and X admits no sojourn nor killing at a. Typical examples of jump type symmetric Markov processes and non-symmetric diffusions on Euclidean domains are given at the end of the paper.

1 Introduction

Let a be a non-isolated point of a topological space E and $X^0 = \{X_t^0, \zeta^0, \mathbf{P}_x^0\}$ be a strong Markov process on $E_0 = E \setminus \{a\}$ which admits no killings inside E_0 and satisfies

$$\varphi(x) := \mathbf{P}_x^0(\zeta^0 < \infty, X_{\zeta^0-}^0 = a) > 0 \qquad \text{for every } x \in E_0.$$

We are concerned with a strong Markovian extension X of X^0 from E_0 to E such that X admits no sojourn at the one-point set $\{a\}$. Natural questions arise: is X uniquely determined by X^0 and how can it be constructed from X^0?

When both X^0 and X are required to be diffusions that are symmetric with respect to a σ-finite measure m on E_0 with $m(\{a\}) = 0$, affirmative answers to these questions were given quite recently in M. Fukushima and H. Tanaka [16]. It is shown in [16] that the entrance law and the absorption rate for the absorbed Poisson point processes of excursions attached to X away from a (due to K. Itô [24] and P.A. Meyer [M]) are uniquely determined by the approaching probability φ to a for X^0 and the measure m, yielding the uniqueness of the extension X that admits no sojourn nor killing at the point a. Conversely such extension X can be constructed from X^0 by piecing together the associated returning excursions and possibly a non-returning one away from a.

The purpose of the present paper is to generalize the stated results of [16] to general standard processes X^0 and X which are not necessarily symmetric but admitting weak dual standard processes $\widehat{X}^0$ and $\widehat{X}$, respectively. We can no longer use the Dirichlet form theory which has played an important role in [16].

Nevertheless, the entrance law and the absorption rate for the absorbed Poisson point process of excursions of X at the point a can still be identified in §2 and §3 in terms of the approaching probabilities to a by X^0 and $\widehat{X}^0$ and m owing to the recent works on the exit system by P.J. Fitzsimmons and R.G. Getoor [12] and by the present authors [5]. It turns out that we must allow the killings of X and $\widehat{X}$ at the point a in order to preserve the duality of X^0 and $\widehat{X}^0$ so that the uniqueness of extensions holds only up to a parameter δ_0 that represents the killing rate of X at a (see Theorem 4.2.).

In §5, we shall construct such an extension X starting from X^0 by piecing together the returning excursions around a and possibly a non-returning excursion from a including a killing at a. X^0 and its dual $\widehat{X}^0$ are assumed to be of no killings inside E_0. The sample path of the constructed process X is cadlag and is continuous at the times t when $X_t = a$. If X^0 is of continuous sample path, then so is X. In this construction, we can proceed along essentially the same line laid in [16] although some natural additional conditions on X^0 and $\widehat{X}^0$ including an off-diagonal finiteness of jumping measures will be required due to the lack of the symmetry and the path continuity. But we shall see that an integrability condition of the α-order approaching probability being imposed on X^0 in [16] can be removed under a fairly general circumstance.

As a typical example of a jump type Markov process, we consider in §6 the case where X^0 is a censored symmetric α-stable process on an open set of $\mathbb{R}^n$ studied by K. Bogdan, K. Burdzy and Z.-Q. Chen [3]. An example is also given on extending non-symmetric diffusions in Euclidean domains. Finally we remark at the end of §6 that the present results on the one point extensions can be applied to obtaining an extension to infinitely many points.

2 Exit System and Point Process of Excursions Around a Point

Let E be a Lusin space (i.e. a space that is homeomorphic to a Borel sunset of a compact metric space), $\mathcal{B}(E)$ be the Borel σ-algebra on E and m be a σ-finite Borel measure on E. We consider a pair of Borel right processes $X = (X_t, \zeta, \mathbf{P}_x)$ and $\widehat{X} = (\widehat{X}_t, \widehat{\zeta}, \widehat{\mathbf{P}}_x)$ on E that are in weak duality with respect to m:

(C.1) $\displaystyle\int_E \widehat{G}_\alpha f(x) g(x) m(dx) = \int_E f(x) G_\alpha g(x) m(dx)$

for every $f, g \in \mathcal{B}^+(E)$ and $\alpha > 0$, where $G_\alpha,\ \widehat{G}_\alpha$ denote the resolvents of $X,\ \widehat{X}$ respectively.

We fix a point $a \in E$ which is regular for $\{a\}$ with respect to X:

(C.2) $\mathbf{P}_a(\sigma_a = 0) = 1$.
Here $\sigma_a = \inf\{t > 0 : X_t = a\}$ with the convention of $\inf \varnothing := \infty$.

Under **(C.1)**, we may and do assume that both X and $\widehat{X}$ are of cadlag paths up to their lifetimes (c. [21, §9]).

Let $E_0 := E \setminus \{a\},\ m_0 := m|_{E_0}$, and

$$\varphi(x) := \mathbf{P}_x(\sigma_a < \infty), \qquad u_\alpha(x) := \mathbf{E}_x\left[e^{-\alpha\sigma_a}\right] \quad \text{for every } x \in E.$$

The corresponding functions for $\widehat{X}$ will be denoted by $\widehat{\varphi}$ and $\widehat{u}_\alpha(x)$, respectively. For $u, v \in \mathcal{B}^+(E_0)$, (u, v) will denote the inner product of u and v in $L^2(E_0; m_0)$, that is, $(u, v) := \int_{E_0} u(x) v(x) m_0(dx)$.

Denote by $X^0 = (X_t^0, \zeta^0, \mathbf{P}_x^0)$ and $\widehat{X}^0 = (\widehat{X}_t^0, \widehat{\zeta}^0, \widehat{\mathbf{P}}_x^0)$ the subprocesses of X and $\widehat{X}$ killed upon leaving E_0, respectively. It is known that they are in weak duality with respect to m_0. The X^0-energy functional $L^{(0)}(\widehat{\varphi} \cdot m_0, v)$ of the X^0-excessive measure $\widehat{\varphi} \cdot m_0$ and an X^0-excessive function v is then well defined by

$$L^{(0)}(\widehat{\varphi} \cdot m_0, v) = \lim_{t \downarrow 0} \frac{1}{t}(\widehat{\varphi} - \widehat{P}_t^0 \widehat{\varphi}, v),$$

where $\widehat{P}_t^0$ is the transition semigroup of $\widehat{X}^0$ (see [16, Lemma 2.1]).

We shall now work with the exit system of X for the point a. To this end, it is convenient to take as the sample space Ω of the process X the space of all paths ω on $E_\Delta = E \cup \Delta$ which are cadlag up to the lifetime $\zeta(\omega)$ and stay at the cemetery Δ after ζ. Thus, $X_t(\omega)$ is just t-th coordinate of ω. Ω is equipped with the minimal completed admissible filtration $\{\mathcal{F}_t,\ t \geq 0\}$ for $\{X_t,\ t \geq 0\}$. The shift operator θ_t is defined by $X_s(\theta_t\omega) = X_{s+t}(\omega),\ s \geq 0$. We also introduce an operator $k_t,\ t \geq 0$, on Ω defined by

$$X_s(k_t\omega) = \begin{cases} X_s(\omega) & \text{if} \quad s < t \\ \Delta & \text{if} \quad s \geq t. \end{cases}$$

We adopt the usual convention that any numerical function of E is extended to E_Δ by setting its value at Δ to be zero.

Let us consider the random time set $M(\omega)$

$$M(\omega) := \overline{\{t \in [0,\infty) : X_t(\omega) = a\}}, \tag{2.1}$$

where $^-$ indicates the closure in $[0,\infty)$. The random set $M(\omega)$ is closed and homogeneous on $[0,\infty)$.

Define $R_t(\omega) := t + \sigma_a(\theta_t\omega)$ for every $t > 0$ and $L(\omega) := \sup\{s > 0 : s \in M(\omega)\}$, with the convention that $\sup\emptyset := 0$. The connected components of the open set $[0,\infty) \setminus M(\omega)$ are called the excursion intervals. The collection of the strictly positive left end points of excursion intervals will be denoted by $G(\omega)$. We can easily see that

$$t \in G(\omega) \quad \text{if and only if} \quad R_{t-}(\omega) < R_t(\omega),$$

and in this case $R_{t-}(\omega) = t$. In particular, $L(\omega) \in G(\omega)$ whenever $L(\omega) < \infty$. We further define the operator i_t, $t \ge 0$, on Ω by $i_t = k_{\sigma_a} \circ \theta_t$. Then

$$\{i_s\omega : s \in G\} \qquad \text{and} \qquad \{i_s\omega : s \in G, R_s < \infty\}$$

are by definition the collection of excursions and the collection of returning excursions respectively of the path ω away from F, while $i_{L(\omega)}(\omega) = \theta_{L(\omega)}(\omega)$ is the non-returning excursion whenever $L(\omega) < \infty$.

Note that those excursions belong to the excursion space W specified by

$$W = \{k_{\sigma_a}\omega : \omega \in \Omega, \sigma_a(\omega) > 0\}, \tag{2.2}$$

which can be decomposed as

$$W = W^+ \cup W^- \cup \{\partial\} \tag{2.3}$$

with

$$W^+ = \{w \in W : \sigma_a < \infty\} \quad \text{and} \quad W^- = \{w \in W : \sigma_a = \infty \text{ and } \zeta > 0\}.$$

Here ∂ denotes the path identically equal to Δ.

The unit mass $\delta_{\{a\}}$ concentrated at the point a is smooth in the sense of [11] because $\{a\}$ is not semipolar by the assumption **(C.2)**. Hence there is a unique positive continuous additive functional (PCAF in abbreviation) $\ell = \{\ell_t, t \ge 0\}$ of X with Revuz measure $\delta_{\{a\}}$. Clearly ℓ is supported by $\{a\}$ and any PCAF of X supported by $\{a\}$ is a constant multiple of ℓ. We call ℓ the local time of X at the point a.

Since the point a is assumed to be regular for $\{a\}$, $\{t \ge 0 : X_t = a\}$ has no isolated points, and the equilibrium 1-potential $\mathbf{E}_x[e^{-\sigma_a}]$ is regular in the sense of [2, Definition IV.3.2] because $\mathbf{E}^x[e^{-\sigma_a}] = c\,\mathbf{E}_x\left[\int_0^\infty e^{-t} d\ell_t\right]$ on E for some $c > 0$. Thus according to [26, §9] (see also [1,8,12] and [20]), there exists a unique σ-finite measure $\mathbf{P}^*$ on Ω carried by $\{\sigma_a > 0\}$ and satisfying

$$\mathbf{P}^* \left[1 - e^{-\sigma_a}\right] < \infty \tag{2.4}$$

such that

$$\mathbf{E}_x \left[\sum_{s \in G} Z_s \cdot \varGamma \circ \theta_s\right] = \mathbf{P}^*(\varGamma) \cdot \mathbf{E}_x \left[\int_0^\infty Z_s d\ell_s\right] \qquad \text{for } x \in E, \tag{2.5}$$

for every non-negative predictable process Z and every non-negative random variable $\varGamma$ on $\varOmega$. Here $\mathbf{E}^*$ is the expectation under the law $\mathbf{P}^*$. The pair $(\mathbf{P}^*, \ell)$ is the predictable version of the exist system for a originated in Maisonneuve [26, §9]. The measure $\mathbf{P}^*$ is Markovian with respect to the transition semigroup of X. We are particularly concerned with the σ-finite measure $\mathbf{Q}^*$ on the space of excursions W induced from $\mathbf{P}^*$ by $\mathbf{Q}^*(\varGamma) = \mathbf{E}_*(\varGamma \circ k_{\sigma_a})$. The measure $\mathbf{Q}^*$ is Markovian with respect to the semigroup $\{P_t^0, t \geq 0\}$ of X^0 and satisfies

$$\mathbf{E}_x \left[\sum_{s \in G} Z_s \cdot \varGamma \circ i_s\right] = \mathbf{Q}^*[\varGamma] \cdot \mathbf{E}_x \left[\int_0^\infty Z_s d\ell_s\right], \qquad x \in E, \tag{2.6}$$

for every non-negative predictable process Z_s and every non-negative random variable $\varGamma$ on W.

We define for $f \in \mathcal{B}^+(E)$

$$\nu_t(f) := \mathbf{Q}^*[f(X_t)] = \mathbf{E}^*[f(X_t); t < \sigma_a], \qquad t > 0.$$

By the Markov property of $\mathbf{Q}^*$, we readily see that $\{\nu_t : t > 0\}$ is an entrance law for X^0: $\nu_t P_s^0 = \nu_{t+s}$.

Proposition 2.1. (i) $\{\nu_t\}_{t>0}$ *is the unique X^0-entrance law characterized by*

$$\widehat{\varphi} \cdot m_0 = \int_0^\infty \nu_t \, dt. \tag{2.7}$$

Moreover $\nu_t(E_0)$ is finite for each $t > 0$.

(ii) $\mathbf{Q}^*[W^-] = L^{(0)}(\widehat{\varphi} \cdot m_0, 1 - \varphi)$.

Proof. (i). We put $\check{\nu}_\alpha(f) = \displaystyle\int_0^\infty e^{-\alpha t} \nu_t(f) dt$. Then, for $f \in \mathcal{B}_b^+(E)$ and for $v \in C_b(E)$ vanishing at a, we have, using **(C.1)**, (2.6) and the Revuz formula [21, (2.13)],

$$\begin{aligned}
&(\widehat{u}_\alpha, v)\widehat{G}_\alpha f(a) = (\widehat{G}_\alpha f - \widehat{G}_\alpha^0 f, v) = (f, G_\alpha v - G_\alpha^0 v) \\
&= \mathbf{E}_{f \cdot m} \left[\int_{\sigma_a}^\infty e^{-\alpha t} v(X_t) 1_{M^c}(t) dt\right] = \mathbf{E}_{f \cdot m} \left[\sum_{s \in M} \int_s^{s + \sigma_a \circ \theta_s} e^{-\alpha t} v(X_t) dt\right] \\
&= \mathbf{E}_{f \cdot m} \left[\sum_{s \in M} e^{-\alpha s} \int_0^{\sigma_a} e^{-\alpha t} v(X_t) dt \circ \theta_s\right] = \check{\nu}_\alpha(v) \mathbf{E}_{f \cdot m} \left[\int_0^\infty e^{-\alpha s} d\ell_s\right] \\
&= \check{\nu}_\alpha(v) \widehat{G}_\alpha f(a).
\end{aligned}$$

Hence

$$\widehat{u}_\alpha \cdot m_0 = \check{\nu}_\alpha, \tag{2.8}$$

from which (2.7) follows by letting $\alpha \downarrow 0$. Since $\widehat{\varphi} \cdot m_0$ is a purely excessive measure of X^0, the uniqueness follows (cf. [20]). The finiteness of ν_t follows from (2.4).

(ii) By (i) and [5, Lemma 3.1], $L^{(0)}(\widehat{\varphi} \cdot m_0, v) = \lim_{t\downarrow 0} \nu_t(v)$ for any X^0-excessive function v. Hence

$$\begin{aligned} L^{(0)}(\widehat{\varphi} \cdot m_0, 1-\varphi) &= \lim_{t\downarrow 0} \mathbf{Q}^*[(1-\varphi)(X_t)] = \lim_{t\downarrow 0} \mathbf{Q}^*[1_{\sigma_a=\infty} \circ \theta_t;\ t<\zeta \wedge \sigma_a] \\ &= \mathbf{Q}^*[W^-]. \end{aligned}$$

□

Remark 1. In the next section, we shall identify $\mathbf{Q}^*$ with the characteristic measure $\mathbf{n}$ of the absorbed Poisson point process of excursions associated with ℓ. Proposition 2.1. was first proved by Fukushima-Tanaka [16] for $\mathbf{n}$ in the case that X is an m-symmetric diffusion by making use of the Dirichlet form of X. In a recent paper of Fitzsimmons-Getoor [12], various properties of some basic quantities for the exit system of a one point set including those in the above proposition have been obtained in the most general setting that X is just a Borel right process with an excessive measure m, in which case $\widehat{X}$ can be taken to be a dual left continuous moderate Markov process. But the present proof, taken from a recent paper by Chen-Fukushima-Ying [5], is simpler under the condition **(C.1)** as far as Proposition 2.1. is concerned.

The next proposition is taken from Fitzsimmons-Getoor [12, (2.10) and (2.17)]. Recall that $L(\omega) := \sup M(\omega)$.

Proposition 2.2. *Put* $\delta = \mathbf{P}^*(\sigma_a = \infty)$. *Then the followings are true:*
(i) $\mathbf{P}_a(\ell_\infty > t) = \exp(-\delta t), \qquad t > 0.$
(ii) $\mathbf{P}_a(L < \infty) = 0$ *or* 1 *according to* $\delta = 0$ *or* $\delta > 0$.

Let $\{\tau_t, t \geq 0\}$ be the right continuous inverse of $\ell = \{\ell_t, t \geq 0\}$, that is,

$$\tau_t := \inf\{s \geq 0 : \ell_s > t\}, \tag{2.9}$$

with the convention that $\inf \emptyset = \infty$. Since ℓ is supported by a, we have (cf. [4, §5]) $\mathbf{P}_a$-a.s.

$$\tau_{\ell_t} = R_t \qquad \text{for every } t \geq 0.$$

We see from the above that, after removing from Ω a $\mathbf{P}_a$-negligible set,

$$L(\omega) < \infty \quad \text{if and only if } \ell_\infty(\omega) < \infty,$$

and in this case,

$$\ell_\infty(\omega) = \ell_L(\omega), \quad \tau_{\ell_\infty -}(\omega) = L(\omega) \quad \text{and} \quad \tau_{\ell_\infty}(\omega) = \infty.$$

Hence, if we let

$$J_\ell(\omega) := \{s \in (0, \infty) : \tau_{s-}(\omega) < \tau_s(\omega)\},$$

then

$$J_\ell(\omega) := \{\ell_t : t \in G(\omega)\} \tag{2.10}$$

and $s \in J_\ell(\omega)$ implies that $s = \ell_t(\omega)$ for some $t \in G(\omega)$ with $\tau_{s-}(\omega) = R_{t-}(\omega) = t$ and $\tau_s(\omega) = R_t(\omega)$.

In particular, $\ell_\infty(\omega) \in J_\ell(\omega)$ whenever it is finite.

Finally the W-valued point process $\mathbf{p} = \mathbf{p}(\omega)$ associated with the local time ℓ is introduced by

$$\mathcal{D}_{\mathbf{p}(\omega)} = J_\ell(\omega) \qquad \text{and} \qquad \mathbf{p}_s(\omega) = i_{\tau_{s-}}\omega \ \ \text{for } s \in \mathcal{D}_{\mathbf{p}(\omega)}. \tag{2.11}$$

Note that $\{\mathbf{p}_s(\omega) : s \in \mathcal{D}_{\mathbf{p}(\omega)}\} \subset W$ and $\{\mathbf{p}_s(\omega) : s \in \mathcal{D}_{\mathbf{p}(\omega)},\ \tau_s < \infty\} \subset W^+$ is the collections of excursions and of the returning excursions away from a, respectively, while $\mathbf{p}_{\ell_\infty}(\omega)(= \theta_L(\omega)) \in W^- \cup \{\partial\}$ is the non-returning excursion whenever $\ell_\infty(\omega) < \infty$ or, equivalently, $L(\omega) < \infty$.

The counting measure of $\mathbf{p}$ is defined by

$$n_{\mathbf{p}}((s,t], \Lambda) = \sum_{u \in \mathcal{D}_{\mathbf{p}} \cap (s,t]} 1_\Lambda(\mathbf{p}_u), \qquad \Lambda \in \mathcal{B}(W), \tag{2.12}$$

and $n_{\mathbf{p}}(t, \Lambda) = n_{\mathbf{p}}((0,t], \Lambda)$ is then $\mathcal{F}_{\tau_t}$-adapted as a process in $t \geq 0$.

Using (2.10), we now make the time substitute in the relation (2.6) to obtain

$$\mathbf{E}_a \left[\sum_{s \in J_\ell} Z_{\tau_{s-}} \cdot \Gamma \circ i_{\tau_{s-}} \right] = \mathbf{Q}^*[\Gamma] \cdot \mathbf{E}_a \left[\int_0^{\ell_\infty} Z_{\tau_s} ds \right]. \tag{2.13}$$

Inserting the predictable process $Z_u = 1_{(0, \tau_{t-}]}(u)$, we arrive at the formula holding for the counting measure of the point process $\mathbf{p}$ associated with ℓ:

$$\mathbf{E}_a[n_{\mathbf{p}}(t, \Lambda)] = \mathbf{Q}^*[\Lambda] \cdot \mathbf{E}_a[t \wedge \ell_\infty] \qquad \text{for every } t \geq 0 \text{ and } \Lambda \in \mathcal{B}(W). \tag{2.14}$$

This formula will be utilized in the next section.

3 Characteristic Measure of Absorbed Poisson Point Process

In this section, we continue to work with the setting in §2 and investigate properties of the point process $(\mathbf{p}_s, \mathcal{D}_{\mathbf{p}})$ defined by (2.11) for the local time

$\ell = \{\ell_t, t \geq 0\}$ at the point a. By the observation made after (2.11), it then holds that

$$\ell_\infty = T \quad \text{where} \quad T = \inf\{s > 0 : \mathbf{p}_s \in W^- \cup \{\partial\}\}. \tag{3.1}$$

In view of Proposition 2.2., T is exponentially distributed with parameter $\delta = \mathbf{P}^*(\sigma_a = \infty)$.

Lemma 3.1. *Under measure $\mathbf{P}_a$, $\mathbf{p}$ is an absorbed Poisson point process with absorption time T in Meyer's sense* ([M]), *that is,*

$$\begin{aligned}\mathbf{P}_a\left(n_{\mathbf{p}}((r+s_1, r+t_1], \Lambda_1) \in H_1, \cdots, n_{\mathbf{p}}((r+s_n, r+t_n], \Lambda_n) \in H_n \,\middle|\, \mathcal{F}_{\tau_r}\right) \\ = 1_{\{T>r\}}\mathbf{P}_a\left(n_{\mathbf{p}}((s_1, t_1], \Lambda_1) \in H_1, \cdots, n_{\mathbf{p}}((s_n, t_n], \Lambda_n) \in H_n\right) \\ +1_{\{T \leq r\}} 1_{H_1}(0) \cdots 1_{H_n}(0),\end{aligned} \tag{3.2}$$

for any $s_1 < t_1, \cdots, s_n < t_n$, $H_1, \cdots, H_n \subset \mathbb{Z}_+$, $r > 0$, $\Lambda_1, \cdots, \Lambda_n \in \mathcal{B}(W)$.

Proof. The proof is the same as in [M, §2] although [M] considered only the conservative case. In fact, the identity $\tau_{r+u} = \tau_r + \tau_u \circ \theta_{\tau_r}$ implies $n_{\mathbf{p}}((r + s, r+t], \Lambda) = n_{\mathbf{p}}((s,t], \Lambda) \circ \theta_{\tau_r}$ and consequently we see from (3.1) and the strong Markov property of X that the left hand side of (3.2) (with $n = 1$) equals

$$\begin{aligned}\mathbf{P}_{X_{\tau_r}}(n_{\mathbf{p}}((s,t], \Lambda) \in H) = 1_{\{T>r\}}\mathbf{P}_a(n_{\mathbf{p}}((s,t], \Lambda) \in H) \\ +1_{\{T \leq r\}}\mathbf{P}_\Delta(n_{\mathbf{p}}((s,t], \Lambda) \in H),\end{aligned}$$

whose last factor is equal to $1_H(0)$. □

By virtue of [M, §1], there is on a certain probability space $(\widetilde{\Omega}, \widetilde{\mathbf{P}})$ a W-valued Poisson point process $\widetilde{\mathbf{p}} = \{\widetilde{\mathbf{p}}, s > 0\}$ with domain $\mathcal{D}_{\widetilde{\mathbf{p}}}$ satisfying the following property.

Let $\widetilde{T} = \inf\{s > 0 : \widetilde{\mathbf{p}}_s \in W^- \cup \{\partial\}\}$ and consider the stopped process $\{\overline{\mathbf{p}}_s, s > 0\}$:

$$\overline{\mathbf{p}}_s = \widetilde{\mathbf{p}}_s \quad \text{for} \quad s \in \mathcal{D}_{\overline{\mathbf{p}}} = \mathcal{D}_{\widetilde{\mathbf{p}}} \cap (0, \widetilde{T}]. \tag{3.3}$$

Then the point process $\{\mathbf{p}_s, s > 0\}$ under $\mathbf{P}_a$ and $\{\overline{\mathbf{p}}_s, s > 0\}$ under $\widetilde{\mathbf{P}}$ are equivalent in law.

Let us denote by $\mathbf{n}$ the characteristic measure of the W-valued Poisson point process $\{\widetilde{\mathbf{p}}, s > 0\}$.

Theorem 3.2. *It holds that*

$$\mathbf{n} = \mathbf{Q}^*. \tag{3.4}$$

Therefore $\mathbf{n}$ is a σ-finite measure on W with $\mathbf{n}(\sigma_a > t) < \infty$ for every $t > 0$, and $\mathbf{n}$ is Markovian with respect to the transition semigroup $\{P_t^0, t \geq 0\}$ of X^0. The X^0-entrance law $\{\nu_t, t > 0\}$ of $\mathbf{n}$ defined by

$$\nu_t(f) = \mathbf{n}(f(X_t); t < \sigma_a), \quad t > 0, \quad f \in \mathcal{B}^+(E)$$

is characterized by

$$\int_0^\infty \nu_t \, dt = \widehat{\varphi} \cdot m_0. \tag{3.5}$$

Define δ_0 *by*

$$\delta_0 = \mathbf{n}(\zeta = 0). \tag{3.6}$$

Then $\widetilde{T}$ *is exponentially distributed with parameter* $L^{(0)}(\widehat{\varphi} \cdot m_0, 1 - \varphi) + \delta_0$*:*

$$\widetilde{\mathbf{P}}(\widetilde{T} > t) = \exp\left(-t\left(L^{(0)}(\widehat{\varphi} \cdot m_0, 1-\varphi) + \delta_0\right)\right) \qquad \text{for every } t > 0. \tag{3.7}$$

Moreover, $\nu_t(E_0) < \infty$ *for each* $t > 0$ *and* $L^{(0)}(\widehat{\varphi} \cdot m_0, 1 - \varphi) < \infty$.

Proof. Since $\{\widetilde{\mathbf{p}}_s : s \in \mathcal{D}_{\widetilde{\mathbf{p}}}, \ \widetilde{\mathbf{p}}_s \in W^+\}$ and $\widetilde{T}$ are independent, we have by (3.1)

$$\mathbf{E}_a[n_{\mathbf{p}}(t, \Lambda)] = \widetilde{\mathbf{E}}\left[\sum_{u \in \mathcal{D}_{\widetilde{\mathbf{p}}} \cap (0, t \wedge \widetilde{T}]} 1_\Lambda(\widetilde{\mathbf{p}}_u)\right] = \mathbf{n}(\Lambda) \cdot \widetilde{\mathbf{E}}[t \wedge \widetilde{T}] = \mathbf{n}(\Lambda) \cdot \mathbf{E}_a[t \wedge \ell_\infty],$$

which compared with (2.14) leads us to (3.4).

Identities (3.5) and (3.7) are the consequences of Proposition 2.1. as

$$\mathbf{Q}^*(W^- \cup \{\partial\}) = \mathbf{Q}^*(W^-) + \mathbf{Q}^*(\{\partial\}) = L^{(0)}(\widehat{\varphi} \cdot m_0, 1 - \varphi) + \delta_0.$$

Then σ-finiteness of $\mathbf{n}$ and the last statement follow from (2.4). □

4 Duality Preserving One-Point Extension

Let E be a locally compact separable metric space, a be a non-isolated point of E and m be a σ-finite measure on $E_0 := E \setminus \{a\}$. Contrarily to the preceding two sections, we shall start in this section with two given strong Markov processes X^0 and $\widehat{X}^0$ on E_0 that are in weak duality with respect to m_0 and have no killings inside E_0. We are concerned with their possible duality preserving extensions X and $\widehat{X}$ to E that admit no sojourn at a. It turns out that we need to allow X and $\widehat{X}$ have killings at a in order to guarantee their weak duality but they are unique up to a parameter δ_0 that represents the killing rate of X at a.

We shall assume that we are given two Borel standard processes $X^0 = (X_t^0, \mathbf{P}_x^0, \zeta^0)$ and $\widehat{X}^0 = (\widehat{X}_t^0, \widehat{\mathbf{P}}_x^0, \widehat{\zeta}^0)$ on E_0 satisfying the next three conditions.

(A.1) X^0 and $\widehat{X}^0$ are in weak duality with respect to m_0; that is, for every $\alpha > 0$ and $f, g \in \mathcal{B}^+(E_0)$,

$$\int_{E_0} \widehat{G}^0_\alpha f(x) g(x) m_0(dx) = \int_{E_0} f(x) G^0_\alpha g(x) m_0(dx),$$

where G^0_α and $\widehat{G}^0_\alpha$ are the resolvent of X^0 and $\widehat{X}^0$, respectively.

(A.2) X^0 and $\widehat{X}^0$ are approachable to $\{a\}$ but admit no killings inside E_0: for every $x \in E_0$,

$$\mathbf{P}^0_x\left(\zeta^0 < \infty,\, X^0_{\zeta^0-} = a\right) > 0 \quad \text{and} \quad \mathbf{P}^0_x\left(\zeta^0 < \infty,\, X^0_{\zeta^0-} \in E_0\right) = 0, \tag{4.1}$$

$$\widehat{\mathbf{P}}^0_x\left(\widehat{\zeta}^0 < \infty,\, \widehat{X}^0_{\widehat{\zeta}^0-} = a\right) > 0 \quad \text{and} \quad \widehat{\mathbf{P}}^0_x\left(\widehat{\zeta}^0 < \infty, \widehat{X}^0_{\widehat{\zeta}^0-} \in E_0\right) = 0. \tag{4.2}$$

Here for a Borel set $B \subset E$, the notation "$X^0_{\zeta^0-} \in B$" means that the left limit of X^0_t at $t = \zeta^0$ exists under the topology of E and takes values in $B \subset E$. We use the same convention for $\widehat{X}$.

We shall use the same notations as in [16]: for $x \in E_0$ and $\alpha > 0$,

$$\varphi(x) := \mathbf{P}^0_x\left(\zeta^0 < \infty, X^0_{\zeta^0-} = a\right) \text{ and } u_\alpha(x) := \mathbf{E}^0_x\left[\, e^{-\alpha\zeta^0} : X^0_{\zeta^0-} = a\right]. \tag{4.3}$$

As in §2, the X^0-energy functional of X^0-excessive measure μ and X^0-excessive function v is denoted by $L^{(0)}(\mu, v)$. The corresponding notations for $\widehat{X}^0$ will be designated by $\widehat{\varphi}$, $\widehat{u}_\alpha$, $\widehat{L}^{(0)}$. We use (u, v) to denote the inner product between u and v in $L^2(E_0, m_0)$, that is, $(u, v) = \int_{E_0} u(x)v(x)m_0(dx)$.

We say that a strong Markov process X (resp. $\widehat{X}$) on E is an extension of X^0 (resp. $\widehat{X}^0$) if the subprocess on E_0 of X (resp. $\widehat{X}$) killed upon hitting the point a is identical in law to X^0 (resp. $\widehat{X}^0$).

Let us now consider two Borel right processes $X = (X_t, \mathbf{P}_x, \zeta)$ and $\widehat{X} = (\widehat{X}_t, \widehat{\mathbf{P}}_x, \widehat{\zeta})$ on E satisfying the next four conditions.

(1) X and $\widehat{X}$ are in weak duality with respect to a σ-finite measure m on E with $m|_{E_0} = m_0$.
(2) X and $\widehat{X}$ are extensions of X^0 and $\widehat{X}^0$ respectively.
(3) The point a is regular for itself with respect to X:

$$\mathbf{P}_a(\sigma_a = 0) = 1,$$

where $\sigma_a = \inf\{t > 0 : X_t = a\}$ is the hitting time of a by X.
(4) X admits no sojourn at the point a, that is,

$$\mathbf{P}_x\left(\int_0^\infty 1_{\{a\}}(X_s)ds = 0\right) = 1 \qquad \text{for every } x \in E.$$

Under **(1)**, we can and do assume that both X and $\widehat{X}$ possess cadlag paths up to their lifetimes.

Proposition 4.1. *Assume that the above conditions* **(1)**, **(2)**, **(3)** *and* **(4)** *hold. Then*

(i) *The measure m does not charging on $\{a\}$: $m(\{a\}) = 0$*
(ii) *X admits no jumping from E_0 to the point a: for every $x \in E_0$,*

$$\mathbf{P}_x\left(X_{t-} \in E_0,\ X_t = a \ \textit{ for some } t \in (0,\zeta)\right) = 0, \tag{4.4}$$

(iii) *X admits no jump from the point a to E_0 in the following sense:*

$$\mathbf{P}_x\left(X_{t-} = a,\ X_t \in E_0 \ \textit{ for some } t \in (0,\zeta)\right) = 0 \quad \text{for q.e. } x \in E. \tag{4.5}$$

Here q.e. *means except on an m-polar set for X.*
(iv) *The one point set $\{a\}$ is not m-polar for X. Let functions φ and u_α be defined as in* (4.3). *Then*

$$\varphi(x) = \mathbf{P}_x(\sigma_a < \infty) \ \textit{and}\ u_\alpha(x) = \mathbf{E}_x\left[e^{-\alpha\sigma_a}\right] \qquad \textit{for } x \in E_0. \tag{4.6}$$

(v) *$u_\alpha,\ \widehat{u}_\alpha \in L^1(E_0, m_0)$ for every $\alpha > 0$.*

Proof. (i). This is immediate from **(1)**, **(4)** for X as

$$\widehat{G}_\alpha f(a) m(\{a\}) = \int_E f(x) G_\alpha 1_{\{a\}}(x) m(dx) = 0 \qquad \text{for every } f \in \mathcal{B}^+(E).$$

(ii). It follows from (4.1) and **(2)** that

$$\mathbf{P}_x\left(X_{\sigma_a -} \in E_0,\ \sigma_a < \infty\right) = 0 \qquad \text{for every } x \in E_0. \tag{4.7}$$

For any open set O that has a positive distance from $\{a\}$, let $\{\sigma_a^n, n \geq 0\}$, $\{\eta^n, n \geq 0\}$ be the stopping times defined by

$$\eta^0 = 0,\ \sigma_a^0 = \sigma_a,\ \eta^n = \sigma_a^{n-1} + \sigma_O \circ \theta_{\sigma_a^{n-1}},\ \sigma_a^n = \eta^n + \sigma_a \circ \theta_{\eta^n} \tag{4.8}$$

with an obvious modification after one of them becomes infinity. Clearly the time set

$$\{t \in (0, \zeta(\omega)) : X_{t-}(\omega) \in O,\ X_t(\omega) = a\} \subset \{\sigma_a^n(\omega);\ n = 0, 1, 2, \cdots\}.$$

Thus it follows from the strong Markov property of X and (4.7) that for every $x \in E_0$,

$$\mathbf{P}_x\left(\text{there is some } t > 0 \text{ such that } X_{t-} \in O,\ X_t = a\right) = 0.$$

Letting O increase to E_0 establishes (4.4).

(iii). Clearly, property (ii) also holds for $\widehat{X}$:

$$\widehat{\mathbf{P}}_x\left(\widehat{X}_{t-}\in E_0,\ \widehat{X}_t=a\ \text{ for some } t\in(0,\widehat{\zeta})\right)=0 \qquad \text{for every } x\in E_0. \quad (4.9)$$

We combine the above with a time reversal argument based on the stationary Kuznetsov process $(\mathbf{P}, Z_t, \alpha<t<\beta)$ associated with X and $\widehat{X}$ as was formulated in [21, §10] : the σ-finite measure $\mathbf{P}$ on a path space $D((-\infty,\infty),E_\Delta)$ with a random birth time α and a random death time β is stationary under the time shift of the path, and furthermore, if we put

$$\widehat{Z}_t=Z_{(-t)-}\ \text{ for }\ t\in\mathbb{R}, \qquad \widehat{\alpha}=-\beta \quad\text{and}\quad \widehat{\beta}=-\alpha,$$

then $\{Z_t,\,0\le t<\beta\}$ (resp. $\{\widehat{Z}_t,\,0\le t<\widehat{\beta}\}$) on $\{Z_0\in E\}$ (resp. $\{\widehat{Z}_0\in E\}$) is a copy of $\{X_t,\,0\le t<\zeta\}$ (resp. $\{\widehat{X}_t,\,0\le t<\widehat{\zeta}\}$) under $\mathbf{P}_m$(resp. $\widehat{\mathbf{P}}_m$). We shall use the formula (10.5) of [21, §10] which express a precise meaning of this property.

Consider the set

$$\Lambda=\{Z_{t-}=a \text{ and } Z_t\in E_0,\ \text{for some } t\in(\alpha,\beta)\}.$$

Then

$$\Lambda=\{\widehat{Z}_{t-}\in E_0 \text{ and } \widehat{Z}_t=a,\ \text{for some } t\in(\widehat{\alpha},\widehat{\beta})\},$$

and thus $\Lambda=\bigcup_{r\in\mathbb{Q}^+}\Lambda_r$ with

$$\Lambda_r=\{\widehat{\alpha}<r<\widehat{\beta},\ \widehat{Z}_{t-}\in E_0,\ \widehat{Z}_t=a\ \text{ for some } t>r\}.$$

According to (10.5) of [21, §10], $\mathbf{P}(\Lambda_r)$ is equal to the integral of the left hand side of (4.7) with respect to m for each rational r. Therefore $\mathbf{P}(\Lambda)=0$.

Denote by $h(x)$ the function of $x\in E$ appearing in the left hand side of (4.5). By (10.5) of [21, §10] again, we have

$$\begin{aligned}\int_E h(x)m(dx)&=\mathbf{P}\left(Z_{t-}=a \text{ and } Z_t\in E_0,\ \text{for some } t\in(0,\beta),\ \alpha<0<\beta\right)\\ &\le \mathbf{P}(\Lambda)=0.\end{aligned}$$

Consequently, $h=0$ m-a.e. and hence q.e. on E because h is X-excessive (cf. [5, §2]).

(iv). On account of [2, p. 59] (see also [21, Proposition 15.7] when E is a Lusin space),

$$\mathbf{P}_x(0<\sigma_a'<\sigma_a)=0, \quad \text{where } \sigma_a'=\inf\{t: X_{t-}=a\}, \quad x\in E.$$

On the other hand, **(A.2)** and **(2)** imply for $\zeta^0=\sigma_a\wedge\zeta$ that for $x\in E_0$,

$$\mathbf{P}_x(\sigma_a<\sigma_a')\le\mathbf{P}_x(\sigma_a<\infty, X_{\sigma_a-}\ne a)\le\mathbf{P}_x(\zeta^0<\infty, X_{\zeta^0-}\in E_0)=0.$$

Hence $\mathbf{P}_x(\sigma_a = \sigma'_a) = 1$ and

$$\varphi(x) = \mathbf{P}_x(\zeta^0 < \infty, X_{\zeta^0-} = a) = \mathbf{P}_x(\sigma_a < \infty) \quad \text{for } x \in E_0.$$

In particular,

$$\mathbf{P}_m(\sigma_a < \infty) = \int_{E_0} \varphi(x) m(dx) > 0$$

by **(A.2)** and therefore $\{a\}$ is not m-polar for X.

(v). By the strong Markov property of $\widehat{X}$,

$$\widehat{G}_\alpha f(x) = \widehat{G}^0_\alpha f(x) + \widehat{u}_\alpha(x)\widehat{G}_\alpha f(a), \quad x \in E.$$

We can take a non-negative m-integrable function f on E such that $\widehat{G}_\alpha f(a) > 0$. Then

$$\widehat{G}_\alpha f(a)(\widehat{u}_\alpha, 1) \leq (\widehat{G}_\alpha f, 1) = (f, G_\alpha 1) \leq \frac{1}{\alpha}(f, 1) < \infty,$$

yielding the m_0-integrability of $\widehat{u}_\alpha$. Similar, we have $u_\alpha \in L^1(E_0, m_0)$. □

Theorem 4.2. *Assume that X and $\widehat{X}$ are two Borel right processes on E satisfying conditions* **(1)**, **(2)**, **(3)** *and* **(4)** *in this section. Let $\{G_\alpha, \alpha > 0\}$ and $\{\widehat{G}_\alpha, \alpha > 0\}$ denote the resolvents of X and $\widehat{X}$, respectively. Then there exist constants $\delta_0 \geq 0$, $\widehat{\delta}_0 \geq 0$ such that*

$$L^{(0)}(\widehat{\varphi} \cdot m_0, 1 - \varphi) + \delta_0 = \widehat{L}^{(0)}(\varphi \cdot m_0, 1 - \widehat{\varphi}) + \widehat{\delta}_0, \tag{4.10}$$

and for every $f \in \mathcal{B}^+(E)$ and $\alpha > 0$,

$$G_\alpha f(a) = \frac{(\widehat{u}_\alpha, f)}{\alpha(\widehat{u}_\alpha, \varphi) + L^{(0)}(\widehat{\varphi} \cdot m_0, 1 - \varphi) + \delta_0}, \tag{4.11}$$

$$G_\alpha f(x) = G^0_\alpha f(x) + u_\alpha(x) G_\alpha f(a) \qquad \text{for } x \in E_0, \tag{4.12}$$

$$\widehat{G}_\alpha f(a) = \frac{(u_\alpha, f)}{\alpha(u_\alpha, \widehat{\varphi}) + \widehat{L}^{(0)}(\varphi \cdot m_0, 1 - \widehat{\varphi}) + \widehat{\delta}_0}, \tag{4.13}$$

$$\widehat{G}_\alpha f(x) = \widehat{G}^0_\alpha f(x) + \widehat{u}_\alpha(x)\widehat{G}_\alpha f(a) \qquad \text{for } x \in E_0. \tag{4.14}$$

Corollary 4.3. *Borel right processes X and $\widehat{X}$ on E satisfying conditions* **(1)**-**(4)** *of this section are unique in law up to a parameter δ_0 satisfying*

$$\delta_0 \geq \max\left\{\widehat{L}^{(0)}(\varphi \cdot m_0, 1 - \widehat{\varphi}) - L^{(0)}(\widehat{\varphi} \cdot m_0, 1 - \varphi),\, 0\right\}.$$

Proof of Theorem 4.2. In view of conditions **(1)-(4)** of this section and Proposition 4.1., X satisfies the conditions **(C.1)-(C.2)** of §2 so that Theorem 3.2. is applicable to X.

The identity (4.12) is a simple consequence of the strong Markov property of X applied to the hitting time σ_a. In order to show (4.11), we consider the local time $\ell = \{\ell_t, t \geq 0\}$ of X with Revuz measure $\delta_{\{a\}}$ and the W-valued point process $\mathbf{p}$ associated with ℓ defined by (2.11). By Lemma 3.1., $\mathbf{p}$ under $\mathbf{P}_a$ is an absorbed Poisson point process and admits the representation (3.3) in terms of a W-valued Poisson point process $\widetilde{\mathbf{p}}$ defined on some probability space $(\widetilde{\Omega}, \widetilde{\mathbf{P}})$ together with its hitting time $\widetilde{T}$ of $W^- \cup \{\partial\}$.

Let $\mathbf{n}$ be the characteristic measure of $\widetilde{\mathbf{p}}$. Then, for any non-negative predictable process $\{a(t, w, \widetilde{\omega}),\ t \geq 0, w \in W, \widetilde{\omega} \in \widetilde{\Omega}\}$, we have

$$\widetilde{\mathbf{E}}\left[\sum_{s \leq t} a(s, \widetilde{\mathbf{p}}_s, \widetilde{\omega})\right] = \widetilde{\mathbf{E}}\left[\int_{W \times (0,t]} a(s, w, \widetilde{\omega}) \mathbf{n}(dw) ds\right], \tag{4.15}$$

because the compensator of $\widetilde{\mathbf{p}}$ equals $t\,\mathbf{n}(\cdot)$ (cf. [23, §II.3]).

We now proceed along the same line as in [16, Remark 4.2]. The terminal time of $w \in W$ is denoted by $\zeta(w)$: for $w = k_{\sigma_a}(\omega)$ with $\omega \in \Omega$, $\zeta(w) = \sigma_a(\omega)$. We put for $f \in \mathcal{B}^+(E_0)$

$$\check{f}_\alpha(w) = \int_0^{\zeta(w)} e^{-\alpha t} f(w(t)) dt, \quad w \in W, \quad \alpha > 0.$$

Note that $t \mapsto X_t(\omega)$ has only at most countably many discontinuous points. Thus by (2.2) and the condition **(4)**, $M(\omega)$ has zero Lebesgue measure almost surely. So we have $\mathbf{P}_a$-a.s.

$$\begin{aligned} \int_0^\infty e^{-\alpha t} f(X_t) dt &= \sum_{s < \ell_\infty} \int_{\tau_{s-}}^{\tau_s} e^{-\alpha t} f(X_t) dt + \int_{\tau_{\ell_\infty -}}^\infty e^{-\alpha t} f(X_t) dt \\ &= \sum_{s < \ell_\infty} e^{-\alpha \tau_{s-}} \check{f}_\alpha(\mathbf{p}_s) + e^{-\alpha \tau_{L_\infty -}} \check{f}_\alpha(\mathbf{p}_{\ell_\infty}), \end{aligned} \tag{4.16}$$

which is equivalent in law to

$$\sum_{s < \widetilde{T}} e^{-\alpha S(s-)} \check{f}_\alpha(\widetilde{\mathbf{p}}_s^+) + e^{-\alpha S(\tilde{T}-)} \check{f}_\alpha(\widetilde{\mathbf{p}}_{\widetilde{T}}), \quad \text{under } \widetilde{\mathbf{P}}, \tag{4.17}$$

where $\{\widetilde{\mathbf{p}}_s^+, s > 0\}$ is a Poisson point process defined by $\widetilde{\mathbf{p}}_s^+ = \widetilde{\mathbf{p}}_s$ for $s \in \mathcal{D}_{\widetilde{\mathbf{p}}^+} = \{s \in \mathcal{D}_{\widetilde{\mathbf{p}}} : \widetilde{\mathbf{p}}_s \in W^+\}$ and $S(s) = \sum_{r \leq s} \zeta(\widetilde{\mathbf{p}}_r^+)$. The characteristic measure of $\{\widetilde{\mathbf{p}}_s^+, s > 0\}$ is the restriction $\mathbf{n}^+$ of $\mathbf{n}$ on W^+.

First we claim that

$$\widetilde{\mathbf{E}}\left[e^{-\alpha S(s)}\right] = \exp(-\alpha(\widehat{u}_\alpha, \varphi)s). \tag{4.18}$$

Since

$$e^{-\alpha S(s)} - 1 = \sum_{r \leq s} \left\{e^{-\alpha S(r)} - e^{-\alpha S(r-)}\right\} = \sum_{r \leq s} e^{-\alpha S(r-)} \left\{e^{-\alpha \zeta(\mathbf{p}_r^+)} - 1\right\},$$

it follows from (4.15) that

$$\widetilde{\mathbf{E}}\left[e^{-\alpha S(s)}\right] - 1 = -c\int_0^s \widetilde{\mathbf{E}}\left[e^{-\alpha S(r)}\right] dr,$$

with

$$c = \mathbf{n}^+(1-e^{-\alpha\zeta}) = \mathbf{n}(1-e^{-\alpha\zeta};\zeta<\infty) = \mathbf{n}\left\{\alpha\int_0^\zeta e^{-\alpha t}dt;\zeta<\infty\right\}$$
$$= \alpha\int_0^\infty e^{-\alpha t}\mathbf{n}(t<\zeta<\infty)dt.$$

Due to (3.5) (see also (2.8)), we have accordingly

$$c = \alpha\int_0^\infty e^{-\alpha t}\nu_t(\varphi)dt = \alpha(\widehat{u}_\alpha,\varphi),$$

which is finite by Proposition 4.1.(v). The identity (4.18) then follows.

On the other hand, we have from Theorem 3.2. and the basic properties of Poisson point processes,

(i) $\widetilde{T}$ has an exponential distribution with exponent $L^{(0)}(\widehat{\varphi}\cdot m_0, 1-\varphi)+\delta_0$, where δ_0 is defined by (3.6).
(ii) The three objects $\{\widetilde{\mathbf{p}}_s^+, s>0\}$, $\widetilde{T}$ and $\widetilde{\mathbf{p}}_{\widetilde{T}}$ are independent.
(iii) The law of $\widetilde{\mathbf{p}}_{\widetilde{T}}$ is $\bar{\mathbf{n}}^-(W^-\cup\{\partial\})^{-1}\bar{\mathbf{n}}^- = (L^{(0)}(\widehat{\varphi}\cdot m_0,1-\varphi)+\delta_0)^{-1}\bar{\mathbf{n}}^-$, where $\bar{\mathbf{n}}^-$ is the restriction of $\mathbf{n}$ on $W^-\cup\{\partial\}$.

Taking these facts and formula (4.15) for $\widetilde{\mathbf{p}}^+$ into account, we get from (4.16), (4.17) and (4.18),

$$\begin{aligned} G_\alpha f(a) &= \widetilde{\mathbf{E}}\left[\sum_{s<\widetilde{T}} e^{-\alpha S(s-)}\check{f}_\alpha(\widetilde{\mathbf{p}}_s^+) + e^{-S(\widetilde{T}-)}\check{f}_\alpha(\widetilde{\mathbf{p}}_{\widetilde{T}})\right] \\ &= \widetilde{\mathbf{E}}\left[\int_0^{\widetilde{T}} e^{-\alpha(\widehat{u}_\alpha,\varphi)s}ds\right]\mathbf{n}^+(\check{f}_\alpha) \\ &\quad + \widetilde{\mathbf{E}}\left(e^{-\alpha(\widehat{u}_\alpha,\varphi)\widetilde{T}}\right)(L^{(0)}(\widehat{\varphi}\cdot m_0,1-\varphi)+\delta_0)^{-1}\mathbf{n}^-(\check{f}_\alpha) \\ &= \frac{\mathbf{n}^+(\check{f}_\alpha)}{\alpha(\widehat{u}_\alpha,\varphi)+L^{(0)}(\widehat{\varphi}\cdot m_0,1-\varphi)+\delta_0} \\ &\quad + \frac{\mathbf{n}^-(\check{f}_\alpha)}{\alpha(\widehat{u}_\alpha,\varphi)+L^{(0)}(\widehat{\varphi}\cdot m_0,1-\varphi)+\delta_0} \\ &= \frac{\mathbf{n}(\check{f}_\alpha)}{\alpha(\widehat{u}_\alpha,\varphi)+L^{(0)}(\widehat{\varphi}\cdot m_0,1-\varphi)+\delta_0}, \end{aligned}$$

which coincides with the right hand side of (4.11) because we have from Theorem 3.2.

$$\mathbf{n}(\check{f}_\alpha) = \int_0^\infty e^{-\alpha t}\nu_t(f)dt = (\widehat{u}_\alpha, f).$$

(4.13) can be obtained analogously.

Under the weak duality assumption **(1)**, the denominators of (4.11) and (4.13) must be equal. Since $(\widehat{u}_\alpha, \varphi) = (u_\alpha, \widehat{\varphi})$ (see the first two equations in the proof of Lemma 5.8.), we must have the identity (4.10). □

In the above proof, we did not use the property of X having no jumps from the point a to E_0, which is proved in Proposition 4.1.(iii). But this property reflects on the following property of the characteristic measure $\mathbf{n}$ of the absorbed Poisson point process $\mathbf{p}$ considered in the above proof.

Proposition 4.4. $\mathbf{n}\{w(0) \neq a\} = 0.$

Proof. By (4.5), we have $\mathbf{E}_a(\sum_{s\in G} 1_\Lambda \circ i_s) = 0$ for $\Lambda = \{w(0) \neq a\}$ and we get $\mathbf{n}(\Lambda) = \mathbf{Q}^*(\Lambda) = 0$ from (2.6) and (3.4). □

Remark 2. In this section, we have assumed that E is a locally compact separable metric space. But all assertions in this section remain valid for a general Lusin space E except that the identities (4.6), (4.12), (4.14) hold only for q.e. $x \in E_0$ rather than for every $x \in E_0$, because we need to replace the usage of [2, p. 59] by [21, (15.7)] in the proof of (4.6). The uniqueness statement in Corollary 4.3. should be modified accordingly in the Lusin space case.

We also note that the expression (4.11) of the resolvent has been obtained in [12] by a different method for a general right process X and its excessive measure m, in which case $\widehat{X}$ can be taken to be a dual moderate Markov process. But the present proof is more useful in the next section.

5 Extending Markov Process via Poisson Point Processes of Excursions

As in §4, let E be a locally compact separable metric space and a be a fixed non-isolated point of E and m_0 be a σ-finite measure on $E_0 := E \setminus \{a\}$ with $\mathrm{Supp}[m_0] = E$. We extend m_0 to a measure m on E by setting $m(\{a\}) = 0$. Note that m could be infinity on a compact neighborhood of a in E. Let $E_\Delta = E \cup \{\Delta\}$ be the one point compactification of E. When E is compact, Δ is added as an isolated point.

5.1 Excursion Laws in Duality

We shall assume that we are given two Borel standard processes $X^0 = \{X_t^0, \mathbf{P}_x^0, \zeta^0\}$ and $\widehat{X}^0 = \left\{\widehat{X}_t^0, \widehat{\mathbf{P}}_x^0, \widehat{\zeta}^0\right\}$ on E_0 satisfying the following conditions.

(A.1) X^0 and $\widehat{X}^0$ are in weak duality with respect to m_0, that is, for every $\alpha > 0$, and $f, g \in \mathcal{B}^+(E_0)$,

$$\int_{E_0} \widehat{G}^0_\alpha f(x) g(x) m_0(dx) = \int_{E_0} f(x) G^0_\alpha g(x) m_0(dx),$$

where G^0_α and $\widehat{G}^0_\alpha$ are the resolvents of X^0 and $\widehat{X}^0$, respectively.
(A.2) X^0 and $\widehat{X}^0$ satisfy, for every $x \in E_0$,

$$\begin{aligned} &\mathbf{P}^0_x \left(\zeta^0 < \infty,\ X^0_{\zeta^0-} = a\right) > 0, \\ &\mathbf{P}^0_x \left(\zeta^0 < \infty,\ X^0_{\zeta^0-} \in \{a,\, \Delta\}\right) = \mathbf{P}^0_x(\zeta < \infty), \qquad (5.1) \\ &\widehat{\mathbf{P}}^0_x \left(\widehat{\zeta}^0 < \infty,\ \widehat{X}^0_{\widehat{\zeta}^0-} = a\right) > 0, \\ &\widehat{\mathbf{P}}^0_x \left(\widehat{\zeta}^0 < \infty,\ \widehat{X}^0_{\widehat{\zeta}^0-} \in \{a,\, \Delta\}\right) = \widehat{\mathbf{P}}^0_x(\widehat{\zeta} < \infty).x \qquad (5.2) \end{aligned}$$

Here, as in §4, for a Borel set $B \subset \mathbf{E}_\Delta$, the notation "$X^0_{\zeta^0-} \in B$" means that the left limit of $t \mapsto X^0_t$ at $t = \zeta^0$ exists under the topology of E_Δ and takes values in B.

The first condition in (5.1) (resp. (5.2)) means that X^0 (resp. $\widehat{X}^0$) is approachable to the point a, while the second condition in (5.1) (resp. (5.2)) implies that X^0 (resp. $\widehat{X}^0$) admits no killings inside E_0.

As in §4, we put for $x \in E_0$ and $\alpha > 0$,

$$\varphi(x) := \mathbf{P}^0_x \left(\zeta^0 < \infty,\ X^0_{\zeta^0-} = a\right) \quad \text{and} \quad u_\alpha(x) := \mathbf{E}^0_x \left[\, e^{-\alpha\zeta^0};\, X^0_{\zeta^0-} = a\right]. \qquad (5.3)$$

The corresponding notations for $\widehat{X}^0$ will be designated by $\widehat{\varphi}$ and $\widehat{u}_\alpha$. As in §2, the X^0-energy functional $L^{(0)}(\widehat{\varphi} \cdot m_0, v)$ of the X^0-excessive measure $\widehat{\varphi} \cdot m_0$ and an X^0-excessive function v is well defined. Similarly the $\widehat{X}^0$-energy functional $\widehat{L}^{(0)}(\varphi \cdot m_0, \widehat{v})$ is well defined. The inner product of u, v in $L^2(E_0, m_0)$ will be denoted by (u, v), that is, $(u, v) = \int_{E_0} u(x)v(x)m_0(dx)$. The space of all bounded continuous functions on E_0 will be denoted by $C_b(E_0)$.

We impose some more assumptions:
(A.3) $u_\alpha,\ \widehat{u}_\alpha \in L^1(E_0, m_0)$ for every $\alpha > 0$.

(A.4) $G^0 f(x),\ \widehat{G}^0 f(x),\ x \in E_0$, are lower semi-continuous for any Borel $f \geq 0$. Here G^0 denotes the 0-order resolvent of X^0:

$$G^0 f(x) := \mathbf{E}_x \left[\int_0^\infty f(X_t)dt\right] =\uparrow \lim_{\alpha \downarrow 0} G^0_\alpha f(x)$$

for $x \in E$ and Borel function $f \geq 0$ on E. The 0-order resolvent $\widehat{G}^0$ of $\widehat{X}^0$ is similarly defined.

We note that, if $G^0_\alpha(C_b(E_0)) \subset C_b(E_0),\ \widehat{G}^0_\alpha(C_b(E_0)) \subset C_b(E_0),\ \alpha > 0$, then **(A.4)** is satisfied by the monotone class lemma.

The next condition will be imposed only when X^0 is non-symmetric, namely, when $X^0 \neq \widehat{X}^0$.

(A.5) $\lim_{x\to a} u_\alpha(x) = \lim_{x\to a} \widehat{u}_\alpha(x) = 1,$ for every $\alpha > 0$.

The next condition **(A.6)** will be imposed only when X^0 is not a diffusion, namely, when

$$\mathbf{P}^0_m \left(X^0_{t-} \neq X^0_t \quad \text{for some } t \in (0, \zeta^0)\right) > 0.$$

Note that $\widehat{X}^0$ then has the same property in view of [21, §10]. According to [31, (73.1), (47.10)], the standard process X^0 on E_0 has a Lévy system (N, H) on E_0. That is, $N(x, dy)$ is a kernel on $(E_0, \mathcal{B}(E_0))$ and H is a PCAF of X^0 in the strict sense with bounded 1-potential such that for any nonnegative Borel function f on $E_0 \times (E_0 \cup \{\Delta_0\})$ that vanishes on the diagonal and is extended to be zero outside $E_0 \times E_0$,

$$\mathbf{E}^0_x \left[\sum_{s \le t} f(X^0_{s-}, X^0_s) \right] = \mathbf{E}^0_x \left[\int_0^t \int_{E_0} f(X^0_s, y) N(X^0_s, dy) dH_s \right] \tag{5.4}$$

for every $x \in E_0$ and $t \geq 0$. Similarly, the standard process $\widehat{X}^0$ has a Lévy system $(\widehat{N}, \widehat{H})$. Let μ_H and $\mu_{\widehat{H}}$ be the Revuz measure of the PCAF H of X^0 and the PCAF $\widehat{H}$ of $\widehat{X}^0$ with respect to the measure m_0 on E_0, respectively. Define

$$J_0(dx, dy) := N(x, dy)\mu_H(dx) \quad \text{and} \quad \widehat{J}_0(dx, dy) := \widehat{N}(x, dy)\mu_{\widehat{H}}(dx). \tag{5.5}$$

The measures J_0 and $\widehat{J}_0$ are called the jumping measure of X^0 and $\widehat{X}^0$, respectively. It is known (see [18]) that

$$J_0(dx, dy) = \widehat{J}_0(dy, dx) \qquad \text{on } E_0 \times E_0. \tag{5.6}$$

We now state the condition **(A.6)**.

(A.6) Either $E \setminus U$ is compact for any neighborhood U of a in E, or for any open neighborhood U_1 of a in E, there exists an open neighborhood U_2 of a in E with $\overline{U}_2 \subset U_1$ such that

$$J_0(U_2 \setminus \{a\},\, E_0 \setminus U_1) < \infty \quad \text{and} \quad \widehat{J}_0(U_2 \setminus \{a\},\, E_0 \setminus U_1) < \infty.$$

Throughout this section, we assume that we are given a pair of Borel standard processes X^0 and $\widehat{X}^0$ on E_0 satisfying conditions **(A.1)**, **(A.2)**, **(A.3)**, **(A.4)**, and additionally **(A.5)** in non-symmetric case and **(A.6)** in non-diffusion case. We aim at constructing (see Theorem 5.15.) under these conditions their right process extensions $X,\ \widehat{X}$ to E with resolvents (4.11), (4.13) respectively. Theorem 5.16. will then be concerned with some stronger conditions **(A.1)'** and **(A.4)'** to ensure the quasi-left continuity of the constructed processes so that they become standard.

We note that, if X^0 is an m_0-symmetric diffusion on E_0, then the present conditions **(A.2), (A.3)** are the same as the conditions **(A.1), (A.2), (A.3)** assumed in [16, §4], while the present **(A.4)** is weaker than **(A.4)** of [16, §4] as is noted in the paragraph below **(A.4)**. Therefore the results of this paper extend the construction problem treated in [16, §4] to a more general case. However we shall proceed along the same line as was laid in [16, §4].

In Theorem 5.17. at the end of this section, we shall present a stronger variant **(A.2)'** of the condition **(A.2)** and prove using a time change argument that, under the conditions **(A.1), (A.2)', (A.4)** and additionally **(A.5)** in non-symmetric case and **(A.6)** in non-diffusion case, the integrability condition **(A.3)** holds automatically and therefore can be dropped.

As is shown in [5, Lemma 3.1], the measure $\widehat{\varphi}\cdot m_0$ is X^0-purely excessive and accordingly there exists a unique entrance law $\{\mu_t\}_{t>0}$ for X^0 characterized by

$$\widehat{\varphi}\cdot m_0 = \int_0^\infty \mu_t dt. \tag{5.7}$$

Analogously there exists a unique $\widehat{X}^0$-entrance law $\{\widehat{\mu}_t\}_{t>0}$ characterized by

$$\varphi\cdot m_0 = \int_0^\infty \widehat{\mu}_t dt. \tag{5.8}$$

Further by [5, Lemma 3.1], the Laplace transforms of $\mu_t,\ \widehat{\mu}_t$ satisfy

$$\int_0^\infty e^{-\alpha t}\langle \mu_t, f\rangle dt = (\widehat{u}_\alpha, f) \quad \text{and} \quad \int_0^\infty e^{-\alpha t}\langle \widehat{\mu}_t, f\rangle dt = (u_\alpha, f) \tag{5.9}$$

for every $\alpha > 0$ and $f\in\mathcal{B}^+(E_0)$. On account of the assumption **(A.3)**, we then have that for every $t>0$,

$$\mu_t(E_0)<\infty,\ \ \widehat{\mu}_t(E_0)<\infty, \quad \text{and} \quad \int_0^1 \mu_s(E_0)ds<\infty,\ \int_0^1 \widehat{\mu}_s(E_0)ds<\infty. \tag{5.10}$$

We now introduce the spaces W' and W of excursions by

$$\begin{aligned} W' &= \{w : \text{a cadlag function from } (0,\zeta(w)) \text{ to } E_0 \text{ for some } \zeta(w)\in(0,\infty]\}, \\ W &= \left\{w\in W' : \text{if } \zeta(w)<\infty \text{ then } w(\zeta(w)-) := \lim_{t\uparrow\zeta(\omega)} w(t)\in\{a,\Delta\}\right\}. \end{aligned} \tag{5.11}$$

We call $\zeta(w)$ the *terminal time* of the excursion w.

We are concerned with a measure $\mathbf{n}$ on the space W specified in terms of the entrance law $\{\mu_t,\ t>0\}$ and the transition semigroup $\{P_t^0, t \geq 0\}$ of X^0 by

$$\int_W f_1(w(t_1))f_2(w(t_2))\cdots f_n(w(t_n))\mathbf{n}(dw) = \mathbf{E}_{\mu_{t_1}}\left[\prod_{k=1}^n f_k(X^0_{t_k - t_1})\right]$$
$$= \mu_{t_1} f_1 P^0_{t_2-t_1} f_2 \cdots P^0_{t_{n-1}-t_{n-2}} f_{n-1} P^0_{t_n - t_{n-1}} f_n, \tag{5.12}$$

for any $0 < t_1 < t_2 < \cdots < t_n$, $f_1, f_2, \cdots, f_n \in B_b(E_0)$. Here, we use the convention that $w \in W$ satisfies $w(t) := \Delta$ for $w \in W$ and $t \geq \zeta(w)$, and any function f on E_0 is extended to $E_0 \cup \Delta$ by setting $f(\Delta) = 0$. Further, on the right hand side of (5.12), we employ an abbreviated notation for the repeated operations

$$\mu_{t_1}\left(f_1 P^0_{t_2-t_1}\left(f_2 \cdots P^0_{t_{n-1}-t_{n-2}}\left(f_{n-1} P^0_{t_n-t_{n-1}} f_n\right)\cdots\right)\right).$$

Proposition 5.1. *There exists a unique measure* $\mathbf{n}$ *on the space* W *satisfying* (5.12).

Proof. Let $\mathbf{n}$ be the Kuznetsov measure on W' uniquely associated with the transition semigroup $\{P_t^0, t \geq 0\}$ and the entrance rule $\{\eta_u, u \in \mathbb{R}\}$ defined by

$$\eta_u = 0 \quad \text{for } u \leq 0 \qquad \text{and} \qquad \eta_u = \mu_u \quad \text{for } u > 0,$$

as is constructed in [8, Chap. XIX, §9] for a right semigroup. Because of the present choice of the entrance rule, it holds that the random birth time α for the Kuznetsov process is identically 0 (cf. [20, p. 54]).

On account of the assumption **(A.2)** for the standard process X^0 on E_0, the same method of the construction of the Kuznetsov measure as in [8, Chap. XIX, §9] works in proving that $\mathbf{n}$ is carried on the space W and satisfies (5.12). □

We call $\mathbf{n}$ the *excursion law* associated with the entrance law $\{\mu_t\}$ for X^0. It is strong Markov with respect to the transition semigroup $\{P_t^0, t \geq 0\}$ of X^0. Analogously we can introduce the *excursion law* $\widehat{\mathbf{n}}$ on the space W associated with the entrance law $\widehat{\mu}_t$ for $\widehat{X}^0$.

We split the space W of excursions into two parts:

$$W^+ := \{w \in W : \zeta(w) < \infty \text{ and } w(\zeta-) = a\} \text{ and } W^- := W \setminus W^+. \tag{5.13}$$

For $w \in W^+$, we define time-reversed path $\widehat{w} \in W'$ by

$$\widehat{w}(t) := w((\zeta - t)-) = \lim_{t' \uparrow t} w(\zeta - t'), \qquad 0 < t < \zeta. \tag{5.14}$$

The next lemma asserts that the excursion laws $\mathbf{n}$ and $\widehat{\mathbf{n}}$ restricted to W^+ are interchangeable under this time reversion.

Lemma 5.2. *For any* $t_k > 0$ *and* $f_k \in \mathcal{B}_b(S_0)$, $(1 \le k \le n)$,

$$\mathbf{n}\left\{\prod_{k=1}^{n} f_k(w(t_1+\cdots+t_k));\ W^+\right\} = \mu_{t_1} f_1 P^0_{t_2} f_2 \cdots P^0_{t_{n-1}} f_{n-1} P^0_{t_n} f_n \varphi, \tag{5.15}$$

$$\mathbf{n}\left\{\prod_{k=1}^{n} f_k(w(t_1+\cdots+t_k));\ W^+\right\} = \widehat{\mathbf{n}}\left\{\prod_{k=1}^{n} f_k(\widehat{w}(t_1+\cdots+t_k));\ W^+\right\}. \tag{5.16}$$

Proof. (5.15) readily follows from (5.12) and the Markov property of $\mathbf{n}$. As for (5.16), we observe that, for $\alpha_1, \cdots, \alpha_n > 0$,

$$\begin{aligned}\int_0^\infty \cdots \int_0^\infty e^{-\sum_{k=1}^n \alpha_k t_k} \mathbf{n}\left\{\prod_{k=1}^{n} f_k(w(t_1+\cdots+t_k)); W^+\right\} dt_1\cdots dt_n \\ = \mathbf{n}\{F(w); \zeta<\infty,\ w(\zeta-)=a\},\end{aligned} \tag{5.17}$$

where, with $t+0 := 0$,

$$F(w) = n! \int_{0<t_1<\cdots<t_n<\zeta} \prod_{k=1}^{n} \left\{e^{-\alpha_k(t_k-t_{k-1})} f_k(w(t_k))\right\} dt_1\cdots dt_n.$$

Hence, for (5.16), it suffices to prove for $f_k \in C_b(E_0)$, $1 \le k \le n$,

$$\mathbf{n}\{F(w); \zeta<\infty,\ w(\zeta-)=a\} = \widehat{\mathbf{n}}\{F(\widehat{w}); \zeta<\infty,\ w(\zeta-)=a\}. \tag{5.18}$$

Changing of variables $\zeta - t_k = s_k$ for $0 \le k \le n$ in the following expression

$$F(\widehat{w}) = n! \int_{0<t_1<\cdots<t_n<\zeta} \prod_{k=1}^{n} \left\{e^{-\alpha_k(t_k-t_{k-1})} f_k(w((\zeta-t_k)-))\right\} dt_1\cdots dt_n,$$

where $t_0 := 0$, and noting that

$$s_0 = \zeta \ \text{ and }\ 0<t_1<\cdots<t_n<\zeta \ \text{ if and only if }\ 0<s_n<\cdots<s_1<\zeta,$$

we obtain

$$\begin{aligned}F(\widehat{w}) &= n! \int_{0<s_n<\cdots<s_1<\zeta} \prod_{k=1}^{n} \left\{e^{-\alpha_k(s_{k-1}-s_k)} f_k(w(s_k))\right\} ds_1\cdots ds_n \\ &= n! \int_{0<s_n<\cdots<s_1<\infty} \varGamma_{s_1\cdots s_n}(w) ds_1\cdots ds_n,\end{aligned}$$

where

$$\varGamma_{s_1\cdots s_n}(w) = \prod_{k=2}^{n} \left\{e^{-\alpha_k(s_{k-1}-s_k)} f_k(w(s_k))\right\} \cdot e^{-\alpha_1(\zeta-s_1)} f_1(w(s_1)) 1_{(0,\zeta)}(s_1).$$

On the other hand, we get from (5.10) and the Markov property of $\widehat{\mathbf{n}}$ that

$$
\begin{aligned}
&\widehat{\mathbf{n}}\{\Gamma_{s_1 s_2 \cdots s_n}(w); \zeta < \infty,\ w(\zeta-) = a\} \\
&\quad = \widehat{\mathbf{n}}\Big\{ f_n(w(s_n)) e^{-\alpha_n(s_{n-1}-s_n)} \cdots f_2(w(s_2)) e^{-\alpha_2(s_1-s_2)} \\
&\qquad f_1(w(s_1)) u_{\alpha_1}(w(s_1)); s_1 < \zeta \Big\} \\
&\quad = e^{-\sum_{k=2}^{n} \alpha_k (s_{k-1}-s_k} \widehat{\mu}_{s_n} f_n \widehat{P}^0_{s_{n-1}-s_n} f_{n-1} \widehat{P}^0_{s_{n-2}-s_{n-1}} f_{n-1} \\
&\qquad \cdots \widehat{P}^0_{s_2-s_3} f_2 \widehat{P}^0_{s_1-s_2} f_1 \widehat{u}_{\alpha_1}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\widehat{\mathbf{n}}\{F(\widehat{w}); \zeta < \infty, w(\zeta-) = a\} \\
&\quad = \int_0^\infty ds_n \widehat{\mu}_{s_n} f_n \widehat{G}^0_{\alpha_n} f_{n-1} \widehat{G}^0_{\alpha_{n-1}} \cdots f_3 \widehat{G}^0_{\alpha_3} f_2 \widehat{G}^0_{\alpha_2} f_1 \widehat{u}_{\alpha_1}.
\end{aligned}
$$

In view of (5.8), the weak duality **(A.1)**, (5.15) and (5.17), we arrive at

$$
\begin{aligned}
&\widehat{\mathbf{n}}\{F(\widehat{w});\ \zeta < \infty, w(\zeta-) = a\} \\
&\quad = \Big\langle \varphi \cdot m_0,\ f_n \widehat{G}^0_{\alpha_n} f_{n-1} \widehat{G}^0_{\alpha_{n-1}} \cdots f_3 \widehat{G}^0_{\alpha_3} f_2 \widehat{G}^0_{\alpha_2} f_1 \widehat{u}_{\alpha_1} \Big\rangle \\
&\quad = \Big(f_n \varphi,\ \widehat{G}^0_{\alpha_n} f_{n-1} \widehat{G}^0_{\alpha_{n-1}} \cdots f_3 \widehat{G}^0_{\alpha_3} f_2 \widehat{G}^0_{\alpha_2} f_1 \widehat{u}_{\alpha_1} \Big) \\
&\quad = \big(f_1 G^0_{\alpha_2} f_2 G^0_{\alpha_3} f_3 \cdots G^0_{\alpha_n} f_n \varphi,\ \widehat{u}_{\alpha_1} \big) \\
&\quad = \int_0^\infty e^{-\alpha_1 t_1} \mu_{t_1} f_1 G^0_{\alpha_2} f_2 G^0_{\alpha_3} f_3 \cdots G^0_{\alpha_n} f_n \varphi dt_1 \\
&\quad = \mathbf{n}\{F(w);\ \zeta < \infty \text{ and } w(\zeta-) = a\},
\end{aligned}
$$

the desired identity (5.18). This establishes (5.16). □

Next we define

$$
W_a := \{w \in W :\ w(0+) := \lim_{t \downarrow 0} w(t) = a\}. \tag{5.19}
$$

Lemma 5.3. $\mathbf{n}\{W \setminus W_a\} = 0$ *and* $\widehat{\mathbf{n}}\{W \setminus W_a\} = 0$.

Proof. The preceding lemma implies that

$$
\begin{aligned}
\mathbf{n}\left\{W^+ \setminus W_a\right\} &= \mathbf{n}\left\{W^+ \cap (w(0+) = a)^c\right\} \\
&= \widehat{\mathbf{n}}\left\{W^+ \cap (\widehat{w}(0+) = a)^c\right\} \\
&= \widehat{\mathbf{n}}\left\{W^+ \cap (w(\zeta-) = a)^c\right\} \\
&= 0.
\end{aligned}
$$

We then have for each $t > 0$

$$\mathbf{n}\{\varphi(w(t)); (\zeta > t) \cap (w(0+) = a)^c\} = \mathbf{n}\big\{(W^+ \setminus W_a) \cap (\zeta > t)\big\} = 0.$$

As $\varphi(x) > 0$ for every $x \in E_0$ by the assumption **(A.2)**. we conclude that

$$\mathbf{n}\{(W \setminus W_a) \cap (\zeta > t)\} = 0 \qquad \text{for every } t > 0,$$

and therefore $\mathbf{n}\{(W \setminus W_a)\} = 0$ after letting $t \downarrow 0$. The same property of $\widehat{\mathbf{n}}$ can be shown analogously. □

Lemma 5.4. *For any neighborhood U of a in E, define*

$$\tau_U(w) = \inf\{t > 0 : w(t) \notin U\} \qquad \textit{for } w \in W.$$

Then

$$\mathbf{n}\{\tau_U < \zeta\} < \infty \qquad \textit{and} \qquad \widehat{\mathbf{n}}\{\tau_U < \zeta\} < \infty.$$

Proof. We only give a proof for $\mathbf{n}$. Let V be any neighborhood of a in E. It suffices to show

$$\mathbf{n}(\tau_U < \zeta) < \infty$$

for some neighborhood U of a with $U \subset V$. We choose such U as follows. Let us fix a relatively compact open neighborhood U_1 of a in E. When X^0 is a diffusion, we put $U = V \cap U_1$. When X^0 is not a diffusion and the second condition of **(A.5)** is fulfilled, we take U_2 in the condition for U_1 and put $U = V \cap U_2$.

By virtue of the relation

$$\varphi - u_1 = G_1^0\varphi = G^0 u_1$$

and the assumption **(A.4)**, the function $G_1^0\varphi$ is lower semi-continuous on E_0. Furthermore, since φ is X^0-excessive and strictly positive by assumption **(A.2)**, $G_1^0\varphi$ is moreover strictly positive on E_0. As $\overline{U_1}$ is compact in E,

$$\delta := \frac{1}{2} \inf_{x \in \overline{U}_1 \setminus U} G_1^0\varphi(x) > 0. \tag{5.20}$$

Since $G_1^0\varphi(x) = \displaystyle\int_0^\infty e^{-t}\,\mathbf{P}_x\left(t < \zeta^0 < \infty, X^0_{\zeta^0-} = a\right) dt$, we have

$$\mathbf{P}_x\left(\delta < \zeta^0 < \infty, X^0_{\zeta^0-} = a\right) > \delta \qquad \text{for every } x \in \overline{U}_1 \setminus U. \tag{5.21}$$

We shall use the notation τ_U not only for $w \in W$ but also for the sample path of the Markov process X^0. Using the preceding lemma, we have

$$\mathbf{n}\left\{\tau_U < \zeta^0\right\} = \lim_{\epsilon \downarrow 0} \mathbf{n}\left\{\epsilon < \tau_U < \zeta^0\right\} = \lim_{\epsilon \downarrow 0} \int_U \mu_\epsilon(dx) \mathbf{P}_x^0\left\{\tau_U < \zeta^0\right\} = I + II,$$

where

$$I := \lim_{\epsilon\downarrow 0}\int_U \mu_\epsilon(dx)\mathbf{P}^0_x\left(\tau_U<\zeta^0,\ X^0_{\tau_U}\in \overline{U}_1\setminus U\right),$$
$$II := \lim_{\epsilon\downarrow 0}\int_U \mu_\epsilon(dx)\mathbf{P}^0_x\left(\tau_U<\zeta^0,\ X^0_{\tau_U}\in E_0\setminus U_1\right).$$

From (5.21) and (5.10), it follows that

$$\begin{aligned}
I &\le \overline{\lim}_{\epsilon\downarrow 0}\int_U \mu_\epsilon(dx)\mathbf{E}^0_x\Big[\delta^{-1}\mathbf{P}_{X^0_{\tau_U}}\left(\delta<\zeta^0<\infty, X^0_{\zeta^0-}=a\right);\\
&\qquad\qquad \tau_U<\zeta^0,\ X^0_{\tau_U}\in\overline{U}_1\setminus U\Big]\\
&\le \delta^{-1}\lim_{\epsilon\downarrow 0}\int_{E_0}\mu_\epsilon(dx)\mathbf{P}^0_x(\delta<\zeta^0<\infty, X^0_{\zeta^0-}=a)\\
&\le \delta^{-1}\lim_{\epsilon\downarrow 0}\int_{E_0}\mu_\epsilon(dx)\mathbf{P}^0_x(\delta<\zeta^0)\\
&= \delta^{-1}\lim_{\epsilon\downarrow 0}\mu_{\epsilon+\delta}(E_0)\\
&\le \delta^{-1}\mu_\delta(E_0)<\infty.
\end{aligned}$$

II may not vanish when X^0 is not a diffusion. In this case, let $(N(x,dy),H)$ be the Lévy system of X^0 appearing in the condition **(A.5)**. Note that

$$\begin{aligned}
II &= \lim_{\epsilon\downarrow 0}\int_U \mu_\epsilon(dx)\mathbf{E}^0_x\left[\int_0^{\tau_U} 1_U(X^0_s)N(X^0_s,E\setminus U_1)dH_s\right]\\
&\le \lim_{\epsilon\downarrow 0}\int_{E_0} \mu_\epsilon(dx)\mathbf{E}^0_x\left[\int_0^{\infty} 1_U(X^0_s)N(X^0_s,E\setminus U_1)dH_s\right]\\
&= \lim_{\epsilon\downarrow 0}\int_{E_0}\mu_\varepsilon(dx)\,G^0\mu_K(x)
\end{aligned}$$

where $\mu_K(dx) := 1_U(x)N(x,E_0\setminus U_1)\mu_H(dx)$ is the Revuz measure of the PCAF of X^0

$$K_t := \int_0^t 1_U(X^0_s)N(X^0_s,E\setminus U_1)dH_s, \qquad t\ge 0,$$

and $G^0\mu_K(x) := \mathbf{E}_x\left[K_\infty\right]$. Note that μ_K is a finite measure on E_0 by assumption **(A.5)**. For $\alpha>0$ and $x\in E_0$, we define

$$G^0_\alpha\mu_K(x) := \mathbf{E}_x\left[\int_0^\infty e^{-\alpha t}dK_t\right].$$

Observe that $\alpha G^0_\alpha G^0 \mu_K$ increases to $G^0\mu_K$ as $\alpha \uparrow \infty$. We have, by (5.7), the identity $G^0_\alpha G^0 \mu_K = G^0 G^0_\alpha \mu_K$ and [21, (9.3)],

$$\begin{aligned}
\int_{E_0} \mu_\varepsilon(dx)\, G^0\mu_K(x) &= \lim_{\alpha\to\infty} \alpha \int_{E_0} \mu_\varepsilon(dx)\, G^0 G^0_\alpha \mu_K(x) \\
&= \lim_{\alpha\to\infty} \int_0^\infty \langle \mu_\varepsilon P^0_t,\, \alpha G^0_\alpha \mu_K\rangle dt \\
&\le \lim_{\alpha\to\infty} \int_0^\infty \langle \mu_t,\, \alpha G^0_\alpha \mu_K\rangle dt = \lim_{\alpha\to\infty} \langle \widehat{\varphi}\cdot m_0,\, \alpha G^0_\alpha \mu_K\rangle \\
&= \lim_{\alpha\to\infty} \langle \alpha \widehat{G}^0_\alpha \widehat{\varphi},\, \mu_K\rangle = \int_{E_0} \widehat{\varphi}(x)\mu_K(dx) \le \mu_K(E_0) < \infty.
\end{aligned}$$

Hence we get the desired finiteness of II.

When the first condition of **(A.5)** is fulfilled, the first half of the preceding proof is enough if we replace $U,\ \overline{U}_1$ with $V,\ E_0$ respectively. □

Lemma 5.5. $\mathbf{n}(W^-) = L^0(\widehat{\varphi}\cdot m_0, 1-\varphi) < \infty$ *and*
$\widehat{\mathbf{n}}(W^-) = \widehat{L}^0(\varphi\cdot m_0, 1-\widehat{\varphi}) < \infty.$

Proof. Since $\mathbf{n}\left(\zeta > t;\, W^-\right) = \langle \mu_t, 1-\varphi\rangle$, the first identity follows from [5, Lemma 3.1] by letting $t \downarrow 0$. Take a relatively compact neighborhood U of a in E. Since $a \in E$ and Δ is a one-point compactification of E, we have

$$\{\zeta < \infty \text{ and } w(\zeta-) = \Delta\} \subset \{\tau_U < \zeta\}. \tag{5.22}$$

Hence for any $t > 0$,

$$\begin{aligned}
\mathbf{n}\left(W^-\right) &= \mathbf{n}\{\zeta < \infty, w(\zeta-) = \Delta\} + \mathbf{n}\{\zeta = \infty\} \\
&\le \mathbf{n}\{\tau_U < \zeta\} + \mathbf{n}\{\zeta > t\} \\
&= \mathbf{n}\{\tau_U < \zeta\} + \mu_t(E_0),
\end{aligned}$$

which is finite by Lemma 5.4. and (5.10). The second assertion can be shown similarly. □

5.2 Poisson Point Processes on $W_a \cup \{\partial\}$ and a New Process X^a

By Lemma 5.3., the excursion law $\mathbf{n}$ is concentrated on the space W_a defined by (5.19). In correspondence to (5.13), we define

$$W_a^+ := \left\{ w \in W^+ :\ \lim_{t\downarrow 0} w(t) = a \right\} \quad \text{and} \quad W_a^- := \left\{ w \in W^- :\ \lim_{t\downarrow 0} w(t) = a \right\},$$

so that $W_a = W_a^+ + W_a^-$. In the sequel however, we shall employ slightly modified but equivalent definitions of those spaces by extending each w from an E_0-valued excursion to E-valued one as follows:

$$\begin{aligned} W_a = \{w:\ & \text{a cadlag function from } [0,\zeta(w)) \text{ to } E \text{ for some } \zeta(w) \in (0,\infty] \\ & \text{with } w(0) = a,\ w(t) \in E_0 \text{ for } t \in (0,\zeta(w)) \\ & \text{and } w(\zeta(w)-) \in \{a, \Delta\} \text{ if } \zeta(w) < \infty\}. \end{aligned} \tag{5.23}$$

Any $w \in W_a$ with the properties $\zeta(w) < \infty$ and $w(\zeta(w)-) = a$ will be regarded to be a cadlag function from $[0,\zeta(w)]$ to E by setting $w(\zeta(w)) = a$. We further define

$$\begin{aligned} W_a^+ := \{w:\ & \text{a cadlag function from } [0,\zeta(w)] \text{ to } E \text{ for some } \zeta(w) \in (0,\infty) \\ & \text{with } w(t) \in E_0 \text{ for } t \in (0,\zeta(w)) \text{ and } w(0) = w(\zeta(w)) = w(\zeta(w)-) = a\}, \\ W_a^- := W_a \setminus W_a^+. & \end{aligned}$$

The excursion law $\mathbf{n}$ will be considered to be a measure on W_a defined by (5.23). Let us add an extra point ∂ to W_a which represents a specific path constantly equal to Δ. Fix a non-negative constant δ_0 and we assign a point mass δ_0 to $\{\partial\}$ and extend the measure $\mathbf{n}$ on W_a to a measure $\bar{\mathbf{n}}$ on $W_a \cup \{\partial\}$ by

$$\bar{\mathbf{n}}(\Lambda) = \begin{cases} \mathbf{n}(\Lambda) & \text{if} \quad \Lambda \subset W_a \\ \mathbf{n}(\Lambda \cap W_a) + \delta_0 & \text{if} \quad \partial \in \Lambda \end{cases} \tag{5.24}$$

for $\Lambda \subset W_a \cup \{\partial\}$. The restrictions of $\bar{\mathbf{n}}$ to W_a^+ and $W_a^- \cup \{\partial\}$ are denoted by $\mathbf{n}^+$ and $\bar{\mathbf{n}}^-$, respectively.

Let $\mathbf{p} = \{\mathbf{p}_s : s \in \mathcal{D}_{\mathbf{p}}\}$ be a Poisson point process on $W_a \cup \{\partial\}$ with characteristic measure $\bar{\mathbf{n}}$ defined on an appropriate probability space $(\Omega_a, \mathbf{P})$. We then let $\mathbf{p}^+$ and $\mathbf{p}^-$ be the point processes obtained from $\mathbf{p}$ by restricting to W_a^+ and $W_a^- \cup \{\partial\}$ respectively, that is,

$$\mathcal{D}_{\mathbf{p}^+} = \{s \in \mathcal{D}_{\mathbf{p}} : \mathbf{p}_s \in W_a^+\} \qquad \text{and} \qquad \mathcal{D}_{\mathbf{p}^-} = \{s \in \mathcal{D}_{\mathbf{p}} : \mathbf{p}_s \in W_a^- \cup \{\partial\}\}. \tag{5.25}$$

Then $\{\mathbf{p}_s^+, s > 0\}$, $\{\mathbf{p}_s^-, s > 0\}$ are mutually independent Poisson point processes on W_a^+ and $W_a^- \cup \{\partial\}$ with characteristic measures $\mathbf{n}^+$ and $\bar{\mathbf{n}}^-$, respectively. Clearly,

$$\mathbf{p}_s = \mathbf{p}_s^+ \ + \ \mathbf{p}_s^- .$$

Recall that $\zeta(\mathbf{p}_r^+)$ denotes the terminal time of the excursion $\mathbf{p}_r^+$. We define

$$J(s) := \sum_{r \le s} \zeta(\mathbf{p}_r^+) \quad \text{for } s > 0 \qquad \text{and} \qquad J(0) := 0. \tag{5.26}$$

Lemma 5.6. (i) $J(s) < \infty$ a.s. *for* $s > 0$.

(ii) $\{J(s)\}_{s \ge 0}$ *is a subordinator with*

$$\mathbf{E}\left[e^{-\alpha J(s)}\right] = \exp\left(-\alpha(\widehat{u}_\alpha, \varphi)s\right). \tag{5.27}$$

Proof. (i) We write $J(s)$ as $J(s) = I + II$ with

$$I := \sum_{r\le s,\ \zeta(\mathbf{p}_r^+)\le 1} \zeta(\mathbf{p}_r^+) \qquad \text{and} \qquad II := \sum_{r\le s,\ \zeta(\mathbf{p}_r^+)>1} \zeta(\mathbf{p}_r^+).$$

Since $\mathbf{n}^+(\zeta > 1) \le \mu_1(E_0) < \infty$ by (5.10), r in the sum II is finite a.s. and hence $II < \infty$ a.s. On the other hand,

$$\begin{aligned}\mathbf{E}(I) &= s\,\mathbf{n}^+(\zeta;\ \zeta\le 1) \le s\,\mathbf{n}^+(\zeta\wedge 1)\\ &= s\,\mathbf{n}^+\left\{\int_0^1 1_{(0,\zeta)}(t)dt\right\} = s\int_0^1 \mathbf{n}^+(\zeta > t)dt \le s\int_0^1 \mu_t(E_0)dt,\end{aligned}$$

which is finite by (5.10). Hence $I < \infty$ a.s.

(ii) This can be shown exactly in the same way as that for (4.18) in the proof of Theorem 4.2. by using the identity (5.9).

□

In view of Lemma 5.4. and Lemma 5.6., by subtracting a **P**-negligible set from Ω_a if necessary, we may and do assume that the next three properties hold for every $\omega \in \Omega_a$:

$$J(s) < \infty \qquad \text{for every } s > 0, \tag{5.28}$$

$$\lim_{s\to\infty} J(s) = \infty, \tag{5.29}$$

and, for any finite interval $I \subset (0,\infty)$ and any neighborhood U of a in E,

$$\left\{s \in I : \tau_U(\mathbf{p}_s^+) < \zeta(\mathbf{p}_s^+)\right\} \text{ is a finite set.} \tag{5.30}$$

Let T be the first time of occurrence of the point process $\{\mathbf{p}_s^-, s > 0\}$, namely,

$$T = \inf\{s > 0 : s \in \mathcal{D}_{\mathbf{p}^-}\}. \tag{5.31}$$

Since by Lemma 5.5.

$$\bar{\mathbf{n}}^-(W_a^- \cup \{\partial\}) = \mathbf{n}(W_a^-) + \delta_0 = L^0(\widehat{\varphi}\cdot m_0, 1-\varphi) + \delta_0 < \infty,$$

we see that T and $\mathbf{p}_T^-$ are independent and

$$\mathbf{P}(T > t) = e^{-(L(\widehat{\varphi}\cdot m_0,1-\varphi)+\delta_0)t} \quad \text{and} \quad \mathbf{p}_T^- \overset{dist}{=} (L(\widehat{\varphi}\cdot m_0, 1-\varphi) + \delta_0)^{-1}\,\bar{\mathbf{n}}^-. \tag{5.32}$$

We are now in a position to produce a new process $X = \{X_t,\ t \ge 0\}$ out of the point processes of excursions $\mathbf{p}^\pm$.

(i) For $0 \le t < J(T-)$, there is an $s \ge 0$ such that

$$J(s-) \le t \le J(s).$$

We define

$$X_t^a := \begin{cases} \mathbf{p}_s^+(t - J(s-)) & \text{if } J(s) - J(s-) > 0, \\ a & \text{if } J(s) - J(s-) = 0. \end{cases} \tag{5.33}$$

It is easy to see that X^a is well-defined.

(ii) If $\mathbf{p}_T^- \in W_a^-$, then we define

$$\zeta_\omega := J(T-) + \zeta(\mathbf{p}_T^-) \quad \text{and} \quad X_t^a := \mathbf{p}_T^-(t - J(T-)) \quad \text{for} \quad J(T-) \le t < \zeta_\omega. \tag{5.34}$$

(iii) If $\mathbf{p}_T^- = \partial$, then we define

$$\zeta_\omega := J(T-). \tag{5.35}$$

In this way, the E-valued path

$$\{X_t^a,\ 0 \le t < \zeta_\omega\}$$

is well-defined and enjoys the following properties:

$$\begin{aligned} &X_0^a = a, \ \text{ is cadlag in } t \in [0, \zeta_\omega) \text{ and continuous when } X_t^a = a, \\ &\text{and } \ X_{\zeta_\omega -}^a \in \{a, \Delta\} \ \text{ whenever } \ \zeta_\omega < \infty. \end{aligned} \tag{5.36}$$

The second property is a consequence of (5.30). If $\mathbf{p}_T^- \in W_a^-$ and $\zeta_\omega < \infty$, then $X_{\zeta_\omega -}^a = \Delta$. If $T < \infty$, $\mathbf{p}_T^- = \partial$, then $T \notin \mathcal{D}_{\mathbf{p}^+}$ and hence by (5.35), we have $X_{\zeta_\omega -}^a = X_{J(T-)-}^a = a$. Thus the third property holds.

For this process $X^a = \{X_t^a, 0 \le t < \zeta_\omega, \mathbf{P}\}$, let us put

$$G_\alpha f(a) = \mathbf{E}\left[\int_0^{\zeta_\omega} e^{-\alpha t} f(X_t^a) dt\right], \quad \alpha > 0,\ f \in \mathcal{B}(E). \tag{5.37}$$

Similarly we assign a non-negative mass $\widehat{\delta}_0$ to the death path ∂ and extend the measure $\widehat{\mathbf{n}}$ on W_a to a measure $\overline{\widehat{\mathbf{n}}}$ on $W_a \cup \{\partial\}$. By making use of the Poisson point process $\widehat{\mathbf{p}}$ on $W_a \cup \{\partial\}$ with the characteristic measure $\overline{\widehat{\mathbf{n}}}$ on a certain probability space $(\widehat{\Omega}_a, \widehat{\mathbf{P}})$, we can construct a cadlag process $\{\widehat{X}_t^a, 0 \le t < \widehat{\zeta}_{\hat\omega}, \widehat{\mathbf{P}})$ on E quite analogously. The corresponding quantity to (5.37) is denoted by $\widehat{G}_\alpha f(a)$. We can then obtain the first identity of the next proposition exactly in the same way as in the proof of Theorem 4.2. using (5.9), Lemma 5.6. and (5.32). An analogous consideration gives the second identity.

Proposition 5.7. *For $\alpha > 0$ and $f \in \mathcal{B}(E)$, it holds that*

$$G_\alpha f(a) = \frac{(\widehat{u}_\alpha, f)}{\alpha(\widehat{u}_\alpha, \varphi) + L^{(0)}(\widehat{\varphi} \cdot m_0, 1 - \varphi) + \delta_0}. \tag{5.38}$$

$$\widehat{G}_\alpha f(a) = \frac{(u_\alpha, f)}{\alpha(u_\alpha, \widehat{\varphi}) + \widehat{L}^0(\varphi \cdot m_0, 1 - \widehat{\varphi}) + \widehat{\delta}_0}. \tag{5.39}$$

For $\alpha > 0$ and $f \in \mathcal{B}(E)$, define

$$G_\alpha f(x) := G^0_\alpha f(x) + G_\alpha f(a) u_\alpha(x) \qquad \text{for } x \in E_0, \tag{5.40}$$

$$\widehat{G}_\alpha f(x) := \widehat{G}^0_\alpha f(x) + \widehat{G}_\alpha f(a) \widehat{u}_\alpha(x) \qquad \text{for } x \in E_0. \tag{5.41}$$

Lemma 5.8. *$\{G_\alpha, \alpha > 0\}$ and $\{\widehat{G}_\alpha, \alpha > 0\}$ are sub-Markovian resolvents on E. They are in weak duality with respect to m if and only if*

$$L^{(0)}(\widehat{\varphi} \cdot m_0, 1 - \varphi) + \delta_0 = \widehat{L}^{(0)}(\varphi \cdot m_0, 1 - \widehat{\varphi}) + \widehat{\delta}_0. \tag{5.42}$$

Proof. By making use of the resolvent equations for G^0_α, $\widehat{G}^0_\alpha$, their weak duality with respect to m_0 and the equations

$$u_\alpha(x) - u_\beta(x) + (\alpha - \beta) G^0_\alpha u_\beta(x) = 0, \quad \alpha, \beta > 0,\ x \in E_0, \tag{5.43}$$

$$\widehat{u}_\alpha(x) - \widehat{u}_\beta(x) + (\alpha - \beta) \widehat{G}^0_\alpha \widehat{u}_\beta(x) = 0, \quad \alpha, \beta > 0,\ x \in E_0, \tag{5.44}$$

we can easily check the resolvent equations

$$G_\alpha f(x) - G_\beta f(x) + (\alpha - \beta) G_\alpha G_\beta f(x) = 0, \quad x \in E,$$

$$\widehat{G}_\alpha f(x) - \widehat{G}_\beta f(x) + (\alpha - \beta) \widehat{G}_\alpha \widehat{G}_\beta f(x) = 0, \quad x \in E.$$

Moreover we get as in [16, Lemma 2.1] that

$$\begin{aligned}\alpha G_\alpha 1(x) &= \alpha G^0_\alpha 1(x) + u_\alpha(x) \frac{\alpha(\widehat{u}_\alpha, \varphi) + \alpha(\widehat{u}_\alpha, 1 - \varphi)}{\alpha(\widehat{u}_\alpha, \varphi) + L(\widehat{\varphi} \cdot m_0, 1 - \varphi) + \delta_0} \\ &\le 1 - u_\alpha(x) + u_\alpha(x) = 1, \quad x \in E_0,\end{aligned}$$

and similarly, $\alpha G_\alpha 1(a) \le 1$.

The m-weak duality

$$\int_E \widehat{G}_\alpha f(x) g(x) m(dx) = \int_E f(x) G_\alpha g(x) m(dx), \quad f, g \in \mathcal{B}^+(E),$$

holds if and only if the denominators of the right hand sides of (5.38) and (5.39) coincide. Since $(\widehat{u}_\alpha, \varphi) = (u_\alpha, \widehat{\varphi})$ by the above equations for u_α, $\widehat{u}_\alpha$, we get the last conclusion. □

5.3 Regularity of Resolvent Along the Path of X^a

Let $\{U_n\}$ be a decreasing sequence of open neighborhoods of the point a in E such that $U_n \supset \overline{U}_{n+1}$ and $\bigcap_{n=1}^{\infty} U_n = \{a\}$. For $\alpha > 0$ and $0 < \rho < 1$, let

$$A = A_{\alpha,\rho} := \{x \in E_0 : u_\alpha(x) < \rho\}.$$

We then define

$$\sigma_n := \inf\{t > 0 : X_t^0 \in U_n \cap E_0\}, \quad \tau_n := \inf\{t > 0 : X_t^0 \in U_n \cap A\},$$

and $\sigma := \lim_{n\to\infty} \sigma_n$, with the convention that $\inf \emptyset = \infty$. The stopping time σ may be called the approaching time to a of X^0.

The next lemma can be proved exactly in the same way as the proof of [16, Lemma 4.7].

Lemma 5.9. *For any $\alpha > 0$, $\rho \in (0,1)$ and $x \in E_0$,*

$$\lim_{n\to\infty} \mathbf{P}_x^0 \{\tau_n < \sigma < \infty\} = 0. \tag{5.45}$$

Lemma 5.10. *The following are ture.*

(i) *For any $x \in E_0$, $\mathbf{P}_x^0$-a.s. on $\{\sigma < \infty\}$,*

$$\lim_{t\uparrow\sigma} u_\alpha(X_t^0) = 1 \quad \textit{for every } \alpha > 0. \tag{5.46}$$

(ii) $\mathbf{n}(\Lambda \cap W_a^+) = 0$ *where*

$$\Lambda = \left\{ w \in W_a : \exists\alpha > 0,\ \liminf_{t\uparrow\zeta} u_\alpha(w(t)) < 1 \right\}.$$

(iii) $\mathbf{n}(\widehat{\Lambda}) = 0$ *where*

$$\widehat{\Lambda} = \left\{ w \in W_a : \exists\alpha > 0,\ \liminf_{t\downarrow 0} \widehat{u}_\alpha(w(t)) < 1 \right\}.$$

Proof. Let $0 < \rho < 1$. If $\sigma < \infty$ and if $\underline{\lim}_{t\uparrow\sigma} u_\alpha(X_t^0) < \rho$, then for any small $\epsilon > 0$ there exists $t \in (\sigma - \epsilon, \sigma)$ such that $u_\alpha(X_t^0) < \rho$, and so $\tau_n < \sigma$ for all n. Therefore by the preceding lemma

$$\mathbf{P}_x^0 \left(\liminf_{t\uparrow\sigma} u_\alpha(X_t^0) < \rho,\ \sigma < \infty \right) = 0.$$

Since u_α is decreasing in α and ρ can be taken arbitrarily close to 1, we obtain (5.46).

(ii) follows from (i) as

$$\begin{aligned} \mathbf{n}(\Lambda \cap W_a^+) &= \lim_{\epsilon\downarrow 0} \mathbf{n}(\Lambda \cap W_a^+ \cap \{\epsilon < \zeta\}) \\ &= \lim_{\epsilon\downarrow 0} \int_{E_0} \mu_\epsilon(dx) \mathbf{P}_x^0 \left(\liminf_{t\uparrow\sigma} u_\alpha(X_t^0) < 1,\ \sigma < \infty \text{ for every } \alpha > 0 \right) = 0. \end{aligned}$$

(iii) Part (ii) combined with Lemma 5.2. and a similar reasoning as in the proof of Lemma 5.3. leads us to

$$\mathbf{n}(\widehat{\Lambda} \cap W_a^+) = \widehat{\mathbf{n}}(\{\widehat{w} \in \widehat{\Lambda}\} \cap W_a^+) = 0,$$

and also $\mathbf{n}(\widehat{\Lambda}) = 0$. □

Denote by Q^+ the set of all positive rational number and by $C_b(E)$ the space of all bounded continuous functions on E. Let us fix an arbitrary countable subfamily $\mathbf{L}$ of $C_b(E)$. We extend functions $u_\alpha(x)$ and $G^0_\alpha f(x)$ for $f \in C_b(E)$ to be functions on E by setting $u_\alpha(a) = 1$ and $G^0_\alpha f(a) = 0$ respectively. Functions $\widehat{u}_\alpha$ and $\widehat{G}^0_\alpha f$ are similarly extended to E.

As u_α and $G^0_\alpha f$ for a non-negative $f \in C_b(E)$ are α-excessive with respect to the process X^0, it is well-known (cf. [2]) that

$$u_\alpha(X^0_t),\ G^0_\alpha f(X^0_t) \text{ are right continuous in } t \in [0,\zeta) \quad \mathbf{P}^0_x\text{–a.s.} \quad x \in E_0. \tag{5.47}$$

Suppose that X^0 is m_0-symmetric: $X^0 = \widehat{X}^0$. Then $u_\alpha = \widehat{u}_\alpha$ and hence by Lemma 5.10.

$$\mathbf{n}\left(\liminf_{t\downarrow 0} u_\alpha(w(t)) < 1\right) = 0.$$

On account of (5.47) and the inequality $aG^0_\alpha 1(x) \le 1 - u_\alpha(x),\ x \in E$, after subtracting a suitable $\mathbf{n}$-negligible set from W_a if necessary, we may and do assume that, for any $f \in \mathbf{L},\ \alpha \in Q^+$,

$$\begin{aligned}&u_\alpha(w(t)) \text{ and } G^0_\alpha f(w(t)) \text{ are right continuous in } t \in [0,\zeta) \text{ for } w \in W_a,\\ &u_\alpha(w(\zeta-)) = 1\ , G^0_\alpha f(w(\zeta-)) = 0, \text{ for } w \in W^+_a.\end{aligned} \tag{5.48}$$

When X^0 is non-symmetric, $u_\alpha \neq \widehat{u}_\alpha$ and the above argument does not work. However, since we have assumed in this non-symmetric case the condition **(A.5)**, the above property (5.48) holds by Lemma 5.3.

Lemma 5.11. *Let* $0 < \rho < 1$ *and set, for* $\alpha > 0$,

$$\widetilde{W}_\rho = \left\{ w \in W^+_a :\ \sup_{0\le t\le \zeta} \{1 - u_\alpha(w(t))\} > \rho \right\}.$$

Then $\mathbf{n}^+(\widetilde{W}_\rho) < \infty$.

Proof. Define $\delta := -\frac{1}{\alpha}\ \log(1 - \frac{\rho}{2}) > 0$. For any x with $1 - u_\alpha(x) \ge \rho$, we have

$$\begin{aligned}\mathbf{P}^0_x(\sigma > \delta) &\ge \mathbf{E}^0_x\left[1 - e^{-\alpha\sigma}; \sigma > \delta\right] = \mathbf{E}^0_x\left[1 - e^{-\alpha\sigma}\right] - \mathbf{E}^0_x\left[1 - e^{-\alpha\sigma}; \sigma \le \delta\right]\\ &\ge 1 - u_\alpha(x) - (1 - e^{-\alpha\delta}) \ge \rho - (1 - e^{-\alpha\delta}) = \frac{\rho}{2}.\end{aligned}$$

Therefore if we define

$$\tau := \inf\{t > 0 :\ 1 - u_\alpha(w(t)) > \rho\},$$

then for any neighborhood U of a,

$$\mathbf{n}^+(\widetilde{W}_\rho) = \mathbf{n}^+(\tau < \zeta^0) = \lim_{\epsilon\downarrow 0} \mathbf{n}^+(\epsilon < \tau < \zeta^0)$$
$$= \lim_{\epsilon\downarrow 0} \int_{E_0} \mu_\epsilon(dx)\mathbf{P}^0_x(\tau < \zeta^0 < \infty)$$
$$\leq \liminf_{\epsilon\downarrow 0} \int_{E_0} \mu_\epsilon(dx)\mathbf{E}^0_x \left[\left(\frac{2}{\rho}\right)\mathbf{P}^0_{X^0_\tau}(\sigma > \delta); \tau < \zeta^0\right]$$
$$\leq \frac{2}{\rho}\liminf_{\epsilon\downarrow 0} \int_{E_0} \mu_\epsilon(dx)\mathbf{P}^0_x(\sigma > \delta,\ \zeta^0 < \infty)$$
$$\leq \frac{2}{\rho}\lim_{\epsilon\downarrow 0} \int_{E_0} \mu_\epsilon(dx)\mathbf{P}^0_x(\zeta^0 > \delta) + \frac{2}{\rho}\lim_{\epsilon\downarrow 0} \int_{E_0} \mu_\epsilon(dx)\mathbf{P}^0_x(\zeta^0 < \infty, X_{\zeta^0-} = \Delta)$$
$$\leq \frac{2}{\rho}\lim_{\epsilon\downarrow 0} \mu_{\epsilon+\delta}(E_0) + \frac{2}{\rho}\mathbf{n}(\tau_U < \zeta),$$

which is finite in view of (5.10) and Lemma 5.4. □

In last subsection, we have constructed a process $X^a = \{X^a_t, t \in [0, \zeta_\omega)\}$ starting from a out of the Poisson point processes $\mathbf{p}^+$ and $\mathbf{p}^-$ on W^+_a and $W^-_a \cup \{\partial\}$ defined on a probability space $(\Omega, \mathbf{P})$, respectively. A process $\{\widehat{X}^a_t,\ t \in [0, \widehat{\zeta}_{\widehat{\omega}})\}$ can be constructed similarly.

Proposition 5.12. *Let $v(x) = G_\alpha f$ with $f \in C_b(E)$ be defined by* (5.38) *and* (5.40). *Then $v(X^a_t)$ is right continuous in $t \in [0, \zeta_\omega)$ and is continuous when $X_t = a$ for every $f \in \mathbf{L}$ and every $\alpha \in Q^+$ $\mathbf{P}$-a.s. An analogous property holds for $\widehat{X}^a$.*

Proof. We already saw that the functions u_α and $G^0_\alpha f$ for $f \in \mathbf{L},\ \alpha \in \mathbf{Q}^+$, have the property (5.48) along any sample point functions of $\mathbf{p}^+ = \{\mathbf{p}^+_s, s > 0\}$ and $\mathbf{p}^- = \{\mathbf{p}^-_s, s > 0\}$. Moreover, by Lemma 5.11., after subtracting a suitable $\mathbf{P}$-negligible set from Ω if necessary, we can assume that, in addition to the properties (5.28), (5.29) and (5.30), $\mathbf{p}^+$ satisfies the following property for every sample point $\omega \in \Omega$: for any finite interval $I \subset (0, \infty)$ and for any $\rho \in (0, 1)$,

$$\left\{ s \in I : \sup_{0 \leq t \leq \zeta(\mathbf{p}^+_s)} (1 - u_\alpha(\mathbf{p}^+_s(t))) > \rho \right\} \text{ is a finite set.} \tag{5.49}$$

Combining this with the inequality $\alpha G^0_\alpha 1(x) \leq 1 - u_\alpha(x),\ x \in E$, it is not hard to see that $u_\alpha(X^a_t),\ G^0_\alpha f(X^a_t)$ and hence $v(X^a_t)$ enjoy the properties in the statement of the proposition. □

5.4 Constructing a Standard Process X on $E_0 \cup \{a\}$

Combining the given standard process X^0 on E_0 with the process X^a constructed and studied in the last two subsections, we can now construct a right process X on $E := E_0 \cup \{a\}$ whose resolvent coincides with $\{G_\alpha, \alpha > 0\}$ defined by (5.38) and (5.40). We will only do the construction of X. But obviously the analogous procedure allows us to construct out of $\widehat{X}^0$ a right process $\widehat{X}$ on E with resolvent given by (5.39) and (5.41), and these two right processes on E are in weak duality with respect to m if and only if their killing rates δ_0 and $\widehat{\delta}_0$ at a satisfy the relation (4.10).

With the preparations made in the last subsections, we can now just follow the corresponding arguments in [16, §4] without any essential change to construct the desired process X on E.

First, using the approaching time σ to a of X^0 defined in the beginning of the last subsection, we define $P_t f(x)$ for $t > 0, x \in E, f \in \mathcal{B}(E)$, as follows:

$$P_t f(a) := \mathbf{E}\left(f(X_t^a);\ t < \zeta_\omega\right), \tag{5.50}$$

$$P_t f(x) := P_t^0 f(x) + \mathbf{E}_x^0\left[P_{t-\sigma} f(a);\ \sigma \le t\right] \qquad \text{for } x \in E_0. \tag{5.51}$$

Evidently the Laplace transform of P_t equals the resolvent G_α in view of (5.37) and (5.40) and we can see exactly in the same way as the proof of [16, Lemma 4.10] that $\{P_t,\, t \ge 0\}$ is a sub-Markovian transition semigroup on E:

$$P_{t+s} = P_t P_s \quad \text{with} \quad P_t 1 \le 1 \quad \text{for } t, s > 0.$$

Proposition 5.13. (i) $X^a = \{X_t^a, 0 \le t < \zeta_\omega, \mathbf{P}\}$ *is a Markov process on* E *starting from* a *with transition semigroup* $\{P_t, t > 0\}$.

(ii) $\mathbf{P}(\sigma_a = 0,\ \tau_a = 0) = 1$, *where* $\sigma_a = \inf\{t > 0 : X_t^a = a\}$ *and* $\tau_a = \inf\{t > 0 : X_t^a \in E_0\}$.

Proof. The proof of [16, Proposition 4.4] still works to obtain the first assertion (i). The only places to be modified in the proof are to replace $L(m_0, \psi)$ appearing there with $L^0(\widehat{\varphi} \cdot m_0, 1 - \varphi) + \delta_0$ in the present case.

The second assertion (ii) follows from (i) and Proposition 5.12. just as the proof of [16, Lemma 4.12]. □

In §5.1, we have started with a standard process

$$X^0 = \left\{X_t^0,\ 0 \le t < \zeta^0,\ \mathbf{P}_x^0,\ x \in E_0\right\}$$

on E_0, where $\mathbf{P}_x^0,\ x \in E_0$, are probability measures on a certain sample space, say Ω^0.

In §5.2, we have constructed a cadlag process

$$X^a = \{X_t^a(\omega'),\ 0 \le t < \zeta_{\omega'},\ \mathbf{P}\}$$

on E starting from a by piecing together excursions away from a, where $\mathbf{P}$ is a probability measure on another sample space, say Ω', to define the Poisson point process with value in $(W_a \cup \{\partial\},\ \bar{\mathbf{n}})$.

For convenience, we assume that Ω^0 contains an extra path η with $\mathbf{P}^0_x(\{\eta\}) = 0$ for every $x \in E_0$, and we set $\mathbf{P}^0_a = \delta_\eta$, η representing the constant path taking value a identically.

We now define

$$\Omega = \Omega^0 \times \Omega', \qquad \mathbf{P}_x = \mathbf{P}^0_x \times \mathbf{P} \quad \text{for } x \in E. \tag{5.52}$$

Note that $\zeta^0(\omega^0) \le \sigma(\omega^0)$ and $\zeta^0(\omega^0) = \sigma(\omega^0)$ when $\sigma(\omega^0) < \infty$. For $\omega = (\omega^0, \omega') \in \Omega$, let us define $X_t = X_t(\omega)$ as follows:

(1) When $\omega^0 \in \Omega^0 \setminus \{\eta\}$,

$$X_t(\omega) = \begin{cases} X^0_t(\omega^0) & 0 \le t < \zeta^0(\omega^0) \le \sigma(\omega^0) \le \infty \\ X^a_{t-\sigma(\omega^0)}(\omega') & \sigma(\omega^0) \le t < \sigma(\omega^0) + \zeta_{\omega'},\ \text{if } \sigma(\omega^0) < \infty. \end{cases} \tag{5.53}$$

(2) When $\omega^0 = \eta$,

$$X_t(\omega) = X^a_t(\omega') \quad \text{for } 0 \le t < \zeta_{\omega'}. \tag{5.54}$$

The lifetime $\zeta(\omega)$ of $X_t(\omega)$ is defined by

$$\zeta(\omega) = \begin{cases} \zeta^0(\omega^0) & \text{if } \sigma(\omega^0) = \infty, \\ \sigma(\omega^0) + \zeta_{\omega'} & \text{if } \sigma(\omega^0) < \infty. \end{cases} \tag{5.55}$$

Combining Proposition 5.13.(i) with the Markov property of $\{X^0_t,\ t \ge 0,\ \mathbf{P}^0_x,\ x \in E_0\}$, we readily get as in [16, Lemma 4.13] the next lemma:

Lemma 5.14. *$X = \{X_t,\ 0 \le t < \zeta,\ \mathbf{P}_x,\ x \in E\}$ is a Markov process on E with transition semigroup $\{P_t, t \ge 0\}$ defined by* (5.50) *and* (5.51)*.*

The resolvent $\{G_\alpha,\ \alpha > 0\}$ of the Markov process X is defined by

$$G_\alpha f(x) = \mathbf{E}_x\left[\int_0^\infty e^{-\alpha t} f(X_t)dt\right], \qquad x \in E,\ \alpha > 0,\ f \in \mathcal{B}(E). \tag{5.56}$$

The resolvent of X^0 is denoted by G^0_α.

Theorem 5.15. *The process X enjoys the following properties:*

(i) *X is a right process on E. Its sample path $\{X_t,\ 0 \le t < \zeta\}$ is cadlag on $[0, \infty)$, continuous when $X_t = a$ and satisfies*

$$X_{\zeta-} \in \{a, \Delta\} \quad \text{when} \quad \zeta < \infty.$$

(ii) *The point a is regular for itself with respect to X in the sense that for the hitting time $\sigma_a = \inf\{t > 0 : X_t = a\}$*

$$\mathbf{P}_a(\sigma_a = 0) = 1.$$

(iii) X^0 *is identical in law with the subprocess of* X *killed upon hitting* a.
(iv) *The resolvent* $G_\alpha f$ *admits the expression* (5.38) *and* (5.40) *for* $f \in \mathcal{B}(E)$.
(v) *If* X^0 *is a diffusion on* E_0*, then* X *is a diffusion on* E.

Proof. (iv) follows from Lemma 5.14. and a statement next to (5.51).
(i). On account of **(A.1)**, we may assume that

$$X_t^0(\omega^0) \text{ is cadlag in } t \in [0, \zeta^0(\omega^0)) \quad \text{and}$$
$$X^0_{\zeta^0(\omega^0)-}(\omega^0) \in \{a \cup \Delta\} \text{ when } \zeta^0(\omega^0) < \infty,$$

for every $\omega^0 \in \Omega^0$. We have already chosen Ω' in a way that $\{X_t^a(\omega'), 0 \leq t < \zeta_{\omega'}\}$ has the property (5.36). Hence the sample path $t \mapsto X_t(\omega)$ has the stated property in (i).

Take a countable linear subspace $\mathbf{L}$ of $C_b(E)$ such that, for any open set $G \subset E$, there exist functions $f_n \in \mathbf{L}$ increasing to I_G. We then see from the expression (5.40) of $G_\alpha f$, (5.47) and Proposition 5.12. that, for any $v = G_\alpha f$ with $f \in \mathbf{L},\ \alpha \in Q^+$,

$$v(X_t) \text{ is right continuous in } t \in [0, \zeta) \quad \mathbf{P}_x\text{-a.s. for } x \in E.$$

Therefore X is strong Markov by [2, p. 41].
(ii) follows from Proposition 5.13.(ii).
(iii) and (v) are also evident from the construction of X. □

The right process X in the above theorem becomes a standard process if either condition **(A.1)** or **(A.4)** is replaced by the following stronger counterpart, respectively:

(A.1)' X^0 and $\widehat{X}^0$ are standard processes on E_0 in weak duality with respect to m and

$$\text{every semipolar set is } m\text{-polar for } X^0. \tag{5.57}$$

(A.4)' For any $\alpha > 0$, $u_\alpha,\ \widehat{u}_\alpha \in C_b(E_0)$ and

$$G_\alpha^0(C_b(E_0)) \subset C_b(E_0),\ \widehat{G}_\alpha^0(C_b(E_0)) \subset C_b(E_0).$$

We note that condition (5.57) is automatically satisfied if X^0 is m-symmetric or more generally if the Dirichlet form of X^0 on $L^2(E_0; m_0)$ is sectorial (cf. [4]). **(A.4)'** implies **(A.4)** as we noted right after the statement of the latter. Recall that a right process is called a standard process if it is quasi-left continuous up to the lifetime.

Theorem 5.16. (i) *Suppose that the standard processes* X^0 *and* $\widehat{X}^0$ *on* E_0 *satisfy* **(A.1)**, **(A.2)**, **(A.3)**, **(A.4)'** *and additionally* **(A.5)** *in non-symmetric case and* **(A.6)** *in non-diffusion case. Then the right process* X *on* E *in* Theorem 5.15. *is quasi-left continuous up to the lifetime.*

(ii) *Suppose that the standard processes X^0 and $\widehat{X}^0$ on E_0 satisfy* **(A.1)'**, **(A.2)**, **(A.3)**, **(A.4)** *and additionally* **(A.5)** *in non-symmetric case and* **(A.6)** *in non-diffusion case. Then the right process X on E in* Theorem 5.15 *is quasi-left continuous up to the lifetime for X-q.e. starting point $x \in E$.*

Proof. (i) If condition **(A.4)'** is satisfied, then along any cadlag path of X^0, we trivially have

$$\lim_{s \uparrow t} u_\alpha(X_s^0) = u_\alpha(X_{t-}^0) \quad \text{and} \quad \lim_{s \uparrow t} G_\alpha^0 f(X_s^0) = G_\alpha^0(X_{t-}) \qquad \text{for } t \in (0, \zeta^0), \tag{5.58}$$

for any $\alpha > 0$ and $f \in C_b(E_0)$. Combining this with Lemma 5.10.(i) and Lemma 5.11., we easily see as in the proofs of Proposition 5.12. and Theorem 5.15.(i) that

$$\lim_{s \uparrow t} G_\alpha f(X_s) = G_\alpha f(X_{t-}), \quad t \in (0, \zeta), \quad \mathbf{P}_x\text{-a.s.} \tag{5.59}$$

for any $x \in E$ and for any $\alpha > 0$, $f \in C_b(E)$, from which the quasi-left continuity of X follows.

(ii) Here we use the terminologies adopted in [5]. From condition **(A.1)'**, we can deduce as in [5, Lemma 2.2] that (5.58) holds $\mathbf{P}_x^0$-a.s. for X^0-q.e. $x \in E_0$ for each $\alpha > 0$ and each $f \in C_b(E_0)$. In particular, there exists a Borel set $B \subset E_0$ with $m(B) = 0$ such that $E_0 \setminus B$ is X^0-invariant and (5.58) holds $\mathbf{P}_x^0$-a.s. for any $x \in E_0 \setminus B$ and for any $\alpha \in Q^+$, $f \in \mathbf{L}$, where $\mathbf{L}$ is a countable subfamily of $C_b(E_0)$.

Let us observe that the set $E \setminus B$ is invariant for X of Theorem 5.15. Since the restriction of X^0 to the Lusin space $E_0 \setminus B$ is a standard process again, the entrance law $\{\mu_t, t > 0\}$ uniquely characterized by the equation (5.7) is carried by $E_0 \setminus B$ for every $t > 0$ and accordingly the excursion law $\mathbf{n}$ of Proposition 5.1. is carried by the path space (5.23) with E, E_0 being replaced by $E \setminus B$, $E_0 \setminus B$ respectively. Hence $E \setminus B$ is X-invariant by the construction of X.

Now we can see by the same reasoning as in the proof of (i) that (5.59) holds for any $x \in E \setminus B$ and for any $\alpha \in Q^+$, $f \in \mathbf{L}$. Taking $\mathbf{L}$ as in the proof of Theorem 5.15.(i), we conclude that X is quasi-left continuous for every starting point $x \in E \setminus B$. □

To formulate the last theorem in this section, we need the following stronger variant **(A.2)'** of the condition of **(A.2)**:

(A.2)' For every $x \in E_0$,

$$\mathbf{P}_x^0(\zeta^0 < \infty, X_{\zeta^0-}^0 = a) > 0, \qquad \mathbf{P}_x^0(X_{\zeta^0-}^0 \in \{a, \Delta\}) = 1,$$
$$\widehat{\mathbf{P}}_x^0(\widehat{\zeta}^0 < \infty, \widehat{X}_{\widehat{\zeta}^0-}^0 = a) > 0, \qquad \widehat{\mathbf{P}}_x^0(\widehat{X}_{\widehat{\zeta}^0-}^0 \in \{a, \Delta\}) = 1.$$

Theorem 5.17. *We assume that $m_0(U \cap E_0) < \infty$ for some neighborhood U of a in E. Suppose that the pair of standard processes X^0 and $\widehat{X}^0$ on E_0 satisfy the conditions* **(A.1)**, **(A.2)'**, **(A.4)** *and additionally* **(A.5)** *in non-symmetric case and* **(A.6)** *in non-diffusion case. Then the integrability condition* **(A.3)** *is fulfilled by X^0 and $\widehat{X}^0$.*

Proof. Note that the condition **(A.3)** holds if $m_0(E_0) < \infty$. When $m_0(E_0) = \infty$, let $\gamma(x)$ be a continuous function on E_0 such that $0 < \gamma(x) \le 1$ on E_0, $\gamma(x) = 1$ on $U \cap E_0$ and $\int_{E_0} \gamma(x) m_0(dx) < \infty$. Define for $t > 0$,

$$\tau_t := \inf\left\{ s > 0 : \int_0^s \gamma(X_r^0)dr > t \right\}$$

and

$$\widehat{\tau}_t := \inf\left\{ s > 0 : \int_0^s \gamma(\widehat{X}_r^0)dr > t \right\}.$$

Then the time changed processes $Y^0 = \{Y_t^0 := X_{\tau_t}^0,\ t \ge 0\}$ and $\widehat{Y}^0 = \{\widehat{Y}_t^0 := \widehat{X}_{\widehat{\tau}_t}^0,\ t \ge 0\}$ are standard processes on E_0 satisfying **(A.1)** with respect to the finite measure $\mu_0 = \gamma(x)m_0(dx)$. Clearly condition **(A.3)** holds for Y^0 and the reference measure μ_0. Note that since $\gamma(x) \le 1$, we have

$$\tau_t \ge t \quad \text{and} \quad \widehat{\tau}_t \ge t \qquad \text{for every } t \ge 0.$$

Let $G_\alpha^{Y^0}$ denote the 0-order resolvent of Y^0. It is easy to check that for any non-negative Borel function f on E_0, $G^{Y^0} f = G^0(\gamma f)$. Therefore Y^0 and $\widehat{Y}^0$ inherit the conditions **(A.2)'**, **(A.4)** and in non-symmetric case **(A.5)** from X^0 and $\widehat{X}^0$.

Let (N, H) be a Lévy system of X^0. Since its defining formula (5.4) remains valid with the constant time t being replaced by any stopping time, it follows from it and a time change that Y^0 has a Lévy system (N, H^{Y^0}), where

$$H_t^{Y^0} = H_{\tau_t} \qquad \text{for every } t \ge 0.$$

According to [10, Theorem 6.2], the correspondence between PCAF and its Revuz measure is invariant under a strictly increasing time change. Therefore the Revuz measure of the PCAF of H^{Y^0} with respect to the measure μ_0 is the same as that μ_H of PCAF H of X^0 with respect to the measure m. Hence Y^0 has the same jumping measure $J_0(dx, dy) := N(x, dy)\mu_H(dy)$ as that of X^0. The same applies to $\widehat{Y}^0$. Therefore Y^0 and $\widehat{Y}^0$ also inherit the condition **(A.6)** from X^0 and $\widehat{X}^0$.

Thus by Theorem 5.15., there are duality preserving standard processes Y and $\widehat{Y}$ on $E = E_0 \cup \{a\}$ extending Y^0 and $\widehat{Y}^0$. Define for $t > 0$,

$$\sigma_t := \inf\left\{ s > 0 : \int_0^s \gamma(Y_r)^{-1}dr > t \right\}$$

and

$$\widehat{\sigma}_t := \inf \left\{ s > 0 : \int_0^s \gamma(\widehat{Y}_r)^{-1} dr > t \right\}.$$

Then $X = \{X_t := Y_{\sigma_t}, t \geq 0\}$ and $\widehat{X} = \{\widehat{X}_t := \widehat{Y}_{\widehat{\sigma}_t}, t \geq 0\}$ is a pair of standard processes on E in weak duality with respect to m. Clearly X and $\widehat{X}$ extend X^0 and $\widehat{X}^0$, they spend zero Lebesgue amount of time at $\{a\}$, and for X and Y, a is a regular point for $\{a\}$. Therefore by Proposition 4.1.(v), X^0 and $\widehat{X}^0$ must have the property **(A.3)**. □

Remark 3. In this section, we have assumed that E is a locally compact separable metric space, a is a non-isolated point of E and Δ is added to E as a one-point compactification. This assumption is used only to have (5.20) and (5.22).

The local compactness assumption on E can be relaxed and be replaced by the following conditions. Let E be a Lusin space and a a non-isolated point of E and m_0 be a σ-finite measure on $E_0 := E \setminus \{a\}$. Let Δ be a cemetery point added to E. Let X^0 and $\widehat{X}^0$ be Borel standard processes on E_0 with lifetimes ζ^0 and $\widehat{\zeta}^0$, respectively.

We say $X^0_{\zeta^0-} = a$ if $\lim_{t \uparrow \zeta^0} X_t = a$ under the topology of E, and $X^0_{\zeta^0-} = \Delta$ if the limit $\lim_{t \uparrow \zeta^0} X_t$ does not exist in the topology of E. The same applies to the process $\widehat{X}^0$.

Let $\{\mathcal{F}^0_t, t \geq 0\}$ be the minimal admissible completed σ-field generated by X^0. We assume X^0 and $\widehat{X}^0$ satisfy the conditions **(A.1)**, **(A.4)'** and additionally **(A.5)** in non-symmetric case and **(A.6)** in non-diffusion case. We also assume, instead of **(A.2)**, that

(A.2)" There is an open neighborhood U_1 of a such that its closure $\overline{U_1}$ is compact in E. Further

$$\zeta^0 \text{ is } \{\mathcal{F}^0_t\}\text{-predictable}, \ \varphi(x) > 0 \text{ on } E_0, \text{ and } \liminf_{x \to a} \varphi(x) > 0, \tag{5.60}$$

$$\widehat{\zeta}^0 \text{ is } \widehat{\mathcal{F}}^0_t\text{-predictable}, \ \widehat{\varphi}(x) > 0 \text{ on } E_0, \text{ and } \liminf_{x \to a} \widehat{\varphi}(x) > 0, \tag{5.61}$$

where φ is defined by (5.3) and $\widehat{\varphi}$ is defined analogously for $\widehat{X}^0$.

We claim that under the above assumptions, all the main results in this section, including Theorem 5.15., remain true. Note that the existence of an open neighborhood U_1 of a with $\overline{U_1}$ being compact in E guarantees the validity of (5.20). So it suffices to show that (5.22) holds almost surely under measure $\mathbf{n}$ for some neighborhood U of a under condition (5.60). As $c := \liminf_{x \to a} \varphi(x) > 0$ and φ is lower semi-continuous by **(A.4)'**, $U := \{x \in E_0 : \varphi(x) > c/2\} \cup \{a\}$ is an open neighborhood of a. On the other hand, for $x \in E_0$, we have $\mathbf{P}^0_x$-a.s. on $\{t < \zeta^0\}$,

$$\varphi(X^0_t) = \mathbf{E}_x \left[1_{\{\zeta^0 < \infty \text{ and } X^0_{\zeta^0-} = a\}} \middle| \mathcal{F}^0_t \right].$$

As ζ^0 is $\{\mathcal{F}_t^0\}$-predictable, it follows that

$$\lim_{t\uparrow\zeta^0}\varphi(X_t) = 1_{\left\{\zeta^0<\infty \text{ and } X^0_{\zeta^0-}=a\right\}} \qquad \mathbf{P}_x\text{-a.s. for every } x\in E_0.$$

Hence

$$\left\{\zeta^0<\infty \text{ and } X^0_{\zeta^0-}=\Delta\right\}\subset\left\{\tau_U^0<\zeta^0\right\} \qquad \mathbf{P}_x\text{-a.s. for every } x\in E_0.$$

Here $\tau_U^0 := \inf\{t>0:\ X_t^0\notin U\}$. This shows that (5.22) almost surely under measure $\mathbf{n}$. Since condition **(A.2)"** is invariant under the strict time change as in the proof of the preceding theorem, condition **(A.3)** is automatically satisfied. This proves our claim.

Note that condition (5.60) is weaker than the following condition

$$\mathbf{P}_x^0\left(\zeta^0<\infty\right)=\mathbf{P}_x^0\left(\zeta^0<\infty,\ X^0_{\zeta-}=a\right) \qquad \text{for every } x\in E_0. \tag{5.62}$$

□

6 Examples and Application

Several basic examples of Theorem 5.15. have been exhibited in [16, §6] when X^0 are symmetric diffusions on E_0 in which cases their extensions X are symmetric diffusions on E by [16, Theorem 4.1] there or by Theorem 5.15.(v) of the present paper. In this section, we first consider a simple case where X^0 is of pure jump type and admits no killings inside E_0. A typical example of such a process is a censored stable process on an Euclidean open set studied in [3]. We then consider the case that X^0 is an absorbing barrier non-symmetric diffusion on an Euclidean domain. As an application, we finally consider an extension of X^0 by reflecting at infinitely many holes (obstacles).

6.1 Extending Censored Stable Processes in Euclidean Domains

Let D be an open *n-set* in $\mathbb{R}^n$, that is, there exists a constant $C_1>0$ such that

$$m(B(x,r))\ge C_1\, r^n \qquad \text{for all } x\in D \text{ and } 0<r\le 1.$$

Here m is the Lebesgue measure on $\mathbb{R}^n$, $B(x,r):=\{y\in\mathbb{R}^n: |x-y|<r\}$ and $|\cdot|$ is the Euclidean metric in $\mathbb{R}^n$. Note that bounded Lipschitz domains in $\mathbb{R}^n$ are open n-set and any open n-set with a closed subset having zero Lebesgue measure removed is still an n-set. For an n-set D (which can be disconnected), consider for $0<\alpha<2$ the Dirichlet space defined by

$$\mathcal{F}=\left\{u\in L^2(D;dx):\int_{D\times D}\frac{(u(x)-u(y))^2}{|x-y|^{n+\alpha}}dxdy<\infty\right\},$$

$$\mathcal{E}(u,v)=\mathcal{A}_{n,\alpha}\int_{D\times D}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+\alpha}}dxdy,\quad u,v\in\mathcal{F},$$

with $\mathcal{A}_{n,\alpha}=\frac{\alpha 2^{\alpha-1}\Gamma(\frac{\alpha+n}{2})}{\pi^{n/2}\Gamma(1-\frac{\alpha}{2})}$. When $D=\mathbb{R}^n$, $(\mathcal{E},\mathcal{F})$ is just the Dirichlet form on $L^2(\mathbb{R}^n,dx)$ of the symmetric α-stable process on $\mathbb{R}^n$.

We refer the reader to [3] for the following facts. The bilinear form $(\mathcal{E},\mathcal{F})$ is a regular irreducible Dirichlet form on $L^2(\overline{D};1_D(x)dx)$ and the associated Hunt process X on $\overline{D}$ may be called a *reflected α-stable process*. It is shown in [6] that X has Hölder continuous transition density functions with respect to the Lebesgue measure dx on $\overline{D}$ and therefore X can be refined to start from every point in $\overline{D}$.

The process $X^0=(X_t^0,\mathbf{P}_x^0,\zeta^0)$ obtained from X by killing upon leaving D is called the *censored α-stable process* in D, which has been studied in detail in [3]. The process X^0 is symmetric with respect to the Lebesgue measure and its Dirichlet form on $L^2(D,dx)$ is given by $(\mathcal{E},\mathcal{F}^0)$, where $\mathcal{F}^0$ is the closure of $C_0^1(D)$ in $\mathcal{F}$ with respect to $\mathcal{E}_1:=\mathcal{E}+(\cdot,\cdot)_{L^2(D,dx)}$. The process X^0 has no killings inside D in the sense that

$$\mathbf{P}_x\left(\zeta^0<\infty \text{ and } X^0_{\zeta^0-}\in D\right)=0\qquad\text{for every } x\in D.$$

Let $\tau_D:=\inf\{t>0:\ X_t\notin D\}$. Note that for $\beta>0$, $u_\beta(x)=\mathbf{E}_x\left[e^{-\beta\tau_D})\right]$ is a β-harmonic function of X^0 and so it is continuous on D (see [3, (3.8)]). For any bounded measurable function f on D, we extend its definition of $\overline{D}$ by defining $f(x)=0$ on ∂D. By [6], $G_\alpha f(x):=\mathbf{E}_x\left[\int_0^\infty e^{-\beta t}f(X_t)dt\right]$ is a continuous function on $\overline{D}$. Applying strong Markov property of X at its first exit time τ_D from D, we have for $G^0_\beta f(x):=\mathbf{E}_x\left[\int_0^{\tau_D}e^{-\beta t}f(X_t)dt\right]$,

$$G^0_\beta f(x)=G_\beta f(x)-\mathbf{E}_x\left[e^{-\beta\tau_D}G_\beta f(X_{\tau_D})\right]\qquad\text{for } x\in D.$$

Since $x\mapsto\mathbf{E}_x\left[e^{-\beta\tau_D}G_\beta f(X_{\tau_D})\right]$ is a β-harmonic function of X^0 and thus it is continuous on D, we conclude that $G^0_\beta f$ is continuous on D. Hence the conditions **(A.1)** and **(A.4)'** in §5 are always satisfied for censored α-stable process in any open n-set D. In view of [15, §5.3], a Lévy system of X^0 is given by $(N(x,dy),dt)$ with

$$N(x,dy)=2\mathcal{A}_{n,\alpha}\,|x-y|^{-(n+\alpha)}dy$$

and the condition **(A.6)** of §5 is clearly satisfied.

Note that if D_1 is an open subset of D, then X and its subprocess killed upon leaving D_1 have the same class of m-polar sets in D_1. If a closed set $\Gamma\subset\partial D$ has a locally finite and strictly positive d-dimensional Hausdorff measure when $n\geq 2$ and is non-empty when $n=1$, then by [3, Theorem 2.5 and Remark 2.2(i)]

$$\varphi_\Gamma(x):=\mathbf{P}^0_x(\zeta^0<\infty,\ X^0_{\zeta^0-}\in\Gamma)>0\qquad\text{for every } x\in D\qquad(6.1)$$

if and only if $\alpha>n-d$ when $n\geq 2$ and $\alpha>1$ when $n=1$.

In the following $D \subset \mathbb{R}^n$ is a proper open n-set, Γ is a closed subset of ∂D that satisfies the Hausdorff dimensional condition proceeding (6.1). The topology on $D^* = D \cup \{a\}$ will be defined in the following three special cases separately.

(i) D is an open n-set, $\Gamma = \partial D$, and $\alpha \in (n-d,\, n)$. Let D^* be the one point compactification of D. Note that $\varphi(x) = 1$ on D with D is bounded, and $0 < \phi < 1$ on D when D is unbounded with compact boundary.

(ii) D is an n-open set having disconnected boundary ∂D. A prototype is a bounded domain D with one or several holes in its interior. Suppose that $\partial D = \Gamma \cup \Gamma_2$, where Γ and Γ_2 are non-trivial disjoint open subsets of ∂D, with Γ being compact and satisfying the Hausdorff dimensional condition proceeding (6.1) and $\alpha \in (n-d,\, n)$. In this case, $0 < \varphi_\Gamma(x) \le 1$ for $x \in D$. We prescribe a topology on D^* as follows. A subset $U \subset D^*$ containing the point $\{a\}$ is a neighborhood of a if there is an open set $U_1 \subset \mathbb{R}^d$ containing Γ_1 such that $U_1 \cap D = U \setminus \{a\}$. In other words, $D^* = D \cup \{a\}$ is obtained from D by identifying Γ into one point $\{a\}$.

(iii) $\alpha > 1 = n$, $D = (0, \infty)$ and $\Gamma = \{0\}$. In this case $\varphi_\Gamma(x) = 1$. $D^* = [0, \infty)$.

In every case, condition **(A.2)'** in §5 is fulfilled. Indeed the first half of **(A.2)'** follows from (6.1). Its second half can be also verified although the proof will be spelled out elsewhere. Consequently, condition **(A.3)** is automatically satisfied by Theorem 5.17.. Therefore, in each case, we can construct the extension X on D^* of X^0 on D satisfying the properties of Theorem 5.15. by means of the Poisson point process around $\{a\}$. X is a standard process by Theorem 5.16. but admits no jump from D to a nor from a to D.

In case **(iii)**, X coincides with the process on $[0, \infty)$ considered in the beginning of this section and may be called a reflecting α-stable process. But it differs from the two closely related processes on $[0, \infty)$ that are defined by the symmetric α-stable process x_t on $\mathbb{R}$ as

$$X_t^{(1)} = \begin{cases} x_t & t < \sigma_0 \\ x_t - \inf_{\sigma_0 \le s \le t} x_s & t \ge \sigma_0 \end{cases}, \qquad X_t^{(2)} = |x_t|,$$

and investigated in detail by S. Watanabe [W], because both $X^{(1)}$ and $X^{(2)}$ admit jumps from $(0, \infty)$ to 0.

Note that given an open n-set with disconnected boundary, extensions in case **(i)** and **(ii)** can be different. For example for $D = \{x \in \mathbb{R}^n : 1 < |x| < 2\}$ with $\Gamma := \{x \in \mathbb{R}^n : |x| = 1\}$, the process X in case **(ii)** is transient and gets "birth" only when X^0 approaches Γ, while in case **(i)**, the extension process is conservative and gets "birth" when X^0 approaches ∂D.

6.2 Extending Non-Symmetric Diffusions in Euclidean Domains

Let D be a proper domain in $\mathbb{R}^n$ and m be the Lebesgue measure on D. Assume that ∂D is regular for Brownian motion, or, equivalently, for $\frac{1}{2}\Delta$. Let

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\nabla\cdot(a\nabla) + b\cdot\nabla \\ &= \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial}{\partial x_j}\Big) + \sum_{i=1}^{n} b_i\frac{\partial}{\partial x_i},\end{aligned}$$

where $a : \mathbb{R}^n \to \mathbb{R}^d \otimes \mathbb{R}^n$ is a measurable, symmetric $(n \times n)$-matrix-valued function which satisfies the uniform elliptic condition

$$\lambda^{-1} I_{n\times n} \le a(\cdot) \le \lambda I_{n\times n}$$

for some $\lambda \ge 1$ and $b = (b_1, \cdots, b_n) : \mathbb{R}^n \to \mathbb{R}^n$ are measurable functions which could be singular such that

$$1_D|b|^2 \in \mathbf{K}(\mathbb{R}^n), \qquad \sum_{i=1}^{n}\frac{\partial b_i}{\partial x_i} = 0 \text{ on } D.$$

Here $\mathbf{K}(\mathbb{R}^n)$ denote the Kato class functions on $\mathbb{R}^n$. We refer the reader to [7] for its definition. We only mention here that $L^p(\mathbb{R}^n, dx) \subset \mathbf{K}(\mathbb{R}^n)$ for $p > n/2$.

Let X^0 be the diffusion in D with infinitesimal generator $\mathcal{L}$ with Dirichlet boundary condition on ∂D. It is clearly that X^0 has a weak dual diffusion $\widehat{X}^0$ in D with respect to the Lebesgue measure m on D whose generator is $\mathcal{L}^*$, the dual operator of $\mathcal{L}$ with Dirichlet boundary condition on ∂D so that X^0 satisfies condition **(A.1)**. The conditions **(A.4)', (A.5)** are satisfied by [7, Lemma 5.7 and Theorem 5.11]. Condition **(A.2)'** is also satisfied. Its first half is clear and the proof of the second half will be spelled out elsewhere. So condition **(A.3)** is automatically satisfied by Theorem 5.17. and we can apply Theorem 5.15. to construct a weak duality preserving diffusion extension X of X^0 to $D^* := D \cup \{a\}$, where the topology on D^* can be prescribed as in the three special cases **(i)**-**(iii)** in §6.1.

6.3 Extending by Reflection at Infinitely Many Holes

In this paper, we restrict ourself to consider duality preserving one-point extension of standard processes X^0 and $\widehat{X}^0$. The method of this paper allows us to do finite many points $\{a_1, \cdots, a_n\}$ or countably infinite many points $\{a_1, \cdots, a_n, \cdots\}$ extensions, with an obviously modified conditions on a_j's and with no killings at nor direct jumps between $\{a_1, a_2, \cdots\}$, provided that X^0 is symmetric (that is, $X^0 = \widehat{X}^0$). One way to do it is to do one-point extension one at a time. We leave the details to the interested reader.

Thus, for example, consider a domain $D \subset \mathbb{R}^n$ whose complement $\mathbb{R}^n \setminus D$ consists of a countable number of strictly disjoint, non-accumulating compact holes $\{K_1, K_2, \cdots\}$. Let $D^* := D \cup \{a_1, a_2, \cdots\}$ be the topological space obtained by shrinking each set K_i to a point a_i and adding all of them to D. Let $D_0^* = D$ and for each $i \geq 1$, we define $D_i^* := D_{i-1}^* \cup \{a_i\}$, the space obtained by adding K_i to D_{i-1}^* as one point just as in **(ii)** of §6.1. Given an appropriate symmetric Markov process X^0 on D, for $i \geq 1$, the extension X^i to D_i^* can be constructed from X^{i-1} on D_{i-1}^* by means of Theorem 5.15 with $\delta_0 = 0$. The extension X of X^0 on D to $D^* := D \cup \{a_1, a_2, \cdots\}$ is obtained as the limit of X^i's. The process X is then symmetric on D^* and its Dirichlet form may be described in terms of the Feller measure for X^0 on D studied in detail in [4, 13, 25].

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